# A FEW POINTS ON THE STABILITY OF THE SOLAR SYSTEM

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**Abstract.** The secular equations which were used to exhibit the chaotic behaviour of the solar system (Laskar, 1989) are established here in a Hamiltonian framework. The integration of the former secular equations over 400 Myr showed that the two resonant arguments $2(\varpi_4^\bullet - \varpi_3^\bullet) - (\Omega_4^\bullet - \Omega_3^\bullet)$ and $(\varpi_1^\bullet - \varpi_5^\bullet) - (\Omega_1^\bullet - \Omega_2^\bullet)$ given in (Laskar, 1990) present both transitions from libration to circulation. During the circulation of the first argument, temporary libration of the argument $(\varpi_4^\bullet - \varpi_3^\bullet) - (\Omega_4^\bullet - \Omega_3^\bullet)$ is observed, revealing resonance overlap beetween these two resonances, which explains the existence of a large chaotic zone for the motion of the solar system.

## 1. Introduction

In the last 3 years, our vision of the motion of the solar system has changed a lot and the idea of a "stable" solar system suffered many outrages. In their integration of the motion of the outer planets over 875 Myr, Sussman and Wisdom (1988) found a chaotic behaviour for the motion of Pluto, with a Lyapounov exponent of $1/20\ \mathrm{Myr}^{-1}$. Using a different approach, based on perturbation techniques and huge algebraic manipulations, I was able to construct an averaged system of second order with respect to the masses which represents the secular evolution of the orbits of the 8 main planets of the solar system (Laskar, 1985, 1986, 1987, 1988). The integration of this secular system over 200 Myr revealed that the motion of the solar system is chaotic with a Lyapounov exponent reaching $1/5\ \mathrm{Myr}^{-1}$ (Laskar, 1989). Over such time span, this chaotic behaviour affects mainly the inner planets. It was identified as the result of two secular resonances among the inner planets which have not been identified in any previous theories of the motion of the planets (Laskar, 1990). In the present paper, I would like to developped a few points which may be helpful for a better understanding of this chaotic behaviour.

## 2. Analytical Averaging

### 2.1. Planetary theory in Poincaré canonical heliocentric variables

Although the original computations were made with usual non canonical formalism, it is my strong feeling that the canonical heliocentric variables introduced by Poincaré (1896) are more suitable for the elaboration of such analytical work. The use of these variables should not change the conclusions and implications of the computation which were already done, but they should be essential for a better understanding of the dynamical implications of the equations, or for eventual higher-order computations.

Let $P_0, P_1, \ldots, P_n$ be $n+1$ bodies of masses $m_0, m_1, \ldots, m_n$ in gravitational interaction, and let $O$ be their center of mass. For every body $P_i$, we shall denote $\mathbf{u}_i = \vec{OP_i}$. In the barycentric reference frame with origine $O$, the Newton equations of motion form a differential system of order $6(n+1)$ and can be written for each $i$



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$$\frac{d^2\mathbf{u}_i}{dt^2} = -G\sum_{j\neq i} m_j \frac{\mathbf{u}_i - \mathbf{u}_j}{\Delta_{ij}^3}.$$

where $\Delta_{ij} = \| \mathbf{u}_i - \mathbf{u}_j \|$, and $G$ is the constant of gravitation. A canonical set of coordinates is $(\mathbf{u}_i, \tilde{\mathbf{u}}_i = m_i \dot{\mathbf{u}}_i)_{i=1,n}$ and in this set of coordinates the Hamiltonian will be

$$H = \frac{1}{2}\sum_{i=0}^{n} \frac{\| \tilde{\mathbf{u}}_i \|^2}{m_i} - G \sum_{0\leq i<j} \frac{m_i m_j}{\Delta_{ij}}.$$

The reduction of the center of mass is achieved generally in the planetary case by using the non-canonical heliocentric variables $(\mathbf{r}_i, \dot{\mathbf{r}}_i)$ with $\mathbf{r}_i = \mathbf{u}_i - \mathbf{u}_0$ and $\dot{\mathbf{r}}_i = d\mathbf{r}_i/dt$. Actually, canonical heliocentric coordinates are very easy to introduce: Let $\mathbf{r}_0 = \mathbf{u}_0$, the linear transformation $A$ which transforms the variables $(\mathbf{u}_i)$ into the variables $(\mathbf{r}_i)$ can be extended to a canonical transformation

$$(\mathbf{u}_i, \tilde{\mathbf{u}}_i) \longrightarrow (\mathbf{r}_i, \tilde{\mathbf{r}}_i) = \left(A\mathbf{u}_i, {}^t A^{-1}\tilde{\mathbf{u}}_i\right)$$

which leads to the very simple expressions

$$\begin{aligned} \tilde{\mathbf{r}}_0 &= \tilde{\mathbf{u}}_0 + \tilde{\mathbf{u}}_1 + \cdots + \tilde{\mathbf{u}}_n \\ \tilde{\mathbf{r}}_i &= \tilde{\mathbf{u}}_i \qquad \text{for} \qquad i \neq 0 \end{aligned}.$$

The reduction of the center of mass gives $\tilde{\mathbf{r}}_0 = 0$. In these new variables, the expression of the kinetic energy is

$$T = \frac{1}{2}\sum_{i=1}^{n} \frac{\| \tilde{\mathbf{r}}_i \|^2}{m_i} + \frac{1}{2}\frac{\| \tilde{\mathbf{r}}_1 + \tilde{\mathbf{r}}_2 + \ldots + \tilde{\mathbf{r}}_n \|^2}{m_0}.$$

That is,

$$T = T_0 + T_1 \quad \text{with} \quad T_0 = \frac{1}{2}\sum_{i=1}^{n} \| \tilde{\mathbf{r}}_i \|^2 \left[\frac{1}{m_i} + \frac{1}{m_0}\right] \quad \text{and} \quad T_1 = \sum_{0<i<j} \frac{\tilde{\mathbf{r}}_i \cdot \tilde{\mathbf{r}}_j}{m_0}.$$

The computation of the potential is also simple and gives

$$U = U_0 + U_1 \quad \text{with} \quad U_0 = -G\sum_{i=1}^{n} \frac{m_0 m_i}{r_i} \quad \text{and} \quad U_1 = -G\sum_{0<i<j} \frac{m_i m_j}{\Delta_{ij}}$$

with $\Delta_{ij} = \| \mathbf{r}_i - \mathbf{r}_j \|$.

The full Hamiltonian is thus of the form

$$H = H_0 + \varepsilon H_1; \qquad \text{with} \qquad H_0 = T_0 + U_0 \qquad \text{and} \qquad \varepsilon H_1 = T_1 + U_1.$$

$H_0$ is integrable and represents the Hamiltonian of a family of disjoint two body heliocentric problems: the "planets" of mass $m_i$ around the "sun" of mass $m_0$.

