Bull. Aust. Math. Soc. **99** (2019), 369–375
doi:10.1017/S0004972718001168

A NEW MENON'S IDENTITY FROM GROUP ACTIONS

YU-JIE WANG and CHUN-GANG JI✉

(Received 27 August 2018; accepted 23 September 2018; first published online 28 November 2018)

Abstract

Let n be a positive integer. We obtain new Menon's identities by using the actions of some subgroups of $(\mathbb{Z}/n\mathbb{Z})^{\times}$ on the set $\mathbb{Z}/n\mathbb{Z}$. In particular, let p be an odd prime and let α be a positive integer. If H_k is a subgroup of $(\mathbb{Z}/p^{\alpha}\mathbb{Z})^{\times}$ with index $k = p^{\beta}u$ such that $0 \leqslant \beta < \alpha$ and $u \mid p-1$, then

$$\sum_{x\in H_k}(x-1, p^{\alpha}) = \frac{\varphi(p^{\alpha})}{k}\Big(1 + k(\alpha-\beta) + u\frac{p^{\beta}-1}{p-1}\Big),$$

where $\varphi(n)$ is the Euler totient function.

2010 *Mathematics subject classification*: primary 11A25; secondary 11B13.

Keywords and phrases: arithmetical sums, Menon's identity, group action.

1. Introduction

In [4], Menon proved a classical identity: for any positive integer n,

$$\sum_{x\in(\mathbb{Z}/n\mathbb{Z})^{\times}}(x-1, n) = \varphi(n)\tau(n),$$

where $(\mathbb{Z}/n\mathbb{Z})^{\times}$ is the group of units of $\mathbb{Z}/n\mathbb{Z}$, $\varphi(n)$ is the Euler totient function and $\tau(n)$ is the divisor function. In [8], Sury proved that, for every $r \geqslant 2$,

$$\sum_{\substack{x_1\in(\mathbb{Z}/n\mathbb{Z})^{\times}\\ x_2,\ldots,x_r\in\mathbb{Z}/n\mathbb{Z}}}(x_1-1, x_2, \ldots, x_r, n) = \varphi(n)\tau_{r-1}(n),$$

where $\tau_{r-1}(n) = \sum_{d\mid n} d^{r-1}$. There are many generalisations of Menon's identity; see [1–3, 5, 6, 9, 10].

The key tool in proving these results is the Cauchy–Frobenius–Burnside lemma concerning group actions.

This work was partially supported by the Grant No. 11471162 from NNSF of China and the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20133207110012).

LEMMA 1.1 (Cauchy–Frobenius–Burnside lemma, [7]). *Let G be a finite group acting on a finite set X and, for each $g \in G$, let $X^g = \{x \in X \mid gx = x\}$ be the set of elements in X that are fixed by g. Denote the set of all orbits of X under the action of G by G/X. Then*

$$\sum_{g\in G} |X^g| = |G| \cdot |G/X|.$$

The generalisations mentioned above are derived from the Cauchy–Frobenius–Burnside lemma for the action of $(\mathbb{Z}/n\mathbb{Z})^\times$ on some fixed sets. We obtain some new Menon's identities by using the action of the subgroups of $(\mathbb{Z}/n\mathbb{Z})^\times$ on the set $\mathbb{Z}/n\mathbb{Z}$.

THEOREM 1.2. *Let p be an odd prime and let α be a positive integer. Suppose that H_k is a subgroup of $(\mathbb{Z}/p^\alpha\mathbb{Z})^\times$ with index k. Then*

$$\sum_{x\in H_k} (x-1, p^\alpha) = \frac{\varphi(p^\alpha)}{k}\Big(1 + k(\alpha-\beta) + u\frac{p^\beta - 1}{p-1}\Big),$$

where $k = p^\beta u$ with $0 \leqslant \beta < \alpha$ and $u \mid (p-1)$.

