ON THE MODULAR REPRESENTATION OF THE SYMMETRIC GROUP
PART V

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1. Introduction. It has been observed **(2)** that the number of p-regular classes of S_n, i.e. the number of classes of order prime to p, is equal to the number of partitions (λ) of n in which no summand is repeated p or more times. For this relation to hold *it is essential that p be prime.* It seems natural to call the Young diagram $[\lambda]$ associated with (λ) *p-regular* if no p of its rows are of equal length, otherwise *p-singular.*

The problem considered here is that of refining the above result to prove (§5) that the number of regular diagrams in a given block is equal **(1; 2; 4; 5; 6)** to the number of modular irreducible representations (indecomposables of the regular representation of S_n) in that block. It is interesting to see that all our machinery is required. For example, the notion of an r-Boolean Algebra associated with a given diagram **(8)**, which seemed somewhat of a curiosity at first, plays a central role. In particular, *complementation* in such an rBA can be interpreted in terms of the core and so has significance for the block as a whole (§§2, 3). Similarly, the construction of a diagram with a given core from a knowledge of its p-quotient (star diagram) **(3, 7)** has to be made explicit (§4). This shows up the underlying number-theoretic basis of the theory in a new and significant light.

Actually, we are laying the foundation here for the establishment of an explicit correspondence between the indecomposables of the regular representation and the modular irreducible representations of S_n. The existence of this correspondence for any finite group was demonstrated by R. Brauer and C. J. Nesbitt in 1937, using very general arguments.

2. The complement of a Young diagram in its r-Boolean Algebra. Consider a Young diagram $[\lambda]$ to (from) which can be added (removed) $d(d^*)$ nodes of class r. Such a diagram belongs to an r-Boolean Algebra **(8)** of dimension $d + d^*$ in which the *complement* of $[\lambda]$ is obtained by removing the d^* r-nodes and adding r-nodes in the d free r-positions of $[\lambda]$. Let us denote this uniquely defined complement by $[\tilde{\lambda}]$. As was shown in **(8)**

2.1 $$d - d^* = \delta,$$

so that the core $[\tilde{\lambda}_0]$ of $[\tilde{\lambda}]$ is obtainable by adding δ r-nodes to $[\lambda_0]$ where δ is the r-defect of $[\lambda_0]$. Also, the *weight* of $[\tilde{\lambda}]$ is the same as that of $[\lambda]$. Two

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diagrams of the same block may belong to different rBA's and d and d^* may be different, but δ *is the same in each case*. We prove the following theorem (cf. (**8**, 4.4)):

2.2 *The p-quotients of* $[\lambda]$ *and its complement* $[\tilde{\lambda}]$ *in the appropriate* rBA *are the same except for the interchange of the r and $(r - 1)$-constituents.*

Proof. We distinguish three cases.

(i) If the ith row of $[\lambda]$ ends in a removable r-node and the jth column in an r-position (**8**) then $h_{ij} \equiv 0 (\text{mod } p)$. Clearly, removing the r-node at the end of the ith row and adding an r-node at the foot of the jth column does not change the (i, j)-hook length so that $\tilde{h}_{ij} = h_{ij} \equiv 0 \pmod p$. It does however change the residue class of the hook from r to $r - 1$. Similarly, if the ith row ends in an r-position and the jth column in a removable r-node the length of the (i, j) hook remains the same but its class is changed from $r - 1$ to r.