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$\varepsilon H_1$ is the perturbing part of the Hamiltonian resulting from the planets interactions and $\varepsilon$ is a small parameter of same order as the planetary masses. It should be stressed that the expressions of the complete Hamiltonian in these variables are more symetrical and simpler than in Jacobi coordinates. Once the planetary problem is expressed as the perturbation of an integrable problem (here a union of two body problems) of Hamiltonian $H_0$, the coordinates which are convenient are coordinates which form angle-action coordinates for the integrable unperturbed problem. The Hamiltonian can then be expanded with respect to Poincaré elliptical variables $(\lambda, \Lambda, \xi, \eta, p, q)$ in a convenient form for planetary equations(Laskar, 1991). For each planet $P_j$, $\lambda_j$ is the mean longitude,

$$\Lambda_j = \frac{m_0 m_j}{m_0 + m_j}\sqrt{G(m_0 + m_j)a_j}$$

is its conjugate variable related to the semi major axis and

$$\xi_j - i\eta_j = \sqrt{2\Lambda_j}\sqrt{1 - \sqrt{1 - e_j^2}}\, E^{i\varpi_j};$$

$$p_j - iq_j = \sqrt{2\Lambda_j}\sqrt{\sqrt{1 - e_j^2}(1 - \cos i_j)}\, E^{i\Omega_j}.$$

### 2.2. Averaging

With the use of the canonical heliocentric variables, we have thus managed to put the Hamiltonian on the form

$$H(\lambda, \Lambda, \xi, \eta) = H_0(\Lambda) + \varepsilon H_1(\lambda, \Lambda, \xi, \eta)$$

where $\varepsilon$ is a small parameter of the order of the planetary masses. For simplicity of the notations, $\xi$ denotes the variables $\xi_j, p_j$ and $\eta$ the variables $\eta_j, q_j$. Let us make the canonical change of variables depending of time given by the generating function

$$S(\lambda, \tilde\Lambda, \xi, \tilde\eta) = \lambda \cdot \tilde\Lambda + \xi \cdot \tilde\eta - N\,\tilde\Lambda\, t + \Lambda_0\,\lambda$$

We have thus

$$\begin{aligned} \Lambda &= \tilde\Lambda + \Lambda_0; \qquad & \tilde\lambda &= \lambda - N\,t; \\ \eta &= \tilde\eta; & \tilde\xi &= \xi; \end{aligned}$$

and the Hamiltonian becomes

$$\widetilde{H} = H + \frac{\partial S}{\partial t}$$

that is, in the new variables

$$\widetilde{H}(\tilde\lambda, \tilde\Lambda, \tilde\xi, \tilde\eta, t) = H_0(\tilde\Lambda + \Lambda_0) + \varepsilon H_1(\tilde\lambda + Nt, \tilde\Lambda + \Lambda_0, \tilde\xi, \tilde\eta) - N\tilde\Lambda.$$

We have

$$\frac{d\tilde{\Lambda}}{dt} = -\varepsilon \frac{\partial H_1}{\partial \tilde{\lambda}}.$$

Thus, if we choose $\Lambda_0$ as the constant part of $\Lambda$, the integration constant of $\tilde{\Lambda}$ is zero and $\tilde{\Lambda}$ becomes of order 1 with respect to $\varepsilon$ (this can only be achieved by an iterative scheme). By expanding up to order 2 with respect to $\tilde{\Lambda}$ and by choosing for the value of $N$

$$N = \frac{\partial H_0}{\partial \Lambda}(\Lambda_0)$$

we obtain for the new Hamiltonian

$$\widetilde{H}(\tilde{\lambda}, \tilde{\Lambda}, \tilde{\xi}, \tilde{\eta}, t) = H_0(\Lambda_0) + \varepsilon H_1(\tilde{\lambda} + Nt, \tilde{\Lambda} + \Lambda_0, \tilde{\xi}, \tilde{\eta}) + \frac{\partial^2 H_0}{\partial \Lambda^2}(\Lambda_0) \cdot \tilde{\Lambda}^2 + O(\tilde{\Lambda}^3) \tag{1}$$

The constant part $H_0(\Lambda_0)$ does not affect the solutions and can be suppressed. With this change of variables, which consist to search for a solution in the vicinity of the periodic solutions of the unperturbed motion of given semi major axis (or in an equivalent way, of given mean motion), we have managed to suppress the zero order part of the Hamiltonian, as $\tilde{\Lambda}$ is now considered to be of the same order as the small parameter $\varepsilon$.

## 2.3. The direct method of Poincaré

This method of averaging is described in the *Methodes Nouvelles* and has often being called Krylov-Bogolioubov method. It consists to search a solution in quasi periodic form with respect to the angles $\lambda$ by simple identification, as the existence of such a formal change of variables is already given by the Lindstedt-Poincaré method (Poincaré, 1892). We search directly for a change of variables depending on time of the form $(\tilde{\lambda}, \tilde{\Lambda}, \tilde{\xi}, \tilde{\eta}) \longrightarrow (\hat{\lambda}, \hat{\Lambda}, \hat{\xi}, \hat{\eta})$ given by

$$\begin{aligned} \tilde{\lambda} &= \hat{\lambda} + S_\lambda(\hat{\lambda} + Nt, \hat{\Lambda}, \hat{\xi}, \hat{\eta}) \\ \tilde{\Lambda} &= \hat{\Lambda} + S_\Lambda(\hat{\lambda} + Nt, \hat{\Lambda}, \hat{\xi}, \hat{\eta}) \\ \tilde{\xi} &= \hat{\xi} + S_\xi(\hat{\lambda} + Nt, \hat{\Lambda}, \hat{\xi}, \hat{\eta}) \\ \tilde{\eta} &= \hat{\eta} + S_\eta(\hat{\lambda} + Nt, \hat{\Lambda}, \hat{\xi}, \hat{\eta}) \end{aligned} \tag{2}$$

which suppressed the periodic terms depending of $\hat{\lambda} + Nt$ in the Hamiltonian. For each variable $x = \lambda, \Lambda, \xi, \eta$, $S_x$ is purely periodic with respect to $\hat{\lambda} + Nt$ and expanded as a series $S_x = \varepsilon S_{1x} + \varepsilon^2 S_{2x} + O(\varepsilon^3)$ (the formal existence of such a change of variables is obtained with Lindstedt series). Let us consider the differential system of equations for the Hamiltonian (1)

$$\frac{d\tilde{x}}{dt} = \varepsilon F_x(\tilde{\lambda} + Nt, \tilde{\Lambda}, \tilde{\xi}, \tilde{\eta}); \qquad \text{for } x = \lambda, \Lambda, \xi, \eta \tag{3}$$

with

$$\varepsilon F_\lambda = \varepsilon \frac{\partial H_1}{\partial \tilde{\Lambda}} + 2\frac{\partial^2 H_0}{\partial \Lambda^2}(\tilde{\Lambda}_0)\cdot\tilde{\Lambda} + O(\tilde{\Lambda}^2);$$

$$\varepsilon F_\Lambda = -\varepsilon \frac{\partial H_1}{\partial \tilde{\lambda}}; \qquad \varepsilon F_\xi = \varepsilon \frac{\partial H_1}{\partial \tilde{\eta}}; \qquad \varepsilon F_\eta = -\varepsilon \frac{\partial H_1}{\partial \tilde{\xi}}. \tag{4}$$

