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ON A PROBLEM OF DOOB CONCERNING MULTIPLY SUPERHARMONIC FUNCTIONS

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The following is a well-known result due to A.P. Calderon [2], asserting the existence of non-tangential limits of multiply harmonic functions.

Let $E = E_1 \times E_2 \times \cdots \times E_m$ be the cartesian product of the spaces E_k of points $P_k(x_1^{(k)}, x_2^{(k)}, \cdots, x_n^{(k)})$, and $F(P)$, $P = (P_1, \cdots, P_m) \in E$, be defined and continuous in $x_n^{(k)} > 0$, $k = 1, 2, \cdots, m$, and harmonic in P_k, that is, such that

$$\sum_{i=1}^{n} \frac{\partial^2 F}{(\partial x_i^{(k)})^2} = 0 \quad k = 1, 2, \cdots, m.$$

Let $B_k \subset E_k$ be the space $x_n^{(k)} = 0$, and $B = B_1 \times B_2 \times \cdots \times B_m$ the so-called distinguished boundary of $x_n^{(k)} > 0$, $k = 1, 2, \cdots, m$, and suppose that for every point $Q = (Q_1, Q_2, \cdots, Q_m)$, $Q_i \in B_i$, of a set A of positive measure of B, there exist regions Γ_{kQ}, limited by cones with vertices at the points Q_k and hyperplanes $x_n^{(k)} = \text{const}$ such that the function $F(P)$ is bounded in $\Gamma_Q = \Gamma_{1Q} \times \Gamma_{2Q} \times \cdots \times \Gamma_{mQ}$. Then almost everywhere in A, $F(P)$ has a limit as $P = (P_1, \cdots, P_m)$ tends to $Q = (Q_1, \cdots, Q_m) \in A$ in such a way that all P_k tend to Q_k simultaneously and non-tangentially.

Generalizing the above result in the case of functions of one variable, but on Green spaces, J.L. Doob [4] proved the following.

Let Ω be a Green space and Δ its Martin boundary. Let u and h be two superharmonic functions on Ω, $h > 0$. If, for every $z \in E \subset \Delta$, $\frac{u}{h}$ is bounded below in a set which is not thin at z, then $\frac{u}{h}$ has a finite fine limit at μ_h almost every point of E; where μ_h is the canonical measure corresponding to h in the Riesz-Martin integral representation with measures on $\Omega \cup \Delta$.

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In this paper Doob stated that a similar result concerning the boundary behaviour of multiply superharmonic functions involving the limit of these functions under the product of fine filters is probably true. The object of this note is to prove that Doob's theorem does not generalize to the functions of several variables. In fact, as we shall see below the limits may fail to exist even for some multiply harmonic functions.

Let us consider $\Omega = \Omega_1 \times \Omega_2$ where Ω_1 and Ω_2 are half planes (say the region below the x-axis of the plane) or equivalently (for our purpose) unit discs. There is a well-known example due to Saks-Zygmund and others [7, 8] of a positive multiply harmonic function u on Ω which has the property that for every (P^1, P^2) in the distinguished boundary $u(x^1, x^2)$ diverges as $x^i \longrightarrow P^i$ $(i = 1, 2)$ non-tangentially and independent of each other. (This function is constructed with the Abel means of an unbounded positive function [9] the indefinite integral which is of not strongly differentiable.) See also [3]. With the above example it is enough to show that if $u > 0$ is a multiply harmonic function on Ω such that $u(x^1, x^2)$ has a limit α as $x^i \longrightarrow P^i$ following the fine filters $\mathscr{F}_{P^i}$ in Ω_i corresponding to P^i $(i = 1, 2)$, where (P^1, P^2) is on the distinguished boundary of Ω, then $u(x^1, x^2)$ tends to α as $x^i \longrightarrow P^i$ non-tangentially and independent of each other. We shall, in fact, prove a more general theorem. Before doing so, let us recall the following.

A Stolz domain ζ with vertex at P (on the boundary) and radius ρ in a ball or half-space $\omega \subset R^n$, is the connected open set which is the intersection of the open ball with centre at Pand radius ρ with an angular domain (if $n = 2$) or a cone of revolution $(n > 2)$ with vertex at P the closure of which (with the exception of P) is contained in ω. Another Stolz domain ζ' with the same vertex P is said to be sharper than ζ if $\zeta' \subset \zeta$, except for P, in a neighbourhood of P.

