ON THE CONTINUOUS SPECTRA OF SINGULAR BOUNDARY VALUE PROBLEMS

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1. Introduction. Suppose that $p(t) > 0$, that both $p(t)$ and $f(t)$ are continuous functions on the half-line $0 \leqslant t < \infty$, and that λ denotes a real parameter. Only real-valued functions will be considered in this paper. Let the differential equation

$$(1) \qquad L(x) + \lambda x = 0, \quad \text{where } L(x) \equiv (px')' - fx,$$

be of the limit-point type (**3**, p. 238), so that (1) and a linear homogeneous boundary condition

$$(2_\alpha) \qquad x(0)\cos\alpha + x'(0)\,p(0)\sin\alpha = 0, \qquad 0 \leqslant \alpha < \pi,$$

determine a boundary value problem on $0 \leqslant t < \infty$ for every fixed α. Let $\rho_\alpha(\lambda)$ denote the unique continuous monotone basis function on $-\infty < \lambda < \infty$, normalized by $\rho_\alpha(0) = 0$, determining the eigendifferentials associated with the continuous spectrum, C_α (**3**, pp. 238-251).

It is known that the set S' consisting of the set of cluster points of the spectrum, S_α, is independent of α (**3**, p. 251). Furthermore, in the standard examples of equations (1), the set C_α is independent of α; if, for example, $f(t)$ is periodic, (**4**). The question was raised by Weyl (**3**, p. 252) as to whether the continuous spectrum is invariant under change of the boundary condition (2_α), that is, as to whether the set C_α is always independent of α. Although this question will remain unanswered in this paper, except under a special assumption, it still seems to be of interest to compare the various existing basis functions $\rho_\alpha(\lambda)$, belonging to different values α. Except in explicit, special cases (cf., e.g., **3**, p. 264; **2**, p. 59), very little seems to be known in this connection. A contribution to some knowledge in this direction is contained in the following:

THEOREM (*). *Let $p(t) > 0$ and $f(t)$ be continuous on $0 \leqslant t < \infty$ and suppose that (1) is of the limit-point type. Suppose that there exist a fixed interval Δ and two distinct boundary conditions (2_{α_1}) and (2_{α_2}), $\alpha_1 \neq \alpha_2$, such that Δ is in each of the sets C_{α_1} and C_{α_2} and such that the basis function $\rho_{\alpha_1}(\lambda)$ is an absolutely continuous function of $\rho_{\alpha_2}(\lambda)$ on the interval Δ. Then*

(i) *the interval Δ is in the continuous spectrum C_α for every boundary condition (2_α), $0 \leqslant \alpha < \pi$; and*

(ii) *the basis function $\rho_{\alpha_1}(\lambda)$ is an absolutely continuous function of every basis function $\rho_\alpha(\lambda)$ on the interval Δ $(0 \leqslant \alpha < \pi)$.*

Henceforth, for simplicity in notation, let $\rho_k(\lambda) = \rho_{\alpha_k}(\lambda)$ for $k = 1, 2$. It follows from (*) that, for any basis function $\rho_1(\lambda)$ which is strictly increasing

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on an interval Δ, there are only two possibilities: on the fixed interval Δ, either $\rho_1(\lambda)$ is an absolutely continuous function of *every* basis function $\rho_\alpha(\lambda)$ (indeed, the interval Δ is in the continuous spectrum of every boundary value problem (1), (2_α) in the case (i)) or $\rho_1(\lambda)$ is not an absolutely continuous function of any (other, existing) basis function $\rho_\alpha(\lambda)$ for which Δ is in C_α.