(ii) If the ith row of $[\lambda]$ ends in a removable r-node and the jth column in an $(r + 1)$-node below which *no* r-node can be added as in Figure 1,

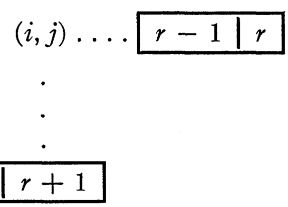

FIG. 1

then removing the r-node yields $\tilde{h}_{i,j-1} = h_{i,j} \equiv 0 \pmod p$. Similarly, if the th column ends in a removable r-node and *no* r-node can be added at the end of the ith row as in Figure 2,

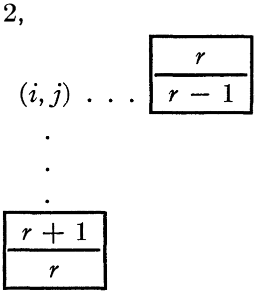

FIG. 2

then removing this r-node yields $\tilde{h}_{i-1,j} = h_{i,j} \equiv 0 \pmod p$. Conversely, we may think of *adding* an r-node in the appropriate r-positions in Figures 1 and 2 to yield similar conclusions.

(iii) In this third case we have to consider (i, j)-hooks such that *no* r-node can be added or removed from the end of the ith row or the jth column. The situation is as indicated in Figure 3.

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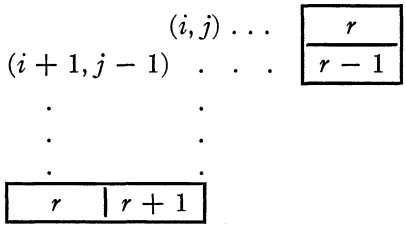

FIG. 3

Taken in conjunction with case (ii) it is clear that if the r-constituent of $[\lambda]_p$ receives a contribution from $h_{ij} \equiv 0 \pmod p$ in case (iii), then also the $(r-1)$-constituent of $[\tilde{\lambda}]_p$ will receive a contribution from $h_{i+1,j-1} \equiv 0 \pmod p$ and *vice versa*. All three cases are concordant, in that an $h \equiv 0 \pmod p$ remains fixed in case (i), or moves one place in its row or column in case (ii), or diagonally in case (iii), so that the r and $r-1$ constituents of $[\lambda]_p$ and $[\tilde{\lambda}]_p$ are interchanged, the others remaining unaltered.

2.3 *Example.* If $[\lambda] = [7, 5, 4^3, 2^2, 1^2]$ for $p = 3$, $r = 1$, then $d = 2$, $d^* = 2$ so that

$$[\lambda]_3 = [3^3], [1], \text{—},$$

where the constituents of $[\lambda]_3$ are associated with the residue classes 0, 1, 2 respectively. We have $[\tilde{\lambda}] = [8, 6, 4^3, 2, 1^2]$ and

$$[\tilde{\lambda}]_3 = [1], [3^3], \text{—}.$$

Each of the changes described in cases (i), (ii), (iii) is illustrated.

3. Regular Young diagrams. Consider the 0-element $[\lambda^0]$ of an rBA for which $d^* = 0$ and let us think of adding an r-node in each of the possible $d = \delta$ r-positions **(8)**, by 2.1. It is possible that the addition of an r-node will lead to a singular diagram; if so, we shall call the corresponding r-position a *singular* position.

Corresponding to a given singular position P there exists a *regular* position P' at which an r-node can be added as indicated in Figure 4, the resulting diagram being regular.

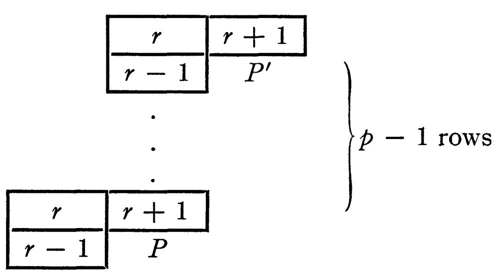

FIG. 4

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Of course P' may itself be a singular position as in the Example 3.1 (whose corresponding regular position is $P'' = (3, 3)$). But the sequence of singular positions must eventually yield a regular position, since adding an r-node in the first row of $[\lambda]$ cannot yield a singular diagram.