Let $G_{y_0}(\cdot)$ be the Green's function of ω with pole at $y_0 \in \omega$ and if X is on the boundary, K_x the minimal harmonic function on ω with pole at X and such that $K_x(y_0) = 1$. For every $\lambda \in R^+$, let

$$\sigma_\lambda = \left\{ y \in \omega : \frac{K_X(y)}{G_{y_0}(y)} \geq \lambda \right\}.$$

The sets σ_λ form a filter which is finer than the trace of all neighbourhoods of X on ω. *A* set $E \subset \omega$ is thin (resp. semi-thin) at X if $R_{K_X}^{E \cap \sigma_\lambda}$ (resp. $R_{\lambda G_{y_0}}^{E \cap \sigma_\lambda}$) tends to zero as $\lambda \longrightarrow +\infty$. The sets $E \subset \omega$ such that $\omega - E$ is thin (resp.

semi-thin) at X form the fine filter $\mathscr{F}_X$ (resp. semifine filter $\mathscr{F}''_X$) corresponding to X. Evidently $\mathscr{F}''_X$ is finer than $\mathscr{F}_X$. A finite valued function on an open subset of $\Omega_1 \times \Omega_2$ is said to be multiply harmonic there if it is continuous and harmonic in each variable for every fixed value of the other.

THEOREM 1. *Let u and h be multiply harmonic functions defined on Ω with $h>0$ and $\frac{u}{h}$ lower bounded in $\zeta_1 \times \zeta_2$ where ζ_1 $(i=1,2)$ is a Stolz domain $\subset \Omega_i$ with vertex at P^i. Then every adherent value α of $\frac{u}{h}$ in the product $\zeta_1' \times \zeta_2'$ of two sharper Stolz domains ζ_1' (with vertex P^1) and ζ_2' (with vertex P^2), is also an adherent value of $\frac{u}{h}$ following the product filter $\mathscr{F}''_{P^1} \times \mathscr{F}''_{P^2}$.*

From the above theorem, we deduce the following.

THEOREM 2. *Let u and h be multiply harmonic on Ω such that $h>0$ and $\frac{u}{h}$ is lower bounded in the intersection with Ω of a neighbourhood of (P^1, P^2), a point on the distinguished boundary. If $\frac{u(x^1,x^2)}{h(x^1,x^2)} \longrightarrow \alpha$ as $x^1 \longrightarrow P^1$ following $\mathscr{F}_{P^1}$ and $x^2 \longrightarrow P^2$ following $\mathscr{F}_{P^2}$. Then $\frac{u(x^1,x^2)}{h(x^1,x^2)} \longrightarrow \alpha$ as x^i converges non-tangentially to P^i $(i=1,2)$.*

The theorem above is a generalization of a result of M. Brelot and J.L. Doob [1], in the case when u and h are harmonic functions of one variable on a half space satisfying similar conditions. In proving the first theorem we make use of their proof in the case of functions of one variable.

However, a generalisation of Doob's theorem to functions of several variables can be proved, but the limits are those following the "fine" filters [5] canonical to the structure of the multiply harmonic functions. More precisely, we have to consider "fine" filters corresponding to the minimal positive multiply harmonic functions; and these filters are finer than the product of the fine filters corresponding to the minimal positive harmonic functions. (The minimal positive multiply harmonic functions are of the form $h_1 h_2 \cdots h_n$ where h_i's are minimal positive on the respective coordinate spaces [6].) These and other related results will be considered in a forthcoming paper.

Proofs of the Theorem 1 and 2.

The second theorem is an immediate consequence of the first. Suppose the hypotheses of the second theorem hold good. Since the filters $\mathscr{F}'_{P^i}$ are finer than $\mathscr{F}_{P^i}$ $(i=1,2)$, $\frac{u(x^1,x^2)}{h(x^1,x^2)}$ tends to α when $x^i \longrightarrow P^i$ following

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$\mathscr{F}''_{P^i}$ $(i=1,2)$. Hence any adherent value of $\frac{u(x^1,x^2)}{h(x^1,x^2)}$ in the product of any two Stolz domains with vertices at P^1 and P^2 necessarily coincides with α. It follows that $\frac{u(x^1,x^2)}{h(x^1,x^2)}$ tends to α as $x^i \longrightarrow P^i$ $(i=1,2)$ non-tangentially. Before proving the first theorem, let us recall the following two results. The first of them is Harnack's inequality and the second is implicitly contained in [1, p. 403].

1° Let B be a ball of radius p and centre z_0 in R^n $(n \geqq 2)$. Let $B\alpha$ be a concentric ball of radius $\alpha p (\alpha < 1)$. Then, there is a function θ of α, (and n) such that whatever be the positive harmonic function v on B, for all $z \in B\alpha$,

$$\frac{1}{\theta(\alpha)} \leqq \frac{v(z)}{v(z_0)} \leqq \theta(\alpha).$$

Further $\theta(\alpha)$ tends to 1 as α tends to 0.