THEOREM 1.3. *Let α and l be two integers such that $\alpha \geqslant 3$ and $0 \leqslant l \leqslant \alpha - 2$. Let H_{2^l} be a subgroup of $(\mathbb{Z}/2^\alpha\mathbb{Z})^\times$ with index 2^l. Then*

$$\sum_{x\in H_{2^l}} (x-1, 2^\alpha) = \begin{cases} 2^{\alpha-1}(\alpha - l + 1) & \text{for } H_{2^l} = \langle 5^{2^{l-1}}\rangle, \\ 2^{\alpha-1}(\alpha-l) + 2^{\alpha-l-1} & \text{for } H_{2^l} = \langle -1\rangle \times \langle 5^{2^l}\rangle \text{ or } \langle -5^{2^{l-1}}\rangle. \end{cases}$$

Let $n > 1$ be a positive integer such that $n = \prod_{i=1}^t p_i^{\alpha_i}$, where $p_1 < p_2 < \cdots < p_t$ are primes and $\alpha_i \geqslant 1$ for $i = 1, \ldots, t$. The Chinese remainder theorem gives the isomorphism

$$(\mathbb{Z}/n\mathbb{Z})^\times \cong \prod_{i=1}^t (\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z})^\times.$$

Let H_{k_i} be a subgroup of $(\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z})^\times$ with index k_i, that is, $[(\mathbb{Z}/p_i^{\alpha_i}\mathbb{Z})^\times : H_{k_i}] = k_i$. Consider the subgroups of the form $H = \prod_{i=1}^t H_{k_i}$ and write

$$f(H_{k_i}, p^{\alpha_i}) = \sum_{x\in H_{k_i}} (x-1, p_i^{\alpha_i}).$$

This leads to the following composite result.

THEOREM 1.4. *With the notation introduced above,*

$$\sum_{x\in H} (x-1, n) = \prod_{i=1}^t f(H_{k_i}, p^{\alpha_i}).$$

2. Some lemmas

Let p be a prime and let α be a positive integer. We start with some well-known facts about the quotient ring $\mathbb{Z}/p^\alpha\mathbb{Z}$. Firstly, the ring $\mathbb{Z}/p^\alpha\mathbb{Z}$ is a principal ideal ring with the ideal chain

$$0 \subseteq p^{\alpha-1}(\mathbb{Z}/p^\alpha\mathbb{Z}) \subseteq \cdots \subseteq p(\mathbb{Z}/p^\alpha\mathbb{Z}) \subseteq \mathbb{Z}/p^\alpha\mathbb{Z}.$$

Hence the number of ideals in $\mathbb{Z}/p^\alpha\mathbb{Z}$ is exactly $\alpha + 1$. Secondly,

$$\mathbb{Z}/p^\alpha\mathbb{Z} = \{0\} \cup p^{\alpha-1}(\mathbb{Z}/p^\alpha\mathbb{Z})^\times \cup \cdots \cup p(\mathbb{Z}/p^\alpha\mathbb{Z})^\times \cup (\mathbb{Z}/p^\alpha\mathbb{Z})^\times$$

is a partition of $\mathbb{Z}/p^\alpha\mathbb{Z}$. In particular, $p^i(\mathbb{Z}/p^\alpha\mathbb{Z})^\times$ is the set of the generators of the ideal $p^i(\mathbb{Z}/p^\alpha\mathbb{Z})$ for $i = 0, 1, \ldots, \alpha$.

Lemma 2.1. *Let p be a prime and let α be a positive integer. Then two elements x and y of $\mathbb{Z}/p^\alpha\mathbb{Z}$ are two generators of the same ideal of $\mathbb{Z}/p^\alpha\mathbb{Z}$ if and only if there exists an element $\mu \in (\mathbb{Z}/p^\alpha\mathbb{Z})^\times$ such that $y = \mu x$.*