Clearly, adding all δ r-nodes to $[\lambda^0]$ yields its complement $[\lambda^l]$, and if $[\lambda^0]$ is regular then so also is $[\lambda^l]$, since all singular positions and their corresponding regular positions are filled. A similar argument shows that $[\lambda^l]$ is singular if $[\lambda^0]$ is singular, and conversely.

When we consider an arbitrary element of an rBA for which $0 < d^* < d + d^* = l$ the situation is somewhat different, since if a regular position is occupied by an r-node, then in the complement the corresponding singular position will be occupied and the diagram will be singular. To overcome this difficulty we define a *modified complement* This definition favours regular diagrams, but a similar one would favour singular diagrams.

Definition. If in a regular diagram $[\lambda]$, a singular r-position is vacant while its corresponding regular position is occupied by an r-node, then the *modified* complement of $[\lambda]$ in the appropriate rBA is that diagram obtained from $[\tilde{\lambda}]$ by raising all r-nodes which occupy singular positions to the corresponding regular positions. Clearly, the modified complement of a modfiied complement is the original diagram.

3.1 *Example.* If $[\lambda] = [7, 4, 3, 2, 1^2]$, then in the complement for $p = 3$, $r = 0$, $[\tilde{\lambda}] = [6, 5, 2^3, 1^2]$ the two singular positions $(5, 2)$ and $(7, 1)$ are occupied and the regular position $(3, 3)$ is vacant. Thus the modified complement is obtained by raising these 0-nodes to the corresponding regular positions to yield $[6, 5, 3, 2^2, 1]$.

We may think of complementation in the ordinary sense as a special case of modified complementation, and so state the following theorem:

3.2 *The property that a diagram be regular is invariant under modified complementation in the appropriate rBA.*

We conclude that (cf. **(7**, 5.6**)**):

3.3 *The number of p-regular diagrams of a given weight w is independent of the core.*

Proof. Complementation or modified complementation amounts to adding δ r-nodes to the core. But we know that the result is again a core, and every core can be obtained in this way **(8)**.

4. The explicit construction of a Young diagram with given p-core and p-quotient. Our construction is explicit and merely reverses an argument given elsewhere **(3)**.

Let us call the set of first column hook lengths obtained from $[\lambda_0]$ a *core set,* denoting them

4.1 $$\Gamma_k : c_1, c_2, \dots, c_k, \qquad c_i > c_{i+1}.$$

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If these c's are divided into residue classes, then it is known **(8; 9)** that the zero class is empty and that all residues smaller than the largest in a given class necessarily appear.

It is in general necessary to *extend* Γ_k in the following manner:

4.2 $\Gamma_h : b_1 = c_1+s,\ b_2 = c_2+s, \ldots b_k = c_k+s,\ b_{k+1} = s-1, \ldots b_{h-1} = 1,\ b_h = 0,$

where $k + s = h$; we shall call Γ_h a *basic set*. Again we may divide the elements of Γ_h into residue classes, the zero class now appearing for $h > k$.

We shall denote the p-quotient $[\lambda]_p$ by the set of disjoint diagrams

4.3 $$[{}_0\lambda], [{}_1\lambda], \ldots, [{}_{p-1}\lambda],$$

where one or more constituents may be vacuous. The partition corresponding to a given constituent may be written out in detail thus:

4.4 $$[{}_r\lambda] = [{}_r\lambda_1, {}_r\lambda_2, \ldots, {}_r\lambda_{k_r}],$$

where r designates the residue class of the constituent.