2. Proof of (i) **of** $(*)$. Let $\phi_k(t, \lambda) = \phi(t, \lambda, \alpha_k)$, for $k = 1$ and 2, be solutions of (1) satisfying

(3) $$\phi_k(0, \lambda) = -\sin \alpha_k, \quad p(0)\, \phi'(0, \lambda) = \cos \alpha_k.$$

Then the eigendifferentials are given by

(4) $$d\Phi_k(t, \lambda) = \phi_k(t, \lambda)\, d\rho_k(\lambda),$$

where

$$\int_0^\infty (\delta\Phi_k)^2 dt = \delta\rho_k$$

for arbitrary δ (see **3**, p. 249). Let $\alpha \neq \alpha_1, \alpha_2$ and let $\phi(t, \lambda) = \phi(t, \lambda, \alpha)$ be the solution of (1) satisfying

(5) $$\phi(0, \lambda) = -\sin \alpha, \quad p(0)\, \phi'(0, \lambda) = \cos \alpha.$$

It will be shown that the interval Δ of theorem $(*)$ is in the set C_α. To this end, suppose, if possible, the contrary. Then there exists a subinterval of Δ, say δ, such that δ has no points in common with the (closed) set C_α. Clearly, there exist continuous functions $A_1(\lambda)$ and $A_2(\lambda)$ such that

(6) $$\phi(t, \lambda) = A_1(\lambda)\, \phi_1(t, \lambda) + A_2(\lambda)\, \phi(t, \lambda).$$

Since $\rho_1(\lambda)$ is an absolutely continuous function of $\rho_2(\lambda)$ on Δ, it follows (Radon-Nikodym) that there is a function $B = B(\lambda)$ such that

(7) $$d\rho_1(\lambda) = B(\lambda)\, d\rho_2(\lambda) \text{ on } \Delta.$$

It is clear from (7) that

(8) $$\int_\Delta B^{-1}\, d\rho_1 < \infty.$$

(It is understood, of course, that if B is zero for some values λ, then, in the integrations with respect to ρ_1, the set δ can be replaced by a set δ' such that $B > 0$ on δ' and $\int_{\delta'} d\rho_1 = \int_\delta d\rho_1$.) Next, define the function $M(t)$ by

(9) $$M(t) = \int_\delta B^{-\frac{1}{2}}(\lambda)\, \phi(t, \lambda)\, d\rho_1(\lambda),$$

so that, by (6) and (7),

(10) $$M(t) = \int_\delta A_1 B^{-\frac{1}{2}} \phi_1\, d\rho_1 + \int_\delta A_2 B^{\frac{1}{2}} \phi_2\, d\rho_2.$$

In view of (7) and (8), the inequality $(a + b)^2 \leqslant 2(a^2 + b^2)$, and the properties of the eigendifferentials (4), one has

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(11) $$\int_0^\infty M^2(t)\,dt \leqslant 2\int_\delta (A_1^2 B^{-1} + A_2^2)\,d\rho_1 < \infty,$$

so that $M(t)$ is of class $L^2[0, \infty)$. Moreover, M is differentiable and

(12) $$M(0) = -(\sin\alpha)\int_\delta B^{-\frac{1}{2}}\,d\rho_1, \quad p(0)\,M'(0) = (\cos\alpha)\int_\delta B^{-\frac{1}{2}}\,d\rho_1.$$

Since each of the integrals of (12) is clearly different from zero, $M(t) \not\equiv 0$. It will be shown that $M(t)$ is orthogonal to all eigenfunctions and eigen-differentials belonging to the boundary value problem determined by (1) and (2_α), and a contradiction will thus be obtained.

Let μ denote an eigenvalue on $\delta = [\lambda_1, \lambda_2]$ of the boundary value problem (1) and (2_α). It will be supposed that $\lambda_1 < \mu < \lambda_2$; the treatment in case μ is an end-point will be clear. Let δ_n denote the set of values λ: $[\lambda_1, \mu - 1/n] + [\mu + 1/n, \lambda_2]$ (n large), and define $M_n(t)$ by

(13) $$M_n(t) = \int_{\delta_n} B^{-\frac{1}{2}}\,\phi\,d\rho_1.$$

It will be first be shown that

(14) $$\int_0^\infty M_n(t)\,\xi(t)\,dt = 0,$$

where $\xi(t)$ denotes an eigenfunction belonging to μ.