The substitution of (2) in (3) gives

$$\frac{d\hat{x}}{dt} + \varepsilon \sum_{y_j} \frac{\partial S_{1x}}{\partial \hat{y}_j}\cdot\frac{d\hat{y}_j}{dt} + \varepsilon\frac{\partial S_{1x}}{\partial t} + \varepsilon^2\frac{\partial S_{2x}}{\partial t} = \varepsilon F_x(\hat{\lambda}+Nt,\hat{\Lambda},\hat{\xi},\hat{\eta}) +$$

$$\varepsilon^2 \sum_{y_j}\frac{\partial F_x}{\partial \hat{y}_j}(\hat{\lambda}+Nt,\hat{\Lambda},\hat{\xi},\hat{\eta})\cdot S_{1y_j}(\hat{\lambda}+Nt,\hat{\Lambda},\hat{\xi},\hat{\eta}) + O(\varepsilon^3)$$

for $y_j = \lambda_j, \xi_j, \Lambda_j, \eta_j$. If we choose

$$\frac{\partial S_{1x}}{\partial t} = \left\{F_x(\hat{\lambda}+Nt,\hat{\Lambda},\hat{\xi},\hat{\eta})\right\}_t \tag{5}$$

and

$$\frac{\partial S_{2x}}{\partial t} = \sum_{y_j}\left\{\frac{\partial F_x}{\partial \hat{y}_j}(\hat{\lambda}+Nt,\hat{\Lambda},\hat{\xi},\hat{\eta})\cdot S_{1y_j}(\hat{\lambda}+Nt,\hat{\Lambda},\hat{\xi},\hat{\eta})\right\}_t -$$

$$\sum_{y_j}\frac{\partial S_{1x}}{\partial \hat{y}_j}\cdot\left\langle F_{y_j}(\hat{\lambda}+Nt,\hat{\Lambda},\hat{\xi},\hat{\eta})\right\rangle_t$$

where $\{F\}_t$ is the purely periodic part of $F$ and $\langle F\rangle_t$ the secular part of $F$; we obtain, by neglecting terms of order 3,

$$\frac{d\hat{x}}{dt} = \varepsilon\left\langle F_x(\hat{\lambda}+Nt,\hat{\Lambda},\hat{\xi},\hat{\eta})\right\rangle_t + \varepsilon^2\sum_{y_j}\left\langle\frac{\partial F_x}{\partial \hat{y}_j}(\hat{\lambda}+Nt,\hat{\Lambda},\hat{\xi},\hat{\eta})\cdot S_{1y_j}(\hat{\lambda}+Nt,\hat{\Lambda},\hat{\xi},\hat{\eta})\right\rangle_t \tag{6}$$

This new system is much more simpler as it does not depend of $\hat{\lambda}+Nt$. We have also

$$\frac{d\hat{\Lambda}}{dt} = \varepsilon\left\langle F_\Lambda(\hat{\lambda}+Nt,\hat{\Lambda},\hat{\xi},\hat{\eta})\right\rangle_t + \varepsilon^2\sum_{y_j}\left\langle\frac{\partial F_\Lambda}{\partial y_j}(\hat{\lambda}+Nt,\hat{\Lambda},\hat{\xi},\hat{\eta})\cdot S_{1y_j}(\hat{\lambda}+Nt,\hat{\Lambda},\hat{\xi},\hat{\eta})\right\rangle_t = 0$$

which is Poisson's theorem. It derives from the fact that $F_\Lambda$ is purely periodic an can be verified using the expressions of $F_\Lambda$ (4) and $S_{1y_j}$ (5), and expanding $H_1$ in Fourier series

$$H_1(\tilde{\lambda}+Nt,\tilde{\Lambda},\tilde{\xi},\tilde{\eta}) = \sum_k \alpha_k(\tilde{\Lambda},\tilde{\xi},\tilde{\eta})E^{ik\cdot(\tilde{\lambda}+Nt)}.$$

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$\hat{\Lambda}$ is thus constant up to order 2 and, from the choice of $\Lambda_0$, we have $\hat{\Lambda} = 0$ and $\langle\Lambda\rangle_t = \Lambda_0$. The computation of the secular system of order 2 is thus particularly simple with the direct method. It may not be the case for higher orders, where a more automatic method like Lie series may be used.

## 3. Numerical results

### 3.1. Accuracy of the secular system

Once the secular system is obtained, it can be integrated numerically with a very large stepsize of about 500 yrs, because all the short periods involving the orbital periods of the planets have been removed. I made several numerical integrations of this system, which also includes the averaged effect of relativity and Moon. The integration over 200 Myr backward revealed that the motion of the solar system and especially the motion of the inner solar system is chaotic, with a Lyapounov exponent of about 1/(5 Myr) (Laskar, 1989, 1990). It was then discovered that this chaotic behaviour originated in the presence of essentially two secular resonances among the planets: $\theta = 2(g_4 - g_3) - (s_4 - s_3)$ and $\sigma = (g_1 - g_5) - (s_1 - s_2)$ (Laskar, 1990). The argument of $\theta$ was found to change several times from libration to circulation during the 200 Myr time span, while the argument of $\sigma$ stayed in libration, but with varying amplitude.

Since this first integration, I extended it forward for another 200 Myr and will present here some of the analysis of the output. The main concern, with such an averaged system, is not the accuracy of the numerical integration (integrating with a 500 yr stepsize over 200 Myr requires only 400000 steps), but the precision of the secular system itself, as a truncation is always used. We have to know how good the solutions of the secular system are compared to the averaged solutions of the full equations of Newton. One way to make an evaluation of the accuracy of the secular system is to make comparison with the results of a direct numerical integration of the full equations and such a comparison was made (Laskar, 1986), but until recently, the only numerical integration which was comparable was the numerical ephemeris from JPL DE102 which spans only about 4000 years. Only this year, Quinn *et al*(1991) issued a numerical integration of the full solar system, including effect of the Moon and general relativity which spanned 3Myr. We made the direct comparison of the secular solution La90 from (Laskar, 1990) with a second integration QTD6 made over 6 Myr by Quinn *et al* and found a very good quantitative agreement with the two solutions (Laskar *et al*, 1991). Over this time span of 6 Myr, the curves for the excentricity of the planets were barely distinguishable and, even more, over this timespan it was possible to confirm the existence of secular resonances in the inner solar system. The validity of all the results on the chaotic behaviour of the solar system which were already obtained with the secular theory was thus strengthened. Since, Wisdom, using his new integration scheme, issued an new integration of the solar system over 100 Myr which also confirms these results (Sussman and Wisdom, 1991).