Let p be an odd prime and let α be a positive integer. Then there is a primitive root modulo p^α, that is, the group $(\mathbb{Z}/p^\alpha\mathbb{Z})^\times$ is cyclic. Throughout, an element g of $(\mathbb{Z}/p^\alpha\mathbb{Z})^\times$ denotes a primitive root modulo p^α, that is,

$$(\mathbb{Z}/p^\alpha\mathbb{Z})^\times = \{g, g^2, \ldots, g^{\varphi(p^\alpha)}\} = \langle g\rangle.$$

Hence for each $k|\varphi(p^\alpha)$, there is only one cyclic subgroup $H_k = \langle g^k\rangle$ of $(\mathbb{Z}/p^\alpha\mathbb{Z})^\times$ generated by g^k. Furthermore, $[(\mathbb{Z}/p^\alpha\mathbb{Z})^\times : H_k] = k$. We call k the index of H_k in $(\mathbb{Z}/p^\alpha\mathbb{Z})^\times$.

It is well known that $(\mathbb{Z}/2^\alpha\mathbb{Z})^\times$ is not cyclic for $\alpha \geqslant 3$.

Lemma 2.2. *Let α and l be two integers such that $\alpha \geqslant 3$ and $0 \leqslant l \leqslant \alpha - 2$. Then the subgroup H_{2^l} of $(\mathbb{Z}/2^\alpha\mathbb{Z})^\times$ with index 2^l must be one of*

$$\langle 5^{2^{l-1}}\rangle, \quad \langle -1\rangle \times \langle 5^{2^l}\rangle \quad \textit{or} \quad \langle -5^{2^{l-1}}\rangle.$$

Proof. For $\alpha \geqslant 3$,

$$(\mathbb{Z}/2^\alpha\mathbb{Z})^\times = \langle -1\rangle \times \langle 5\rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{\alpha-2}\mathbb{Z},$$

where $\langle a\rangle$ denotes the cyclic subgroup of $(\mathbb{Z}/2^\alpha\mathbb{Z})^\times$ generated by a. It is clear that 5 has order $2^{\alpha-2}$ modulo 2^α. Hence, for each l with $0 \leqslant l \leqslant \alpha - 2$, if H_{2^l} is a subgroup of $(\mathbb{Z}/2^\alpha\mathbb{Z})^\times$ with index 2^l, then

$$H_{2^l} = \{5^{a_1}, 5^{a_2}, \ldots, 5^{a_s}\} \cup \{-5^{b_1}, -5^{b_2}, \ldots, -5^{b_t}\},$$

where $1 \leqslant a_1 < a_2 < \cdots < a_s = 2^{\alpha-2}$ and $0 \leqslant b_1 < b_2 < \cdots < b_t \leqslant 2^{\alpha-2} - 1$. Put

$$T_1 = \{5^{a_1}, 5^{a_2}, \ldots, 5^{a_s}\} \quad \text{and} \quad T_2 = \{-5^{b_1}, -5^{b_2}, \ldots, -5^{b_t}\}.$$

Then $T_1 \cap T_2 = \emptyset$.

https://doi.org/10.1017/S0004972718001168 Published online by Cambridge University Press

Now we shall show that T_1 is a subgroup of $\langle 5\rangle$. If $1 \leqslant i, j \leqslant s$, then $5^{a_i+a_j} \in H_{2^l}$. If $5^{a_i+a_j} \notin T_1$, then $5^{a_i+a_j} \in T_2$. Hence there exists an element $-5^{b_h} \in T_2$ such that

$$5^{a_i+a_j} \equiv -5^{b_h} \pmod{2^\alpha}.$$

But this implies that $1 \equiv -1 \pmod 4$, which is a contradiction. Hence $5^{a_i+a_j} \in T_1$: that is, T_1 is closed under multiplication. If $5^{a_i} \in T_1$, then

$$5^{a_i} \times 5^{2^{\alpha-2}-a_i} \equiv 1 \pmod{2^\alpha}.$$

Hence $5^{2^{\alpha-2}-a_i} \in H_{2^l}$ and so $5^{2^{\alpha-2}-a_i} \in T_1$. Consequently, T_1 is a subgroup of $\langle 5\rangle$.