In the required construction of a diagram $[\lambda]$ we must extend the core set Γ_k so that the quantities

4.5 $${}_r\lambda_t \cdot p \qquad (t = 1, 2, \ldots, k_r)$$

can be added in order to the k_r largest members in the appropriate residue class of Γ_h. It only remains to determine this residue class for all r, and we do this by setting

4.6 $$r \equiv b_i - h \pmod p.$$

If we denote by g_r the number of elements in a core set which are congruent to r (mod p), then the number of elements in the basic set Γ_h which are congruent to r (mod p) is given by

4.7 $$g(r, s) = g_{r-s} + \left[\frac{s+p-1-r}{p}\right],$$

where the bracket function $[x]$ denotes the largest integer equal to or less than x. For a given $[\lambda]_p$ we have a set of integers $k_{r'}$ $(r' = 0, 1, \ldots, p-1)$, and the choice of s for a given core is determined by the following conditions:

4.8 $$g(r, s)\begin{cases} \geqslant k_{r'}, & r' \not\equiv -h \pmod p, \\ = k_{r'}, & r' \equiv -h \pmod p, \end{cases}$$

where $r' \equiv r - h \pmod p$ by 4.6. That these conditions determine s *uniquely* **(7)** will be illustrated by the following

4.9 *Example.* To construct $[\lambda]$ given $[\lambda_0] = [2, 1^2]$, $[\lambda]_3 = [2, 1], [1^2], [2]$ where the constituents of $[\lambda]_3$ are associated with the residue classes 0, 1, 2 respectively. Table I gives the values of the function $g(r, s)$ and is arranged according to the residue classes $r' \equiv r - h \pmod p$, for $p = 3$.

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Table I

s	h	$g(r', s)$		
		$r' = 0$	$r' = 2$	$r' = 1$
0	3	**0**	1	2
1	4	0	**2**	2
2	5	0	2	**3**
3	6	**1**	2	3
4	7	1	**3**	3
5	8	1	3	**4**
6	9	**2**	3	4
7	10	2	**4**	4
8	11	2	4	**5**

The integers printed in bold type are those which correspond to the *equality* in 4.8. We can use the table to determine s. Clearly the k's of $[\lambda]_3$ are $k_0 = 2$, $k_2 = 1$, $k_1 = 2$, so that $s \geqslant 6$. That $s \not> 6$ follows from the equality part of 4.8. Thus the basic set is

$$10, 8, 7, 5, 4, 3, 2, 1, 0.$$

Corresponding to $r' = 0$ we must add 6 to 3 and 3 to 0; corresponding to $r' = 1$ we must add 3 to 10 and 3 to 7; corresponding to $r' = 2$ we must add 6 to 8. Rearranging, we have the set of first column hook lengths:

$$14, 13, 10, 9, 5, 4, 3, 2, 1,$$

which belongs to $[6^2, 4^2, 1^5]$.

5. The enumeration of p-regular diagrams. As we have remarked, a core set can have no element $\equiv 0 \pmod p$, and moreover every class of elements must contain every integer less than the largest in the class. If a diagram is singular then there will be at least p *successive* integers in the set of first column hook lengths. Conversely this condition is also sufficient for singularity.

5.1 *Any diagram obtained by adding p-hooks to a p-core, while not increasing the number of rows, is necessarily regular.*

Proof. By adding multiples of p to a core set we can never introduce the zero residue class and so have p consecutive first column hook lengths.

The enumeration problem of regular diagrams will be solved by showing how we can enumerate singular diagrams of a given weight w, *provided the core is suitably chosen.* Having established the enumeration in one case then, by 3.3, it applies in all cases. The choice of the core affects the situation in a somewhat subtle manner as we can see by examining first the case $p = 2$. Here

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there is only one type of core and by taking the number of rows to be g we have the corresponding core set to be

$$\Gamma_g : 2g - 1, 2g - 3, \ldots, 3, 1.$$

If we extend Γ_g to Γ_{g+s} where s is *even* it will always be possible to obtain a set of at least two consecutive terms provided $w \leqslant g + 1$, and the diagrams have non-vacuous $[_r\lambda]$, where $r \equiv -g \pmod 2$. If $w > g + 1$ then the diagram represented by

$$[_r\lambda] = [w],$$

$r \equiv -g \pmod 2$, will have no two consecutive terms and so will be regular.