### 3.2. Analysis of the 400 Myr solution

It is important to understand the meaning of analysing the solution over such an extended time span of 400 Myr. I showed that the motion is chaotic with a Lyapounov exponent of 1/(5Myr); it means that any computed solution will not represent the real solar system after about 100 Myr. Indeed, to obtain a solution over such a time span will already require to know all the initial conditions and parameters with an accuracy better than $10^{-10}$, and to take into account all the perturbations of this size, in the model, or in the integration. In particular, this implies to take into account the perturbation of about three dozen of asteroids (Williams, 1984). Extending the time span to 120 Myr will require an accuracy of $10^{-12}$ and to track several hundreds of asteroids.

On the other hand, as shown by the comparisons with numerical integrations, the solution of the secular system provides a good ephemeris of the solar system which can be used over about 10 Myr, as was already stated in (Laskar, 1988). But the reason for analysing the solution over 400 Myr is very different: this is a tool for a first analysis of the phase space in the region of the solution of the solar system. After a few tens of millions years, it does not represent the motion of the solar system, but only its possible motion in the chaotic region where it belongs.

Of course, as stated by Arnold (1963), as the solar system has much more degres of and the diffusion of the orbits can eventually take place in a very large portion of the phase space, but this diffusion can be very slow and not very meaningful over ages comparable to the age of the universe. On the contrary, in each of the projections of the phase space which correspond to the two resonant arguments $\theta$ and $\sigma$, the diffusion of the actions is much more rapid. By the numerical analysis of the fundamental frequencies, which are then used as parameters for the dynamical description of the system, I showed (Laskar, 1990) that the chaotic region is relatively large in the direction of the main frequencies of the inner planets.

This frequency analysis is efficient and, also, clearly reveals the existence of quasi-integrals in the outer solar system and the extent of the chaotic zones. It has been applied since to the standard mapping, in order to demonstrate its power on a much more simple two degrees of freedom problem (Laskar *et al*, 1991), but I understand that although frequencies are very meaningful for the dynamical understanding of the system, actions and physical parameters are more appealing for the physical meaning. This is why I present here some of the plots which were obtained during the analysis of the solution over 400 Myr.

### 3.3. Description of the plots

In figures (1–4) are plotted the eccentricities and inclination for the inner planets solutions over 400 Myr, as well as the associated proper modes. Indeed, due to the linear coupling (and resonant terms in the expansions), the solutions for any of the planets is in first approximation a linear combination of the proper modes (Laskar, 1987) which are obtained presently by the inverse linear transformation, given in (Laskar, 1990). The proper modes are associated to the different planets according to the conventional way of numbering the fundamental frequencies, although in