If $T_2 = \emptyset$, then $H_{2^l} = T_1$ is cyclic. Hence $H_{2^l} = \langle 5^{2^{l-1}}\rangle$.

If $T_2 \neq \emptyset$, we shall show that $T_2 = -5^{b_1}T_1$. It is easy to see that $-5^{b_1}T_1 \subseteq T_2$. If $1 \leqslant j \leqslant t$, then

$$(-5^{2^{\alpha-2}-b_1}) \times (-5^{b_j}) \in T_1.$$

Hence there is an element $5^{a_j} \in T_1$ such that $-5^{b_j} \equiv -5^{b_1} \times 5^{a_j} \pmod{2^\alpha}$. It follows that $T_2 \subseteq -5^{b_1}T_1$. Hence $T_2 = -5^{b_1}T_1$. Let $T_1 = \langle 5^{2^{l_1}}\rangle$, where $l_1 \geqslant l$. Then

$$H_{2^l} = \langle -5^{b_1}\rangle \times \langle 5^{2^{l_1}}\rangle.$$

Since $(-5^{b_1})^2 \in T_1$, we have $|\langle 5^{2^{l_1}}\rangle| = 2^{\alpha-l-2}$ and $2^{l_1-1}|b_1$. Hence $l_1 = l$ and $b_1 = 2^{l-1}k$. If k is even, then $H_{2^l} = \langle -1\rangle \times \langle 5^{2^l}\rangle$. If k is odd, then $H_{2^l} = \langle -5^{2^{l-1}}\rangle$. This completes the proof of Lemma 2.2. □

3. Proof of Theorem 1.2

Let $(\mathbb{Z}/p^\alpha\mathbb{Z})^\times = \langle g\rangle$. Then $H_k = \langle g^k\rangle$. Now we compute the number of orbits of the group H_k acting on the set $\mathbb{Z}/p^\alpha\mathbb{Z}$. Let $x, y \in \mathbb{Z}/p^\alpha\mathbb{Z}$. Then x, y are in the same orbit if and only if $y = g^{kt}x$ for some t with $1 \leqslant t \leqslant \varphi(p^\alpha)/k$. By Lemma 2.1, x and y are two generators of the same ideal. Hence it is enough to compute the number of orbits of H_k acting on the set $p^i(\mathbb{Z}/p^\alpha\mathbb{Z})^\times$ for $i = 0, 1, \ldots, \alpha$. If $i = \alpha$, then $p^\alpha(\mathbb{Z}/p^\alpha\mathbb{Z})^\times = \{0\}$, which is an orbit. Let $0 \leqslant i \leqslant \alpha - 1$. Then each element of $p^i(\mathbb{Z}/p^\alpha\mathbb{Z})^\times$ is of the form p^ig^m, with $1 \leqslant m \leqslant \varphi(p^\alpha)$. Suppose $x = p^ig^m$ and $y = p^ig^n$ are in the same orbit. Then

$$p^ig^m \equiv p^ig^{n+kt} \pmod{p^\alpha} \tag{3.1}$$

for some t with $1 \leqslant t \leqslant \varphi(p^\alpha)/k$. Since g is a primitive root modulo $p^{\alpha-i}$, by (3.1),

$$m - n \equiv kt \pmod{p^{\alpha-i-1}(p-1)}. \tag{3.2}$$

Let $k = p^\beta u$ with $0 \leqslant \beta < \alpha$ and $u|(p-1)$. From (3.2),

$$p^{\min(\beta,\alpha-i-1)}u|(m-n).$$

So the number of orbits of $p^i(\mathbb{Z}/p^\alpha\mathbb{Z})^\times$ is equal to the number of residue classes modulo $p^{\min(\beta,\alpha-i-1)}u$. Hence the number of orbits of $p^i(\mathbb{Z}/p^\alpha\mathbb{Z})^\times$ is $p^{\min(\beta,\alpha-i-1)}u$ and the total number of orbits of the set $\mathbb{Z}/p^\alpha\mathbb{Z}$ is

$$|\mathbb{Z}/p^\alpha\mathbb{Z}/H_k| = 1 + \sum_{i=0}^{\alpha-\beta-1} p^\beta u + \sum_{i=\alpha-\beta}^{\alpha-1} p^{\alpha-i-1}u = 1 + k(\alpha-\beta) + u\frac{p^{\beta+1}-1}{p-1}.$$

https://doi.org/10.1017/S0004972718001168 Published online by Cambridge University Press