The functions ϕ and ξ satisfy the equations

(15) $$L(\phi) + \lambda\phi = 0, \quad L(\xi) + \mu\xi = 0$$

(cf. (1)), and hence for every $T \geqslant 0$,

(16) $$\int_0^T [\xi L(\phi) - \phi\,L(\xi)]\,dt = (\mu - \lambda)\int_0^T \phi\xi\,dt.$$

Moreover, an integration by parts shows that, for any two functions x, y possessing continuous second derivatives on $0 \leqslant t < \infty$,

(17) $$\int_0^T [x\,L(y) - y\,L(x)]\,dt = p(xy' - x'y)\Big|_0^T$$

(**3**, p. 223). An application of Fubini's theorem for the interchange of the order of integration shows that

(18) $$\int_0^T M_n\xi\,dt = \int_{\delta_n}\left(\int_0^T \phi\xi\,dt\right) B^{-\frac{1}{2}}\,d\rho_1,$$

where M_n is defined by (13). Relation (18) implies, as a consequence of (16), (17), and the fact that M_n and ξ satisfy the boundary condition (2_α), that

(19) $$\int_0^T M_n\xi\,dt = \int_{\delta_n} p(T)[\phi'(T, \lambda)\,\xi(T) - \phi(T, \lambda)\,\xi'(T)](\mu - \lambda)^{-1}B^{-\frac{1}{2}}(\lambda)\,d\rho_1(\lambda).$$

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If $A = A(t)$ is defined by

$$A(t) = \int_{\delta_n} \phi(t, \lambda)(\mu - \lambda)^{-1} B^{-\frac{1}{2}}(\lambda)\, d\rho_1(\lambda), \tag{20}$$

it is seen that

$$\int_0^\infty A^2(t)\, dt \leqslant 2 \int_{\delta_n} (A_1^2 B^{-1} + A_2^2)(\mu - \lambda)^{-2}\, d\rho_1 < \infty \tag{21}$$

and that

$$\int_0^\infty (L(A))^2\, dt \leqslant 2 \int_{\delta_n} (A_1^2 B^{-1} + A_2^2)\, \lambda^2 (\mu - \lambda)^{-2}\, d\rho_1 < \infty \tag{22}$$

(**3**, p. 249, and relations (7), (8), and (11) above). Relation (19) can be expressed as

$$\int_0^T M_n \xi\, dt = p(T)[A'(T)\, \xi(T) - A(T)\, \xi'(T)]. \tag{23}$$

It follows from (21) and (22) and the fact that ξ and $L(\xi)$ also belong to class $L^2[0, \infty)$ that the expression on the right side of (23) tends to zero as $T \to \infty$ (**3**, pp. 241–242). Consequently, relation (14) now follows.

Next, it will be shown that

$$\int_0^\infty M(t)\, \xi(t)\, dt = 0. \tag{24}$$

In view of (14), it is sufficient to show that

$$\int_0^\infty (M(t) - M_n(t))^2\, dt \to 0, \quad \text{as } n \to \infty. \tag{25}$$

However,

$$M - M_n = \int_{\mu - 1/n}^{\mu + 1/n} \phi(t, \lambda)\, B^{-\frac{1}{2}}(\lambda)\, d\rho_1(\lambda)$$

and hence

$$\int_0^\infty (M - M_n)^2\, dt \leqslant 2 \int_{\mu - 1/n}^{\mu + 1/n} (A_1^2 B^{-1} + A_2^2)\, d\rho_1. \tag{26}$$

The right side of (26) tends to zero when $n \to \infty$ and relation (24) now follows.

It remains to be shown that $M(t)$ is orthogonal to all of the eigendifferentials of the boundary value problem (1), (2_α). To this end, it is convenient to assume that $\delta = [-\lambda_1, \lambda_1]$, where $\lambda_1 > 0$. (That this may be assumed without loss of generality is clear from the fact that the continuous spectrum is merely translated by a constant γ if f is replaced by $f + \gamma$.) If the set C_α is not empty, then the eigendifferentials are given by

$$N = N(t, J) = \int_J \phi(t, \lambda)\, d\rho(\lambda), \quad \rho(\lambda) \equiv \rho_\alpha(\lambda), \tag{27}$$

where J is an arbitrary (say, closed) λ-interval. Since the closed interval δ contains no points in common with the set C_α, it is sufficient to show that

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$$\int_0^\infty M(t)\, N(t, J)\, dt = 0 \tag{28}$$

for all closed intervals J having no point in common with δ.