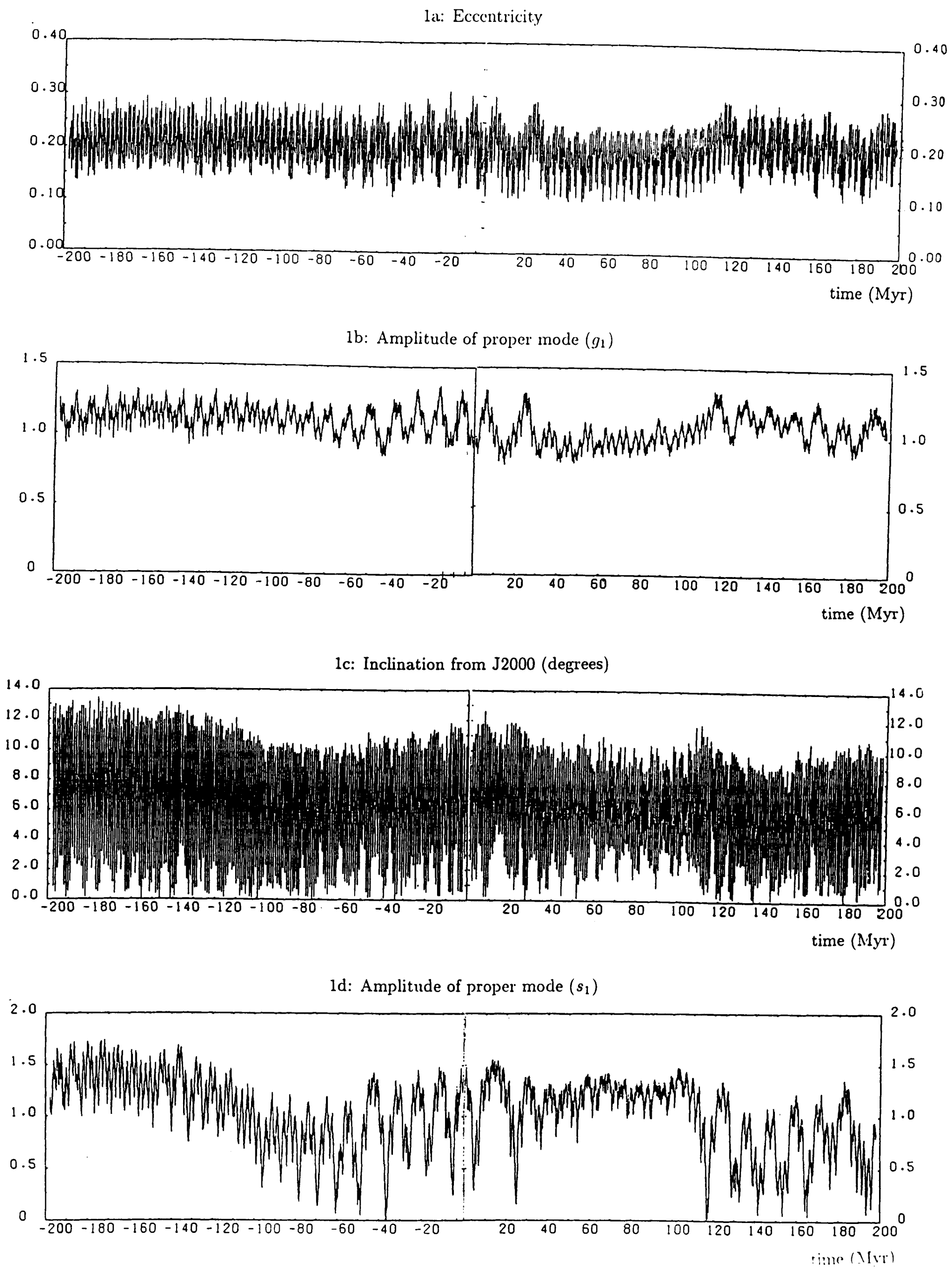


Fig. 1. Mercury

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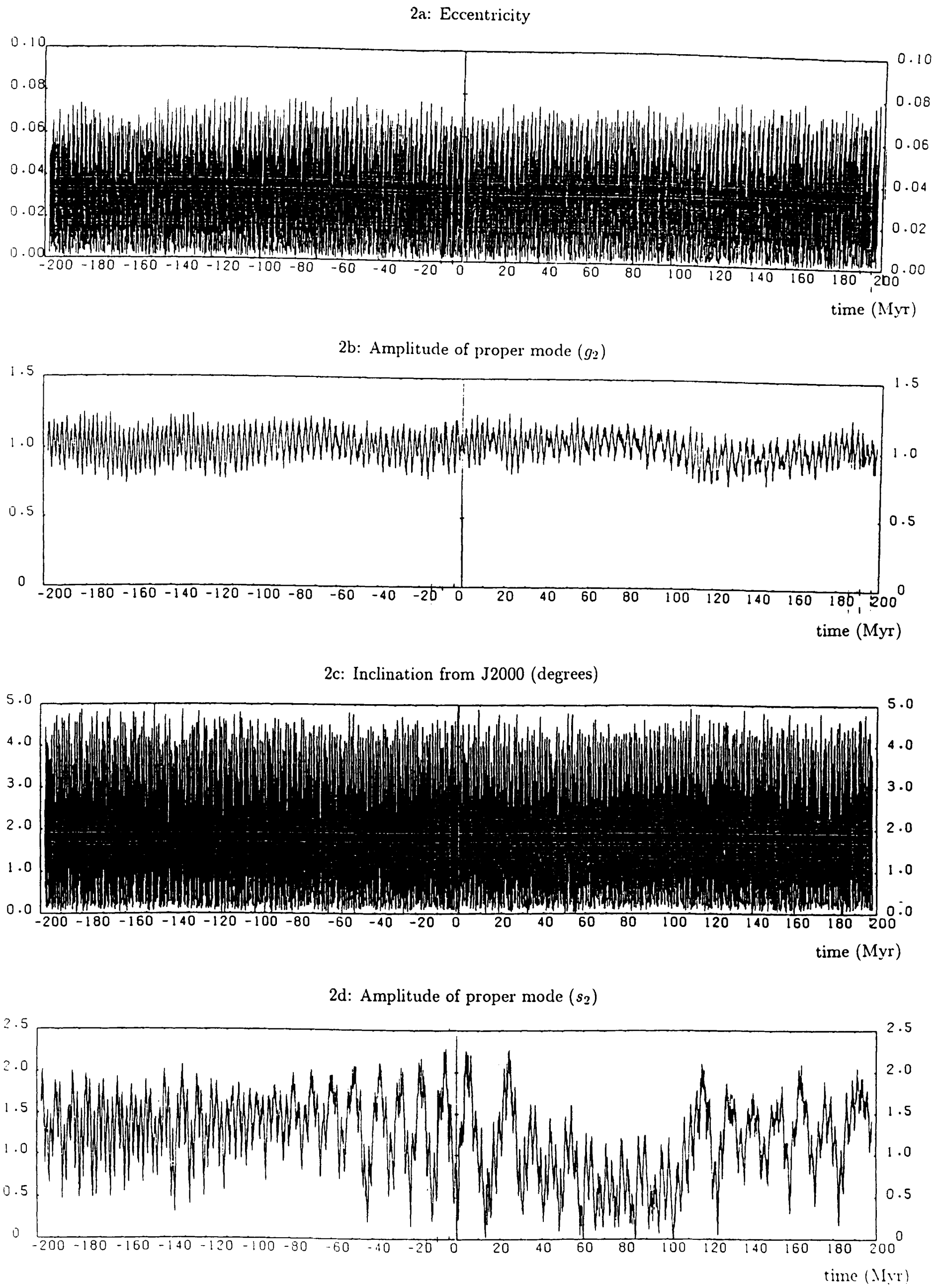
2a: Eccentricity
0.00 0.02 0.04 0.06 0.08 0.10
-200 -180 -160 -140 -120 -100 -80 -60 -40 -20 0 20 40 60 80 100 120 140 160 180 200
time (Myr)
2b: Amplitude of proper mode ($g_2$)
0 0.5 1.0 1.5
-200 -180 -160 -140 -120 -100 -80 -60 -40 -20 0 20 40 60 80 100 120 140 160 180 200
time (Myr)
2c: Inclination from J2000 (degrees)
0.0 1.0 2.0 3.0 4.0 5.0
-200 -180 -160 -140 -120 -100 -80 -60 -40 -20 0 20 40 60 80 100 120 140 160 180 200
time (Myr)
2d: Amplitude of proper mode ($s_2$)
0 0.5 1.0 1.5 2.0 2.5
-200 -180 -160 -140 -120 -100 -80 -60 -40 -20 0 20 40 60 80 100 120 140 160 180 200
time (Myr)

