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$\mathfrak{m}$-BOUNDED EXTENSIONS OF TOPOLOGICAL SPACES

BY

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Introduction. An $\mathfrak{m}$-bounded extension of a topological space is an $\mathfrak{m}$-bounded space which contains the original as a dense subspace. $\mathfrak{m}$-bounded spaces have been studied by Gulden, Fleischman, and Weston [4], Saks and Stephenson [6], and Woods [8]. In [8], Woods showed the existence of a maximal $\mathfrak{m}$-bounded extension of a completely regular Hausdorff space X and characterized it as a subspace of βX.

We begin by examining $\mathfrak{m}$-bounded extensions in general and, as an example, construct the maximal $\mathfrak{m}$-bounded extension of a countably compact, linearly ordered topological space. Wallman $\mathfrak{m}$-bounded extensions, which parallel Wallman compactifications in the sense of Steiner [7], are considered in section two. In the final section we construct a one point $\mathfrak{m}$-bounded extension and, as an application, use it to strengthen a theorem of Glicksberg [3, p. 379] on products of $\mathfrak{m}$-compact spaces.

All hypothesized cardinals will be assumed infinite, and the cardinality of the set A will be designated $|A|$.

1. $\mathfrak{m}$-**Bounded extensions.** A topological space X is said to be $\mathfrak{m}$-*bounded* if for each $A \subseteq X$ with $|A| \leq \mathfrak{m}$ there is a compact subset K of X with $A \subseteq K$.

An $\mathfrak{m}$-*bounded extension* of a space X is a pair $(h, \mathfrak{m}X)$ where $\mathfrak{m}X$ is an $\mathfrak{m}$-bounded space and $h: X \to \mathfrak{m}X$ is a homeomorphism onto a dense subset of $\mathfrak{m}X$. An $\mathfrak{m}$-bounded extension $(h, \mathfrak{m}X)$ of a Tychonoff space is said to be a *maximal* $\mathfrak{m}$-*bounded extension* of X if for each $\mathfrak{m}$-bounded Tychonoff space Y and continuous function $f: X \to Y$ there is a continuous function $F: \mathfrak{m}X \to Y$ such that $f = F \circ h$.

If $(h, \mathfrak{m}X)$ is an $\mathfrak{m}$-bounded extension of X we shall identify X with its homeomorphic image $h[Z]$ in $\mathfrak{m}X$.

THEOREM 1.1. *If X is a Tychonoff space and $\mathfrak{m}$ is an infinite cardinal then there is a unique Tychonoff space $p\mathfrak{m}X$ which is a maximal $\mathfrak{m}$-bounded extension of X. $p\mathfrak{m}X$ can be identified with the set of all points in βX that belong to the βX-closure of some subset of X of cardinality at most $\mathfrak{m}$.*

Proof. Woods [8,1.3]. Wood's proof that *pmX* is $\mathfrak{m}$-bounded actually shows the following.

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LEMMA 1.1. *If Y is a compact space, $X \subseteq Y$ and* $\mathfrak{m}X = \bigcup \{A^- \mid A \subseteq X, |A| \leq \mathfrak{m}\}$ *where the closure is taken in Y, then $\mathfrak{m}X$ is $\mathfrak{m}$-bounded. $\mathfrak{m}X$ will be called the $\mathfrak{m}$-bounded completion of X in Y.*

COROLLARY 1.1. *If Y is a compact Hausdorff space, $X \subseteq Y$, and $\mathfrak{m}X$ the $\mathfrak{m}$-bounded completion of X in Y, then* $\mathfrak{m}X = \bigcap \{Z \mid X \subseteq Z \subseteq Y, \text{ and } Z \text{ is } \mathfrak{m}\text{-bounded}\}$.

Proof. Let $p \in \mathfrak{m}X$ then p is in the Y-closure of some subset $A \subseteq X$ satisfying $|A| \leq \mathfrak{m}$. If $X \subseteq Z \subseteq Y$ and Z is $\mathfrak{m}$-bounded then, since Z is Hausdorff, the Z-closure of A is compact and hence closed in Y. Thus the Y-closure of A is a subset of Z and hence $\mathfrak{m}X \subseteq \bigcap \{Z \mid X \subseteq Z \subseteq Y \text{ and } Z \text{ is } \mathfrak{m}\text{-bounded}\}$. Since $\mathfrak{m}X$ is $\mathfrak{m}$-bounded the equality is clear.