For $x \in H_k$, let $(\mathbb{Z}/p^\alpha\mathbb{Z})^x$ denote the subset of elements of $\mathbb{Z}/p^\alpha\mathbb{Z}$ fixed by x, that is,

$$(\mathbb{Z}/p^\alpha\mathbb{Z})^x = \{y \in \mathbb{Z}/p^\alpha\mathbb{Z} \mid xy \equiv y \pmod{p^\alpha}\}.$$

Then $|(\mathbb{Z}/p^\alpha\mathbb{Z})^x| = (x-1, p^\alpha)$. By the Cauchy–Frobenius–Burnside lemma,

$$\sum_{x\in H_k} |(\mathbb{Z}/p^\alpha\mathbb{Z})^x| = |H_k| \cdot |\mathbb{Z}/p^\alpha\mathbb{Z}/H_k|.$$

Hence

$$\sum_{x\in H_k} (x-1, p^\alpha) = \frac{\varphi(p^\alpha)}{k}\Big(1 + k(\alpha-\beta) + u\frac{p^\beta - 1}{p-1}\Big).$$

This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.3

First, we compute the number of orbits of the group H_{2^l} acting on the set $\mathbb{Z}/2^\alpha\mathbb{Z}$. By

$$\mathbb{Z}/2^\alpha\mathbb{Z} = \{0\} \cup 2^{\alpha-1}(\mathbb{Z}/2^\alpha\mathbb{Z})^\times \cup \cdots \cup 2(\mathbb{Z}/2^\alpha\mathbb{Z})^\times \cup (\mathbb{Z}/2^\alpha\mathbb{Z})^\times$$

and Lemma 2.1, it is enough to compute the number of orbits of H_{2^l} acting on the set

$$2^i(\mathbb{Z}/2^\alpha\mathbb{Z})^\times = \{\pm 2^i 5^a \mid 1 \leqslant a \leqslant 2^{\alpha-2} - 1\}$$

for $i = 0, 1, \ldots, \alpha$. If $i = \alpha$ or $\alpha - 1$, then $|2^i(\mathbb{Z}/2^\alpha\mathbb{Z})^\times| = 1$, that is, $2^i(\mathbb{Z}/2^\alpha\mathbb{Z})^\times$ is an orbit. By Lemma 2.2, there are three distinct subgroups of $(\mathbb{Z}/2^\alpha\mathbb{Z})^\times$ with index 2^l. We discuss each of these cases separately.

Case 1. Suppose that $H_{2^l} = \langle 5^{2^{l-1}} \rangle$ and $0 \leqslant i \leqslant \alpha - 2$. If $x, y \in 2^i(\mathbb{Z}/2^\alpha\mathbb{Z})^\times$ are in the same orbit, then $x, y \in \{2^i 5^a \mid 1 \leqslant a \leqslant 2^{\alpha-2} - 1\}$ or $x, y \in \{-2^i 5^a \mid 1 \leqslant a \leqslant 2^{\alpha-2} - 1\}$. Without loss of generality, $x = 2^i 5^m$, $y = 2^i 5^n \in \{2^i 5^a \mid 1 \leqslant a \leqslant 2^{\alpha-2} - 1\}$. Then there is an integer $t \geqslant 0$ such that