Fig. 2. Venus

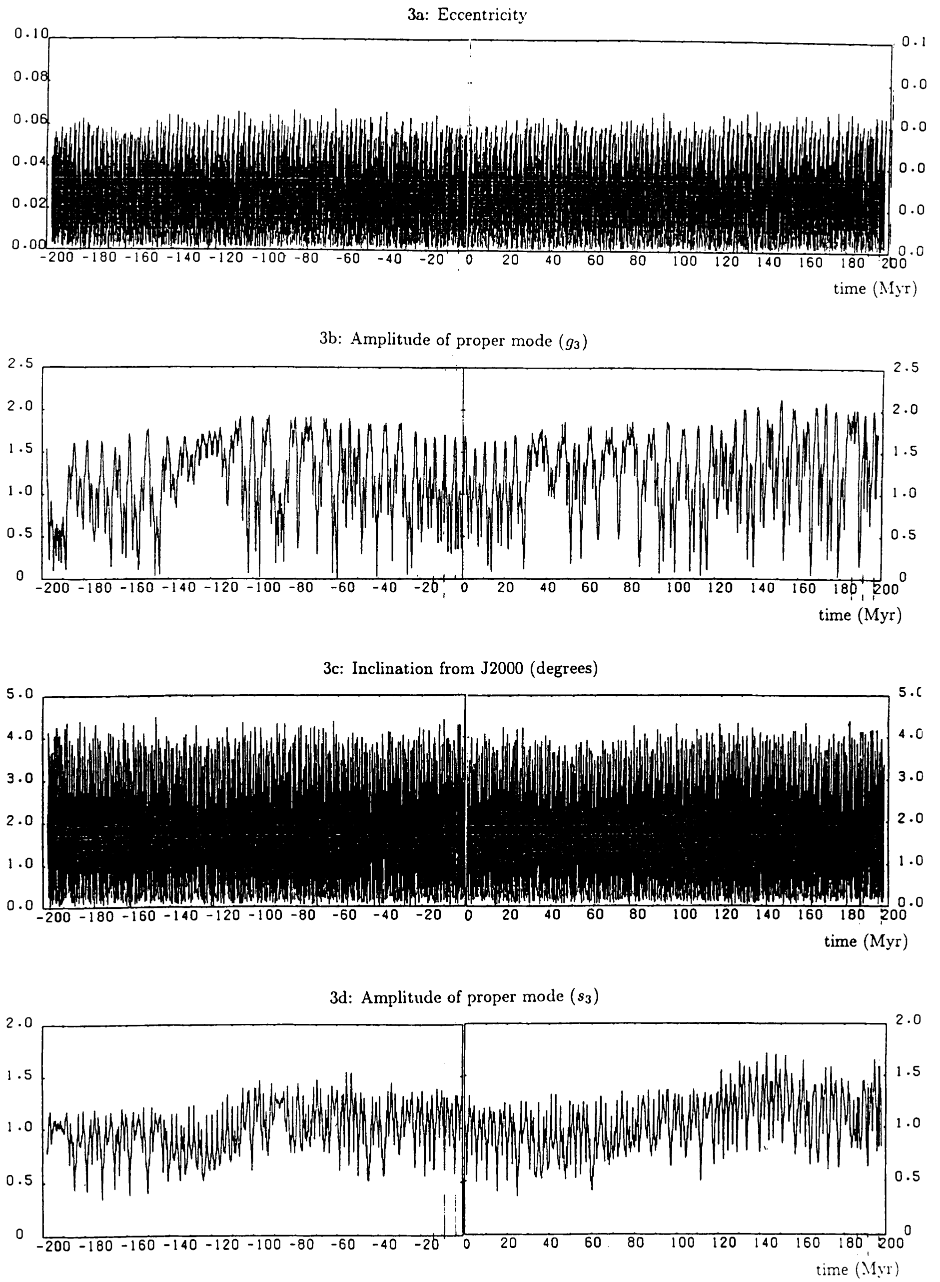


Fig. 3. The Earth

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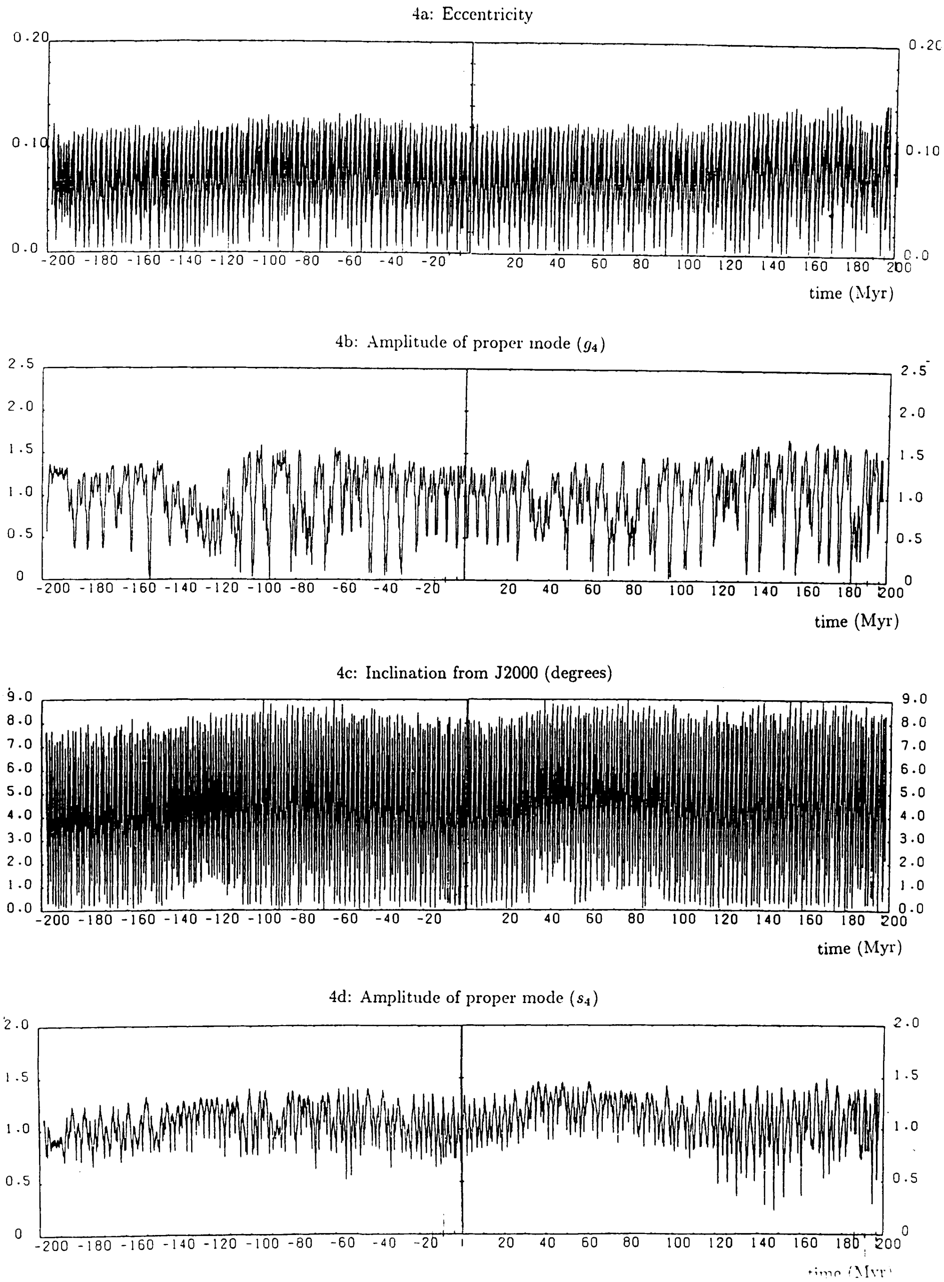