A point x in a space X is said to be a *complete accumulation point* of a set $A(\neq \varnothing) \subseteq X$ if for each open neighbourhood U of x, $|U \cap A| = |A|$. We designate $\{x \in X \mid x \text{ is a complete accumulation point of } A \text{ in } X\}$ by $Ca'_c A$ and $A \cup Ca'_c A$ by $Ca_c A$.

LEMMA 1.2. *Let Y be a compact space, $X \subseteq Y$ and $\mathfrak{m}X$ the $\mathfrak{m}$-bounded completion of X in Y. If* $Z = \bigcup \{Ca_Y A \mid A \subseteq X, 0 \neq |A| \leq \mathfrak{m}\}$ *then $Z = \mathfrak{m}X$.*

Proof. Suppose $y \in \mathfrak{m}X$ then y is in the Y-closure of some $M \subseteq X$ with $|M| \leq \mathfrak{m}$. Let $\mathfrak{n} = \min\{|U \cap M| \mid U \text{ open in } Y, y \in U\}$. Then $0 < \mathfrak{n} \leq \mathfrak{m}$ and $y \in Ca_Y(U \cap M)$ for some U such that $|U \cap M| = \mathfrak{n}$. Thus $\mathfrak{m}X \subseteq Z$. Since $Ca_Y A$ is a subset of the Y-closure of A, $Z \subseteq \mathfrak{m}X$.

DEFINITION. A space X is said to be $\mathfrak{m}$-*compact* if each open cover, of cardinality no more than $\mathfrak{m}$, has a finite subcover.

THEOREM 1.2. *Let X be a Tychonoff space and Y a Hausdorff compactification of X.*
(a) *X is $\mathfrak{m}$-bounded if and only if for each $A \subseteq X$ with $|A| \leq \mathfrak{m}$, $Ca'_Y A \subseteq X$.*
(b) *X is $\mathfrak{m}$-compact if and only if for each $A \subseteq X$ with $|A| \leq \mathfrak{m}$, $X \cap Ca'_Y A \neq \varnothing$.*

Proof. (a) X is $\mathfrak{m}$-bounded if and only if $X = \mathfrak{m}X$
(b) X is $\mathfrak{m}$-compact if and only if each subset of X of cardinality at most $\mathfrak{m}$ has a complete accumulation point (essentially outlined in Kelley [5, Problem 51]).

EXAMPLE. Let X be a linearly ordered set and X^+ its order completion. For each gap u of X (i.e. $u \in X^+ \backslash X$) let $u_1 = u$ if u is the right end-gap, and $u_2 = u$ if u is the left end-gap, otherwise let $u_1 = u \times 1$ and $u_2 = u \times 2$. Let $H = \{u_i \mid i = 1, 2 u \in X^+ \backslash X\}$ and call the elements of H half gaps of X Let $X^{++} = X \cup H$ and extend the order of X to X^{++} in the obvious fashion with $u_1 < u_2$ for each interior gap u of X It is easily seen that the topology induced on X by the interval topology on X^{++} is the interval topology on X.

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For each regular initial ordinal ω_α let ω_α^* denote ω_α with the reverse order. A half gap $u_1(u_2)$ of X is called an ω_α-limit of X if the set of elements of X which precede u_1 (follow u_2) is cofinal (coinitial) with $\omega_\alpha(\omega_\alpha^*)$. The unique ordinal for which u_i is an ω_α-limit will be designated $\omega_{\alpha(u_i)}$.

LEMMA 1.3. *Let X be a linearly ordered topological space and $u_1(u_2)$ a half gap of X with $\omega_{\alpha(u_1)} > \omega_0(\omega_{\alpha(u_2)} > \omega_0)$. If $f: X \to R$ is a continuous function then there is a $z \in X$ so that $f \mid [z, u_1)(f \mid (u_2, z])$ is a constant. (R the real numbers).*

Proof. Essentially in Gillman and Jerison [2, §5.12].