$$2^i 5^m \equiv 2^i 5^n 5^{2^{l-1}t} \pmod{2^\alpha}.$$

This implies that

$$m - n \equiv 2^{l-1} t \pmod{2^{\alpha-i-2}}.$$

Hence the number of orbits of $\{2^i 5^a \mid 1 \leqslant a \leqslant 2^{\alpha-2} - 1\}$ is equal to the number of residue classes modulo $2^{\min(l-1,\alpha-i-2)}$ and the number of orbits of $\{\pm 2^i 5^a \mid 1 \leqslant a \leqslant 2^{\alpha-2} - 1\}$ is $2 \times 2^{\min(l-1,\alpha-i-2)}$. So the total number of orbits of $\mathbb{Z}/2^\alpha\mathbb{Z}$ is

$$|\mathbb{Z}/2^\alpha\mathbb{Z}/H_{2^l}| = 1 + 1 + 2\sum_{i=0}^{\alpha-l-1} 2^{l-1} + 2\sum_{i=\alpha-\beta}^{\alpha-2} 2^{\alpha-l-2} = 2^l(\alpha - l + 1).$$

Case 2. Suppose that $H_{2^l} = \langle -1 \rangle \times \langle 5^{2^l} \rangle$ and $0 \leqslant i \leqslant \alpha - 2$. If x and $y \in 2^i(\mathbb{Z}/2^\alpha\mathbb{Z})^\times$ are in the same orbit, then $x = (-1)^\delta 2^i 5^m$ and $y = (-1)^\eta 2^i 5^n$, where $\delta, \eta \in \{0, 1\}$. Hence there exist integers $t \geqslant 0$ and $\xi \in \{0, 1\}$ such that

$$(-1)^\delta 2^i 5^m \equiv (-1)^\eta 2^i 5^n \cdot (-1)^\xi 5^{2^l \cdot t} \pmod{2^\alpha}.$$

https://doi.org/10.1017/S0004972718001168 Published online by Cambridge University Press

So $(-1)^{\delta}5^m \equiv (-1)^{\eta}5^n \cdot (-1)^{\xi}5^{2^l \cdot t} \pmod{2^{\alpha-i}}$. Since $\alpha - i \geqslant 2$, it follows that $(-1)^{\delta} \equiv (-1)^{\eta} \cdot (-1)^{\xi} \pmod{2^2}$. Thus $(-1)^{\delta} = (-1)^{\eta} \cdot (-1)^{\xi}$ and $5^m \equiv 5^n \cdot 5^{2^l \cdot t} \pmod{2^{\alpha-i}}$. This implies that

$$m - n \equiv 2^l t \pmod{2^{\alpha-i-2}}.$$

So the number of orbits of $2^i(\mathbb{Z}/2^{\alpha}\mathbb{Z})^{\times}$ is equal to the number of residue classes modulo $2^{\min(l,\alpha-i-2)}$. Hence the total number of orbits of $\mathbb{Z}/2^{\alpha}\mathbb{Z}$ is

$$|\mathbb{Z}/2^{\alpha}\mathbb{Z}/H_{2^l}| = 1 + 1 + \sum_{i=0}^{\alpha-l-2} 2^l + \sum_{i=\alpha-\beta-1}^{\alpha-2} 2^{\alpha-l-2} = 2^l(\alpha - l) + 1.$$

Case 3. Suppose that $H_{2^l} = \langle -5^{2^{l-1}} \rangle$ and $0 \leqslant i \leqslant \alpha - 2$. For each element -2^i5^h with $1 \leqslant h \leqslant 2^{\alpha-2} - 1$, there exists an element 2^i5^m with $1 \leqslant m \leqslant 2^{\alpha-2} - 1$ such that

$$(-2^i5^h)(-5^{2^{l-1}}) \equiv 2^i5^m \pmod{2^{\alpha}}.$$

Thus each element of $\{-2^i5^a \mid 1 \leqslant a \leqslant 2^{\alpha-2} - 1\}$ belongs to a certain orbit of the set $\{2^i5^a \mid 1 \leqslant a \leqslant 2^{\alpha-2} - 1\}$. Hence it is enough to consider the group $\langle -5^{2^{l-1}} \rangle$ acting on the set $\{2^i5^a \mid 1 \leqslant a \leqslant 2^{\alpha-2} - 1\}$. Let $x = 2^i5^m$ and $y = 2^i5^n \in \{2^i5^a \mid 1 \leqslant a \leqslant 2^{\alpha-2} - 1\}$ be in the same orbit. Then there is an integer $t \geqslant 0$ such that