Moreover, if s is *odd* all the terms in the extended set are even except some of those at the end so it will be necessary to add (from the top down) at least $g + 2$ multiples of 2 to obtain a set of first column hook lengths, i.e. $w > g + 1$.

By denying singularity we obtain regularity so we have proved the following theorem:

5.3 *For $p = 2$, and $w \leqslant g + 1$ the necessary and sufficient condition that a diagram be regular is that in its 2-quotient the constituent $[_r\lambda]$ be vacuous for $-r \equiv h \equiv g \pmod 2$.*

For $p > 2$ the situation is complicated by the fact that there are $p - 1$ classes of terms in a core set and some of these may be vacuous. Thus to produce a set of p consecutive first column hook lengths in the manner envisaged above the weight w will be limited by the *shortest* residue class in the core set; we call the length g of this shortest class the *grade* of the core. If any class is vacuous, $g = 0$, and $w = 1$ is the only possible case yielding such an enumeration of singular diagrams.

The following theorem includes 5.3 as a special case.

5.4 *If a p-core set Γ_k contains $p - 1$ residue classes, each class containing at least g members, then the necessary and sufficient condition that a diagram $[\lambda_1, \lambda_2, \ldots, \lambda_k]$ of weight $w \leqslant g + 1$ be regular is that $[_r\lambda]$ be vacuous for $-r \equiv h \equiv k \pmod p$.*

Proof. As before we fix attention on the singular diagrams. From our definition of the grade g, we know that there are at most g sets of $p - 1$ consecutive residues in Γ_k. The worst case we need consider is where $[_r\lambda] = [w]$ so that $s = p$. Adding wp to zero we obtain Γ_h which contains the consecutive set

$$wp, wp - 1, \ldots, (w - 1)p + 1,$$

so that $[\lambda]$ is singular. The other extreme case is where $[_r\lambda] = [1^w]$ and we add p to each of w terms which are all congruent to zero $\pmod p$. In this case we have the consecutive terms

$$wp, wp - 1, \ldots, 3, 2, 1,$$

and $[\lambda]$ is certainly singular. All other partitions of w clearly yield at least p consecutive terms in Γ_k.

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If the w nodes are distributed over more than one constituent of $[\lambda]_p$ then, provided $[_r\lambda]$ is not vacuous, the above argument is still applicable and $[\lambda]$ must be singular.

On the other hand if $[_r\lambda]$ is vacuous, $h \equiv k \pmod p$, and since additions must be made at the top of a residue class of Γ_h, $w > g + 1$ as in 5.3, proving that all diagrams under consideration must be regular.

Denoting the number of partition of w, by $\mathfrak{p}_w$, we combine 3.3 and 5.4 and conclude (**1**; **2**; **4**; **5**; **6**) that:

5.5 *The number of p-regular diagrams of weight w having a given core* $[\lambda_0]$ *is equal to*

$$\sum_{w_1 \ldots w_{p-1}} \mathfrak{p}_{w_1}\mathfrak{p}_{w_2} \ldots \mathfrak{p}_{w_{p-1}} \qquad \left(\sum_1^{p-1} w_i = w, 0 \leqslant w_i \leqslant w\right)$$

and so equal to the number of modular irreducible representations of S_n in the corresponding block.

We prove in conclusion the following interesting result:

5.6 *For $w \leqslant g + 1$ the diagrams in a given block are all zero elements* $(d^* = 0)$ *of their respective rBA's, where $-r \equiv k$ (mod p).*

Proof. If $d^* \neq 0$ there is a removable node of class r; this implies the presence of a term of Γ_h congruent to zero (mod p), *followed* by a gap in the set. Since this does not happen for $w \leqslant g + 1$, with $-r \equiv k \pmod p$, we conclude that $d^* = 0$ for every such diagram.