THEOREM 1.3. *If X is an $\aleph_0$-compact linearly ordered topological space then $\beta X = X^{++}$.*

Proof. Since X^{++} is linearly ordered and has no gaps it is compact and Hausdorff. Since X is $\aleph_0$-compact $\omega_{\alpha(u_i)} > \omega_0$ for each halfgap u_i [4, Theorem 3]. Let f: $X \to R$ be a bounded continuous function then, for each $u_1(u_2) \in H$, there is a $z \in X$ and an $r \in R$ so that $f \mid [z, u_1) = r(f \mid (u_2, z] = r)$. Define $\beta f: X^{++} \to R$ by $\beta f \mid X = f$ and $\beta f(u_i) = r$.

COROLLARY. *If X is an $\aleph_0$-compact linearly ordered topological space then $p\mathfrak{m}X = X \cup \{u_i \in H \mid \aleph_{\alpha(u_i)} \le \mathfrak{m}\} \subseteq X^{++}$.*

THEOREM 1.4. *Let X be a linearly ordered topological space. βX is orderable if and only if X is $\aleph_0$-compact.*

Proof. If X is $\aleph_0$-compact then $\beta X = X^{++}$. Suppose X is not $\aleph_0$-compact then it has a countable, closed, discrete subspace C. Since X is normal C is C^*-embedded in X and hence the closure D of C in βX is homeomorphic to βN (N positive integers). The order induced on D by the order on βX gives D an interval topology which is a subset of the relative topology on D as a subspace of βX. Since the interval topology is Hausdorff and the subspace topology is compact they are identical. Thus βN is orderable. But clearly βN is not orderable for if it were then for each $p \in \beta N \backslash N$ $\beta N \backslash p$ would be an $\aleph_0$-compact orderable space which is not $\aleph_0$-bounded. (See [4, Theorem 3].)

2. **Wallman $\mathfrak{m}$-bounded extensions.** The notation and terminology in this section are taken from Steiner [7].

Let $\mathscr{F}$ be a ring of subsets of X, $\mathscr{A}$ an $\mathscr{F}$-ultrafilter on X, and $\mathscr{S} = \{A \subseteq X \mid A \subseteq F \in \mathscr{F}$ implies $F \in \mathscr{A}\}$. Define $\Phi(\mathscr{A}) = \min\{|A| \setminus A \in \mathscr{S}\}$.

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THEOREM 2.1. *Let X be a T_1 space and $\mathscr{F}$ a separating ring of closed subsets of X. If $\mathfrak{m}X$ is the $\mathfrak{m}$-bounded completion of X in $w(X, \mathscr{F})$ then* $\mathfrak{m}X=\{\mathscr{A} \in w(X, \mathscr{F}) \mid \Phi(\mathscr{A}) \leq \mathfrak{m}\}$.

Proof. $\mathscr{A} \in \mathfrak{m}X$ if and only if there is an $A \subseteq X$ such that $|A| \leq \mathfrak{m}$ and $\mathscr{A} \in \mathrm{Cl}\, A$ (closure in $w(X, \mathscr{F})$). But

$$\mathrm{Cl}\, A = \bigcap \{F^* \mid F \in \mathscr{F} \text{ and } A \subseteq F^*\}$$
$$= \bigcap \{F^* \mid F \in \mathscr{F} \text{ and } A \subseteq F\}.$$

Thus $\mathscr{A} \in \mathrm{Cl}\, A$ if and only if $A \subseteq F \in \mathscr{A}$ implies $F \in \mathscr{A}$. Hence $\mathscr{A} \in \mathfrak{m}X$ if and only if $\Phi(\mathscr{A}) \leq \mathfrak{m}$.

COROLLARY 1. *If X is a T_1 space and $\mathscr{F}$ is the collection of all closed subsets of X then* $\mathfrak{m}X=\{\mathscr{A} \in w(X, \mathscr{F}) \mid A^- \in \mathscr{A}$ *for some* $A \subseteq X$ *with* $|A| \leq \mathfrak{m}\}$.