Consider then an interval $J = [\mu_1, \mu_2]$, where $\lambda_1 < \mu_1$. (The case in which $\mu_2 < -\lambda_1$ can be treated similarly and will not be considered separately.) Suppose then that λ is in δ and that μ is in J. It follows from the equations

$$L(\phi(t, \lambda)) + \lambda\phi(t, \lambda) = 0, \quad L(\phi(t, \mu)) + \mu\phi(t, \mu) = 0 \tag{29}$$

and the relations (16) and (17) that

$$\int_0^T \phi(t, \lambda)\, \phi(t, \mu)\, dt = p(T)[\phi'(T, \lambda)\, \phi(T, \mu) - \phi(T, \lambda)\, \phi'(T, \mu)]\, (\mu - \lambda)^{-1}. \tag{30}$$

It follows readily from (30) that

$$\int_0^T M(t)\, N(t, J)\, dt = \sum_{n=0}^{\infty} p(T)[A_n'(T)\, B_n(T) - A_n(T)\, B_n'(T)], \tag{31}$$

where M, N, A_n, and B_n are defined by (9), (27), and

$$A_n(t) = \int_\delta \lambda^n\, B^{-\frac{1}{2}}(\lambda)\, \phi(t, \lambda)\, d\rho_1(\lambda), \quad B_n(t) = \int_J \phi(t, \mu)\, \mu^{-n-1}\, d\rho(\mu). \tag{32}$$

(The interchanges of the order of integration together with the interchange of the summation and integration are readily seen to be justified.) Relation (17) implies

$$p(T)[A_n'(T)\, B_n(T) - A_n(T)\, B_n'(T)] = \int_0^T [B_n\, L(A_n) - A_n\, L(B_n)]\, dt. \tag{33}$$

By the Schwarz inequality,

$$\left| \int_0^T B_n\, L(A_n)\, dt \right| \leqslant \left(\int_0^\infty B_n^{\,2}\, dt \right)^{\frac{1}{2}} \left(\int_0^\infty (L(A_n))^2\, dt \right)^{\frac{1}{2}}. \tag{34}$$

From (32) and the fact that $J = [\mu_1, \mu_2]$,

$$\int_0^\infty B_n^{\,2}\, dt \leqslant \int_J \mu^{-2(n+1)}\, d\rho(\mu) \leqslant \mu_1^{-2(n+1)} \int_J d\rho(\mu) < \infty. \tag{35}$$

Furthermore, relations (6), (7), and (10) imply that

$$L(A_n) = -\int_\delta \lambda^{n+1} A_1 B^{-\frac{1}{2}}\, \phi_1\, d\rho_1 - \int_\delta \lambda^{n+1} A_2 B^{\frac{1}{2}}\, \phi_2\, d\rho_2. \tag{36}$$

Hence (cf. (4)),

$$\begin{aligned} \int_0^\infty (L(A_n))^2\, dt &\leqslant 2 \int_\delta \lambda^{2(n+1)} (A_1^{\,2} B^{-1} + A_2^{\,2})\, d\rho_1 \\ &\leqslant 2\lambda_1^{\,2(n+1)} \int_\delta (A_1^{\,2} B^{-1} + A_2^{\,2})\, d\rho_1 < \infty. \end{aligned} \tag{37}$$

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In particular, the functions B_n and $L(A_n)$ are of class $L^2[0, \infty)$, while a similar analysis shows that A_n and $L(B_n)$ are also in class $L^2[0, \infty)$. Consequently, each term of the summation of (31) satisfies

$$(38) \qquad P(T)[A_n'(T)\, B_n(T) - A_n(T)\, B_n'(T)] \to 0, \qquad \text{as } T \to \infty$$

(**3**, pp. 241–242).

It now follows from (35), (37), and the inequality $\lambda_1\mu_1^{-1} < 1$, that

$$\sum_{n=N}^{\infty} \left| \int_0^T B_n\, L(A_n)\, dt \right| \to 0, \qquad \text{as } N \to \infty,$$

holds uniformly in T $(0 \leqslant T < \infty)$. Similarly,

$$\sum_{n=N}^{\infty} \left| \int_0^T A_n\, L(B_n)\, dt \right| \to 0, \qquad \text{as } N \to \infty,$$

holds uniformly in T $(0 \leqslant T < \infty)$. Hence, the series on the right side of the equation (31) tends to zero as $T \to \infty$ and so (28) follows. Thus the function $M(t)$ of (9) is orthogonal to all eigenfunctions and eigendifferentials of the boundary value problem determined by (1) and (2_α) and, as remarked earlier in this section, a contradiction is obtained. This completes the proof of part (i) of (*).