Consider a core C_1 of grade g having k rows. Clearly, the next succeeding node in the first column would be of class r where $-r \equiv k \pmod p$. If δ such r-nodes are added to C_1 we obtain a core C_2. Successive complementation in this manner yields a *series* of cores. For $p = 3$,

(a) $[2], [3, 1], [3, 1^2], [4, 2, 1^2], [4, 2^2, 1^2], \ldots,$
(b) zero, $[1], [1^2], [2, 1^2], [2^2, 1^2], [3, 2^2, 1^2], \ldots,$

are two such series. We state the following lemmas without proofs.

5.7 *Two distinct series cannot have a core in common.*

5.8 *The grade cannot decrease under complementation.*

It follows from 5.6 that if $w \leqslant g + 1$ for diagrams having C_1 as core, then subsequent complementation with $-r \equiv k \pmod p$ is *ordinary* and the diagrams so obtained have C_2, C_3, etc. as cores. Moreover, regular diagrams go into regular diagrams and singular into singular by 2.2, 5.4 and 5.8. Clearly, the critical class of the p-quotient which is vacuous for regular diagrams by 5.4 is permuted according to the cycle

$$(0, p - 1, p - 2, \ldots 2, 1),$$

under successive complementation in the series.

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5.9 *Example.* To illustrate these ideas we give in Table II the sets Γ_h for 3-singular diagrams with core $[2^2, 1^2]$ for which $\Gamma_k = 5, 4, 2, 1$ and $g = 2$, of weight $w = 3$, and the associated sets of first column hook lengths. The arrangement of these should be monotonic to construct $[\lambda]$, but they are left as they come after the appropriate multiples of p are added to the terms of Γ_h.

Table II

$[\lambda]_3$	Γ_h	First column hook lengths	$[\lambda]$
–, –, [3]	8, 7, 5, 4, 2, 1, 0	8, 7, 5, 4, 2, 1, **9**	$[3^3, 2^2, 1^2]$
–, –, [2,1]	11, 10, 8, 7, 5, 4, 3, 2, 1, 0	11, 10, 8, 7, 5, 4, **9**, 2, 1, **3**	$[2^5, 1^5]$
–, –, $[1^3]$	14, 13, 11, 10, 8, 7, 6, 5, 4, 3 2, 1, 0	14, 13, 11, 10, 8, 7, **9**, 5, 4, **6**, 2, 1, **3**	$[2^2, 1^{11}]$
[1], –, [2]	8, 7, 5, 4, 2, 1, 0	8, **10**, 5, 4, 2, 1, **0**	$[4, 3, 2^3, 1^2]$
[1], –, $[1^2]$	11, 10, 8, 7, 5, 4, 3, 2, 1, 0	11, **13**, 8, 7, 5, 4, **6**, 2, 1, **3**	$[4, 3, 1^8]$
[2], –, [1]	8, 7, 5, 4, 2, 1, 0	8, **13**, 5, 4, 2, 1, **3**	$[7, 3, 1^5]$
$[1^2]$, –, [1]	8, 7, 5, 4, 2, 1, 0	8, **10**, 5, **7**, 2, 1, **3**	$[4, 3^2, 2, 1^3]$
–, [1], [2]	8, 7, 5, 4, 2, 1, 0	**11**, 7, 5, 4, 2, 1, **6**	$[5, 2^4, 1^2]$
–, [1], $[1^2]$	11, 10, 8, 7, 5, 4, 3, 2, 1, 0	**14**, 10, 8, 7, 5, 4, **6** 2, 1, **3**	$[5, 2, 1^8]$
–, [2], [1]	8, 7, 5, 4, 2, 1, 0	**14**, 7, 5, 4, 2, 1, **3**	$[8, 2, 1^5]$
–, $[1^2]$, [1]	8, 7, 5, 4, 2, 1, 0	**11**, 7, **8**, 4, 2, 1, **3**	$[5, 3^2, 1^4]$
[1], [1], [1]	8, 7, 5, 4, 2, 1, 0	**11**, **10**, 5, 4, 2, 1, **3**	$[5^2, 1^5]$

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