COROLLARY 2. *If X is an infinite set with the discrete topology and $\mathfrak{m}X$ is the $\mathfrak{m}$-bounded completion of X in βX then*

$$\mathfrak{m}X = p\mathfrak{m}\, X = \{\mathscr{A} \in \beta X \mid \text{there is an } A \in \mathscr{A} \text{ with } |A| \leq \mathfrak{m}\}.$$

In this case $\mathfrak{m}X$ is an open subset of βX and hence locally compact.

Proof. If $\mathscr{A} \in \mathfrak{m}X$ then $|A| \leq \mathfrak{m}$ for some $A \in \mathscr{A}$. Thus $\mathscr{A} \in A^* \subseteq \mathfrak{m}X$ and A^* is open.

THEOREM 2.2. *A regular space X is $\mathfrak{m}$-bounded if and only if each ultrafilter $\mathscr{F}$ on X with $\Phi(\mathscr{F}) \leq \mathfrak{m}$ converges.*

Proof. Recall that for a regular space to be compact it is sufficient that each filter on a dense subset of X have a nonvoid adherence in X.

Suppose $A \subseteq X$, $|A| \leq \mathfrak{m}$ and $\mathscr{F}$ is a filter on A. There is an ultrafilter $\mathscr{G}$ on X with $\mathscr{F} \subseteq \mathscr{G}$, hence $\Phi(\mathscr{G}) \leq \mathfrak{m}$, and thus $\mathscr{G}$ converges to some point in A^-. Thus $\mathscr{F}$ has a nonvoid adherence in A^- and therefore A^- is compact.

Suppose X is $\mathfrak{m}$-bounded and $\mathscr{F}$ is an ultrafilter on X with $\Phi(\mathscr{F}) \leq \mathfrak{m}$. Then there is an $A \in \mathscr{F}$ with $|A| \leq \mathfrak{m}$ and hence $A \subseteq K$ for some compact subset K of X. Thus $K \in \mathscr{F}$ and $\mathscr{F}$ is convergent.

3. **One point $\mathfrak{m}$-bounded extensions.** Let X be a non $\mathfrak{m}$-bounded Hausdorff space, $\mathscr{T}$ its topology and $\mathscr{S}=\{A^- \mid A \subseteq X, |A| \leq \mathfrak{m}$ and A^- is not compact$\}$. Let $p \notin X$, $X^*=X \cup \{p\}$, and $\mathscr{T}^*=\mathscr{T} \cup \{U \subseteq X^* \mid p \in U,\ U \cap X \in \mathscr{T}$, and $S \backslash U$ is compact for each $S \in \mathscr{S}\}$.

It is easily seen that $\mathscr{T}^*$ is an $\mathfrak{m}$-bounded topology for X^* and that X is dense in X^*. X^* will be called the *one point $\mathfrak{m}$-bounded extension of X.*

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As in the case of the one point compactification of a space X it is important to know conditions for X^* to be Hausdorff.

THEOREM 3.1. *Let X be a non $\mathfrak{m}$-bounded Hausdorff space and X^* its one point $\mathfrak{m}$-bounded extension. The following are then equivalent.*

(a) *X^* is Hausdorff*

(b) *For each $x \in X$ there is an open neighborhood U of x such that for each $A \subseteq X$ with $|A| \leq \mathfrak{m}$, $U^- \cap A^-$ is compact.*

Proof. (a)$\rightarrow$(b): If X^* is Hausdorff then for each $x \in X$ there are disjoint open sets U, V in X^* with $x \in U$ and $p \in V$. Let $A \subseteq X$ with $|A| \leq \mathfrak{m}$ then, since X^* is $\mathfrak{m}$-bounded and Hausdorff, $A^- \backslash V = A^{-*} \backslash V$ is compact. Since $A^- \cap U^- \subseteq A^- \backslash V$, $A^- \cap U^-$ is compact.

(b)$\rightarrow$(a): Let $x \in X$ then there is an open neighborhood U of x satisfying (b). Let $V = X^* \backslash U^-$. If $A \subseteq X$ with $|A| \leq \mathfrak{m}$ then $A^- \backslash V = A^- \cap U^-$ and thus $A^- \backslash V$ is compact. Hence V is open in X^*.