$$2^i5^m \equiv 2^i5^n \cdot (-5^{2^{l-1}})^t \pmod{2^{\alpha}}.$$

It is clear that $t = 2t_1$. Hence

$$m - n \equiv 2^l t_1 \pmod{2^{\alpha-i-2}}.$$

So the number of orbits of $\{2^i5^a \mid 1 \leqslant a \leqslant 2^{\alpha-2} - 1\}$ is equal to the number of residue classes modulo $2^{\min(l,\alpha-i-2)}$. Hence the total number of orbits of $\mathbb{Z}/2^{\alpha}\mathbb{Z}$ is

$$|\mathbb{Z}/2^{\alpha}\mathbb{Z}/H_{2^l}| = 1 + 1 + \sum_{i=0}^{\alpha-l-2} 2^l + \sum_{i=\alpha-\beta-1}^{\alpha-2} 2^{\alpha-l-2} = 2^l(\alpha - l) + 1.$$

By the Cauchy–Frobenius–Burnside lemma,

$$\sum_{x \in H_{2^l}} (x - 1, 2^{\alpha}) = \begin{cases} 2^{\alpha-1}(\alpha - l + 1) & \text{for } H_{2^l} = \langle 5^{2^{l-1}} \rangle, \\ 2^{\alpha-1}(\alpha - l) + 2^{\alpha-l-1} & \text{for } H_{2^l} = \langle -1 \rangle \times \langle 5^{2^l} \rangle \text{ or } \langle -5^{2^{l-1}} \rangle. \end{cases}$$

This completes the proof of Theorem 1.3.

Acknowledgement

The authors would like to thank the referee for helpful comments and valuable suggestions.

References

[1] M. H. Jafari and A. R. Madadi, 'On the number of cyclic subgroups of a finite group', *Bull. Korean Math. Soc.* **54** (2017), 2141–2147.

[2] Y. Li and D. Kim, 'A Menon-type identity with many tuples of group of units in residually finite Dedekind domains', *J. Number Theory* **175** (2017), 42–50.

[3] Y. Li and D. Kim, 'Menon-type identities derived from actions of subgroups of general linear groups', *J. Number Theory* **179** (2017), 97–112.

[4] P. K. Menon, 'On the sum $\sum(a-1,n)[(a,n)=1]$', *J. Indian Math. Soc. (N.S.)* **29** (1965), 155–163.

[5] C. Miguel, 'Menon's identity in residually finite Dedekind domains', *J. Number Theory* **137** (2014), 179–185.

[6] C. Miguel, 'A Menon-type identity in residually finite Dedekind domains', *J. Number Theory* **164** (2016), 43–51.

[7] P. Neumann, 'A lemma that is not Burnside's', *Math. Sci.* **4** (1979), 133–141.

[8] B. Sury, 'Some number-theoretic identities from group actions', *Rend. Circ. Mat. Palermo (2)* **58** (2009), 99–108.

[9] M. Tărnăuceanu, 'A generalization of Menon's identity', *J. Number Theory* **132** (2012), 2568–2573.

[10] X. Zhang and C. G. Ji, 'Sums of generators of ideals in residue class ring', *J. Number Theory* **174** (2017), 14–25.

YU-JIE WANG, School of Mathematical Sciences, Nanjing Normal University,
Nanjing 210023, PR China
e-mail: wangyujie9291@126.com

CHUN-GANG JI, School of Mathematical Sciences, Nanjing Normal University,
Nanjing 210023, PR China
e-mail: cgji@njnu.edu.cn

https://doi.org/10.1017/S0004972718001168 Published online by Cambridge University Press