3. Proof of (ii) **of** (*). Let ϕ be defined as in §2, and let $d\Phi$ denote the eigendifferentials, so that

$$(39) \qquad d\Phi(t, \lambda) = \phi(t, \lambda)\, d\rho(\lambda).$$

Let $M(t) = M_\delta(t)$ be defined by (9) where, now, δ is any interval contained in Δ. Then, by (**3**, pp. 250–251), the function $M_\delta(t)$ has an expansion

$$(40) \qquad M_\delta(t) = \sum c_k\, \phi_k(t) + \int_{-\infty}^{\infty} \phi(t, \lambda)\, d\Gamma(\lambda),$$

where the ϕ_k denote the eigenfunctions of the boundary value problem (1), (2_α) and the c_k and $d\Gamma(\lambda)$ are given by

$$(41) \quad c_k = \int_0^{\infty} M_\delta(t)\, \phi_k(t)\, dt, \quad \delta'\Gamma = \int_0^{\infty} M_\delta(t)\, \delta'\Phi\, dt \ (\delta' \text{ arbitrary}).$$

In view of the uniqueness properties associated with the expansion (40), however, it follows from (40) and (9) that $c_k = 0$ and

$$(42) \qquad B^{-\frac{1}{2}}(\lambda)\, d\rho_1(\lambda) = d\Gamma(\lambda)$$

holds on the interval δ. Thus, provided δ' is contained in δ, relation (42), the second relation of (41), and the Schwarz inequality imply

$$(43) \qquad \int_{\delta'} B^{-\frac{1}{2}}\, d\rho_1 \leqslant \left(\int_0^{\infty} M_\delta^2(t)\, dt \right)^{\frac{1}{2}} \left(\int_0^{\infty} (\delta'\Phi)^2\, dt \right)^{\frac{1}{2}}.$$

Henceforth, it will be convenient to put $\delta' = \delta$. From the properties of the eigendifferentials (39),

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$$(44) \qquad \int_0^\infty (\delta\Phi)^2\, dt = \delta\rho.$$

It follows from (43), (44), (11), and the Schwarz inequality that

$$(45) \qquad \left(\sum \int_\delta B^{-\frac{1}{2}}\, d\rho_1\right)^2 \leqslant 2\left(\sum \int_\delta (A_1{}^2 B^{-1} + A_2{}^2)\, d\rho_1\right)\left(\sum \int_\delta d\rho\right),$$

where the summations are taken over any sequence of intervals δ contained in Δ. Let Z denote any subset of the interval Δ for which

$$\int_Z d\rho(\lambda) = 0.$$

It follows readily from (45) and (8) that

$$\left(\int_Z B^{-\frac{1}{2}}(\lambda)\, d\rho_1(\lambda)\right)^2 \leqslant \text{const.} \int_Z d\rho(\lambda) = 0;$$

hence,

$$\int_Z d\rho_1 = \int_Z B^{\frac{1}{2}} B^{-\frac{1}{2}}\, d\rho_1 = 0.$$

Thus the variation of $\rho_1(\lambda)$ is zero over any set Z over which the variation of $\rho(\lambda)$ is zero. Hence $\rho_1(\lambda)$ is an absolutely continuous function of $\rho(\lambda)$ (that is, by the Radon-Nikodym theorem, there is a function $C(\lambda)$ such that $d\rho_1(\lambda) = C(\lambda)\, d\rho(\lambda)$) and the proof of (ii) of (*) is now complete.

References

1. P. Hartman and A. Wintner, *A separation theorem for continuous spectra*, Amer. J. of Math., *71* (1949), 650–662.

2. E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations* (Oxford, 1946).

3. H. Weyl, *Ueber gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen*, Math. Ann. *68* (1910), 222–269.

4. A. Wintner, *Stability and spectrum in the wave mechanics of lattices*, Phys. Rev. *72* (1947), 81–82.

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