DEFINITION. A space X is said to be *locally $\mathfrak{m}$-bounded* if for each $x \in X$ there is an open neighborhood U of x such that U^- is $\mathfrak{m}$-bounded.

THEOREM 3.2. *Let X be a non $\mathfrak{m}$-bounded regular space and X^* its one point $\mathfrak{m}$-bounded extension. X^* is Hausdorff if and only if X is locally $\mathfrak{m}$-bounded.*

Proof. We shall show that a regular locally $\mathfrak{m}$-bounded space satisfies (b) of Theorem 3.1. Let $x \in X$ and U be an open neighborhood of x so that U^- is $\mathfrak{m}$-bounded. Let V be an open neighborhood of x such that $V^- \subseteq U$. If $A \subseteq X$ with $|A| \leq \mathfrak{m}$ then $V^- \cap A^- \subseteq U^- \cap (U \cap A)^-$. $U^- \cap (U \cap A)^-$ is compact and hence so is $V^- \cap A^-$.

Theorem 3.2 is not true for Hausdorff spaces as is seen by the following example of a locally $\aleph_0$-bounded Hausdorff space which does not satisfy (b) of Theorem 3.1.

EXAMPLE. Let $T = W(\omega_1+1) \times W(\omega_0+1)$ have the product topology where $W(\gamma) = \{\alpha < \gamma\}$. There is a compactification γN of the positive integers N so that $\gamma N \backslash N$ is homeomorphic to $W(\omega_1+1)$ [1, Example 1.1]. Let Y be the quotient space of $T \cup \gamma N$ obtained by identifying each $\alpha \in \gamma N \backslash N$ with $(\alpha, \omega_0) \in T$. The space X is the set Y with the smallest topology containing the quotient topology and $\{U(\alpha, n) \mid 0 < \alpha < \omega_1, 0 < n < \omega_0\}$ where $U(\alpha, n) = \{(\beta, k) \in T \mid \alpha < \beta \leq \omega_1, n < k < \omega_0\} \cup \{(\omega_1, \omega_0)\}$.

Each point of X except (ω_1, ω_0) has a compact neighborhood and $\{U(\alpha, n) \mid 0 < \alpha < \omega_1, 0 < n < \omega_0\}$ is an open neighborhood base of (ω_1, ω_0). $U(\alpha, n)^- = (\alpha, \omega_1] \times (n, \omega_0]$ is $\aleph_0$-bounded for if $C \subseteq U(\alpha, n)^-$ is countable then there is a $\beta < \omega_1$ so that

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$C \subseteq [W(\beta+1)\times W(\omega_0+1)] \cup [\{\omega_1\}\times W(\omega_0+1)]$ which is compact. X does not satisfy (b) of Theorem 3.2 since $N^- \cap U(\alpha, n)^- = (\alpha, \omega_1]\times\{\omega_0\}$ which is not compact.

APPLICATION. In [3, pp. 379–380, Remark (2)] Glicksberg proves the following.

THEOREM 3.3. *The product of at most* $\mathfrak{m}$ *Hausdorff spaces, each* $\mathfrak{m}$*-compact and all but at most one locally compact, is* $\mathfrak{m}$*-compact.*

Using the concept of the one point $\mathfrak{m}$-bounded extension we are able to modify Glicksberg's proof to prove the following theorem.

THEOREM 3.4. *The product of at most* $\mathfrak{m}$ *regular spaces, each* $\mathfrak{m}$*-compact and all but at most one locally* $\mathfrak{m}$*-bounded, is* $\mathfrak{m}$*-compact.*

Proof. Let $\{X_\alpha \mid 1\leq\alpha<\omega_\gamma\}$ be the hypothesized spaces, $X_1^*=X_1$ the exceptional case and $\aleph_\gamma\leq\mathfrak{m}$. For each $\alpha>1$ let X_α^* be the one point $\mathfrak{m}$-bounded extension of X so that $\pi\{X_\alpha^* \mid 1\leq\alpha<\omega_\gamma\}$ is $\mathfrak{m}$-compact. Without further modification carry out the proof of [3, pp. 379–380, Remark (2)].

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