Fig. 4. Mars

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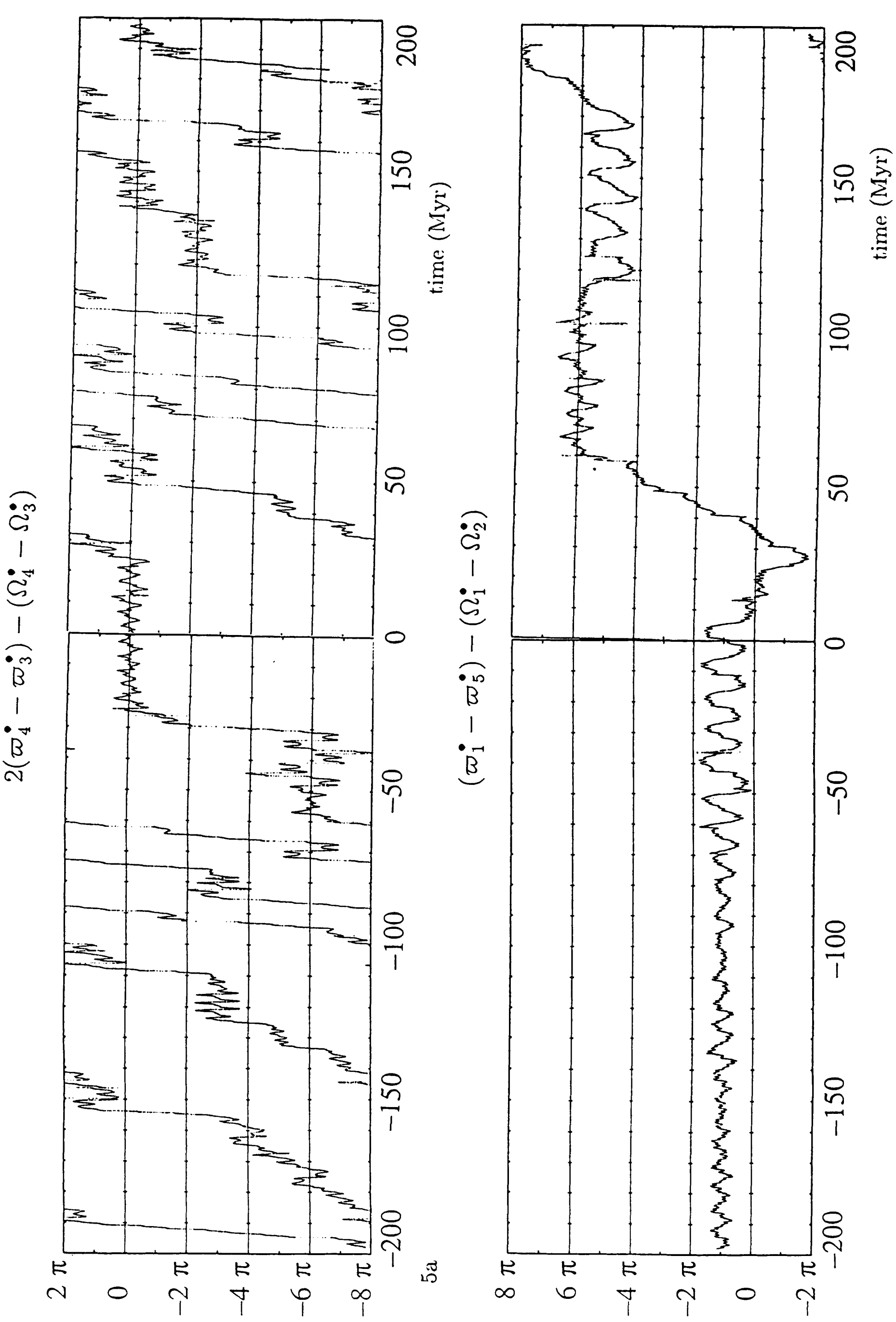


Fig. 5. The secular resonances

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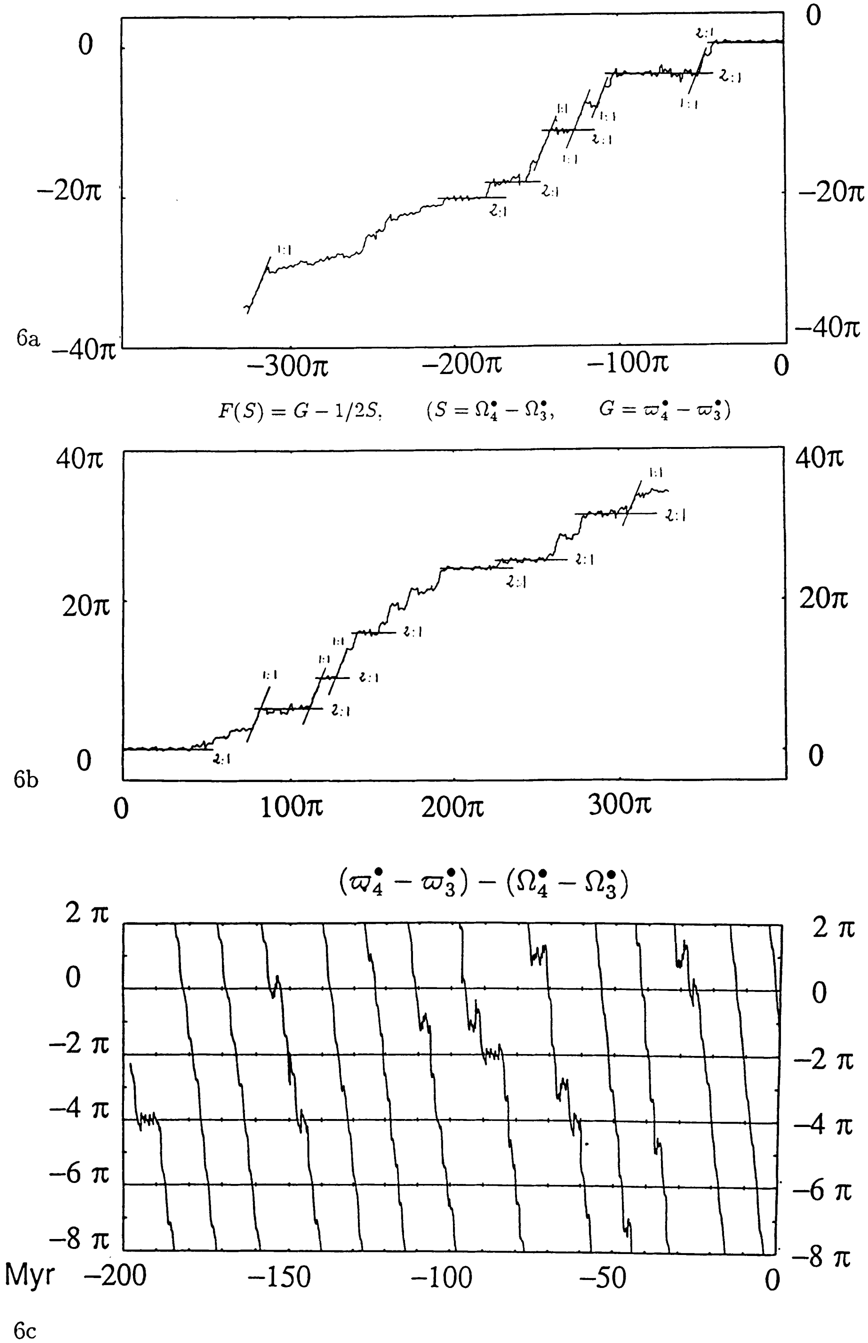


Fig. 6. Resonance overlap

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many cases the corresponding proper mode is not the dominant terms in the solution of the planet.

The equivalent plots are not given for the outer planets, because their solutions are so close to quasi-periodic motion, that a plot will not give any more information than the analytic expansions given in (Laskar, 1990).

The first obvious result is that, on this span of time, we do not assist to a very large catastrophic event, like the burst of eccentricity which occurs in the 3:1 asteroidal resonance (Wisdom, 1983). But let us look more closely to the various solutions.

The solutions for the eccentricity of the Earth and of Venus looks very regular, but this is mainly due to the fact that the leading linear terms in these solutions are related to the proper modes of Jupiter and Saturn, which behave in a very regular way. The chaotic behaviour is much more visible on the amplitudes of the proper modes. On the plot of eccentricity, it is nevertheless visible and manifests itself by the modulation of the maximum amplitude which reaches nearly 0.01 for the Earth. The chaotic effect on the inclination is more visible, as the inclination of the Earth is driven by the proper mode related to $s_3$ which has an important chaotic behaviour (fig 3d). Unfortunately, the plotted inclination is the inclination from J2000 reference plane instead of the invariable plane, which induces a constant offset of the inclination. The chaotic behaviour accounts on this variable for nearly 1 degree over 200 millions years. The behaviour of Venus is very similar. The chaotic effect on the orbit of Mars is more important (fig.4a-d), as it reaches 0.02 in eccentricity and more than 1 degree in inclination. The planet for which the chaotic behaviour is the most important is Mercury (fig 1a-d), as the changes in eccentricity reach 0.05, while changes in inclination reach more than 2 degrees (fig 1c).

These changes observed over 400 Myr are thus very important, as we cannot exclude larger changes over a longer time span. The actual large eccentricity and inclination of Mercury may even be explained by increase from more moderate values through a chaotic route.

### 3.4. Resonance overlap

In (Laskar, 1990), it was found that the chaotic behaviour of the inner solar system mainly originated in the two secular resonances $\theta$ and $\sigma$, related to the resonant arguments $2(\varpi_4^\bullet - \varpi_3^\bullet) - (\Omega_4^\bullet - \Omega_3^\bullet)$ and $(\varpi_1^\bullet - \varpi_5^\bullet) - (\Omega_1^\bullet - \Omega_2^\bullet)$. These arguments are changing from libration to circulation several times over 400 Myr (fig. 5a–b), which is characteristic from chaotic behaviour in the vicinity of the separatrix of a resonance. For the first resonance, the argument is in circulation during a long time and I searched if it enters into another secular resonance during that time. At the begining, several trials were made, but a more systematic approach consists to plot the function $F(s) = g - 1/2s$, where $g = \varpi_4^\bullet - \varpi_3^\bullet$ and $s = \Omega_4^\bullet - \Omega_3^\bullet$, versus $s$ (fig. 6a–c). The temporary libration $\theta$ is thus identified as the horizontal portions of the curve $F(s)$, which can be considered as resonances $2:1$ of the arguments $g$ and $s$. Several portions of this curve can also be identified as being in resonance $1:1$, which mean that we have temporary libration of the argument $(\varpi_4^\bullet - \varpi_3^\bullet) - (\Omega_4^\bullet - \Omega_3^\bullet)$.

All the other parts of the curve seem to have slopes between these two values. The existence of temporary librations of the argument $g - s$ can also be seen by a direct plot (Fig. 6c).

From this analysis, we can conclude that the chaotic zone extends from the resonance $2g - s$ to the resonance $g - s$. We have thus probably resonance overlap of these two resonances (Chirikov, 1979), which explains the presence of a large chaotic zone which is not restricted to the vicinity of the separatrix of a single resonance.

## References


Arnold, V.I.: 1963, 'Small denominators and problems of stability of motion in classical and celestial mechanics', *Russ. Math. Surv.*, **135**, 427 vol 18 N6, 85–192.

Chirikov, B.V.: 1979,'A universal instability of many-dimensional oscillator systems', *Phys. Reports* **52**, 265.

Laskar, J.: 1985, 'Accurate methods in general planetary theory', *Astron. Astrophys.* **144**, 133–146.

Laskar, J.: 1986, 'Secular terms of classical planetary theories using the results of general theory', *Astron. Astrophys.* **157**, 59–70

Laskar, J.: 1987, 'Secular evolution, proper modes and resonances in the inner Solar System', in *Resonances in the Solar System*, M.Sidlichovsky ed., Prague.

Laskar, J.: 1988, 'Secular evolution of the Solar System over 10 million years', *Astron. Astrophys.* **198**, 341–362.

Laskar, J.: 1989, 'A numerical experiment on the chaotic behaviour of the Solar System', *Nature*, **338**, 237–238.

Laskar, J.: 1990, 'The chaotic motion of the solar system. A numerical estimate of the size of the chaotic zones', *Icarus*, **88**,266–291. Laskar, J.: 1991, 'Analytical framework in Poincaré variables for the motion of the solar system', in *Predictability, Stability and Chaos in n-body Dynamical Systems*, A. Roy, ed., Plenum Press.

Laskar, J., Quinn, T., Tremaine, S.: 1991, 'Confirmation of Resonant Structure in the Solar System', *submitted to Icarus.*

Laskar, J., Froeschlé, Cl., Celletti, A.: 1991 'The Measure of Chaos by the Numerical Analysis of the Fundamental Frequencies. Application to the Standard Mapping', *submitted to Physica D.*

Poincaré, H.: 1892, *Méthodes Nouvelles de la Mécanique Celeste* Vol. 1 (Gauthier-Villars, Paris).

Poincaré, H.: 1896, 'Une forme nouvelle des équations du problème des trois corps ', *C. R. Acad. Sc.* **123**, 1031–1035.

Quinn, T.R., Tremaine, S., Duncan, M.: 1991, 'A three million year integration of the Earth's orbit',*Astron. J.***101**, 2287–2305.

Sussman, G.J., Wisdom, J.: 1988, 'Numerical evidence that the motion of Pluto is chaotic'. *Science***241**, 433–437.

Sussman, G.J., Wisdom, J.: 1991, these proceedings

Williams, J.G.: 1984, 'Determining Asteroid Masses from Perturbations on Mars', *Icarus*, **57**,1–13.

Wisdom, J.: 1983, 'Chaotic Behaviour and the origin of the 3/1 Kirkwood Gap', *Icarus*, **56**, 51–74


# Discussion

*P.Goldreich* – What are the implications of your secular theory integrations for qualitative changes over the age of the solar system?

*J.Laskar* – Over the 200 Myr of my integrations, the qualitative changes reflect mostly in the change of amplitudes of the proper modes of the different planets. When you go back to the planets, due to the mixing of proper modes, the main variations of the orbital parameters of the planets mostly cames from the linear

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theory, but it is modulated by the non-linear chaotic component. For the eccentricity of the Earth, it amounts to about 0.01, but in the inclination of Mercury, it goes up to a few degrees in inclination over 200 Myr. The chaotic behaviour of the solar system is thus the probable explanation of the high inclination of Mercury. For longer time spans, it is possible to integrate over a much larger time (my integration took only 6 hours in a supercomputer), but due to the exponential divergence of the orbits, it will not represent the motion of a solar system. A more difficult study would be to obtain a more complete description of the chaotic zone to which our solar system belongs. This requires either heavy numerical computations or more refined analytical studies.

*C. Williams* – How did you the averaging and can you comment on the effect of the small divisors?

*J.Laskar* – The averaging is a second-order averaging with respect to the masses. The small divisors in mean motion, which appear at second order with respect to the masses, are thus taken into account.