2° Let ω be a half space in R^n $(n \geqq 2)$. Let ζ and ζ' be two Stolz domains contained in ω with the same vertex P on the boundary, ζ' sharper than ζ. Let $\{x_n\}$ converge to P, $x_n \in \zeta'$. Let d_n be the distance of x_n from the boundary of ω. Let the balls B_n of radii αd_n and centres x_n be contained in ζ. Then, $\bigcup_{n \geq n_0} B_n$ is not semi-thin at P, for all sufficiently large n_0.

Let us now assume that the hypotheses of the theorem 1 hold good. If k is a lower bound for $\frac{u}{h}$ on $\zeta_1 \times \zeta_2$, then we can consider $u\text{-}kh$ which is a multiply harmonic function >0 on $\zeta_1 \times \zeta_2$. Hence we can suppose that $u>0$ on $\zeta_1 \times \zeta_2$. We can find a sequence (x_n^1, x_n^2) of elements in $\zeta_1^1 \times \zeta_2^1$, converging to (P^1, P^2) such that $\frac{u(x_n^1, x_n^2)}{h(x_n^1, x_n^2)}$ tends to α as $n \longrightarrow \infty$. Let us suppose that $0 < \alpha < +\infty$ (the other cases can be treated with suitable alterations). Given $\delta > 0$, there is a N_1 such that $n \geq N_1$,

$$\alpha - \delta \leqq \frac{u(x_n^1, x_n^2)}{h(x_n^1, x_n^2)} \leq \alpha + \delta.$$

Let α_n^i be the distance of x_n^i from the boundary of Ω_i. We can choose $\beta > 0$, sufficiently small so that, for $n \geq N_2$ (chosen to be $\geq N_1$) the closed balls of radii βd_n^i with centres at x_n^i are contained in ζ_i $(i=1,2)$. Let $\gamma > 0$ and consider the balls $B(x_n^i, d_n^i \beta\gamma)$ with centres at x_n^i and radii $\beta\gamma d_n^i$

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$(i=1,2)$ and $n \geqq N_2$. Now whatever be $x^2 \in B(x_n^2, \beta\gamma d_n^2)$, $u(x^1, x^2)$ is harmonic in x^1 and >0 on $B(x_n^1, \beta d_n^1)$ and for $x^1 \in B(x_n^1, \beta\gamma d_n^1)$, by 1°

$$\frac{1}{\theta(\gamma)} \leqq \frac{u(x^1, x^2)}{u(x_n^1, x^2)} \leqq \theta(\gamma)$$

and a similar inequality is valid for $h(x^1, x^2)$. Using this argument repeatedly and in the other variable too, it can be easily seen that

$$\frac{\alpha-\delta}{(\theta(\gamma))^4} \leqq \frac{u(x^1, x^2)}{h(x^1, x^2)} \leqq (\alpha+\delta)(\theta(\gamma))^4$$

whatever be (x^1, x^2) in $B(x_n^1, \beta\gamma d_n^1) \times B(x_n^2, \beta\gamma d_n^2)$, $(n \geqq N_2)$. Now, since $\theta(\gamma) \longrightarrow 1$ as $\gamma \longrightarrow 0$, given $\varepsilon > 0$, we can choose $\delta > 0$ and $\gamma > 0$ near 0 so that

$$\alpha - \varepsilon \leqq \frac{u(x^1, x^2)}{h(x^1, x^2)} \leqq \alpha + \varepsilon$$

whatever be $(x^1, x^2) \in A = \bigcup_{n \geq N_2} [B(x_n^1, \beta\gamma d_n^1) \times B(x_n^2, \beta\gamma d_n^2)]$. It is now enough to show that $A \cap (F \times F')$ is not void, for arbitrary $F \in \mathscr{F}''_{P^1}$ and $F' \in \mathscr{F}''_{P^2}$. Using (2°), we can choose a subsequence $\{n_i\}_{i=1}^{\infty}$ in such a way that $B(x_{n_i}^1, \beta\gamma d_{n_i}^1) \cap F \neq \phi$ and $F' \cap B(x_{n_i}^2, \beta\gamma d_{n_i}^2) \neq \phi$ for all $i \geqq i_0$, for all suitably large i_0 (so that $n_{i_0} \geqq N_2$). Now, since $A \cap (F \times F') \supset (F \times F') \bigcap_{i \geq i_0} [B(x_{n_i}^1, d_{n_i}^1 \beta\gamma) \times B(x_{n_i}^2, \beta\gamma d_{n_i}^2)]$ we have that $A \cap (F \times F') \neq \phi$. This completes the proof.

Remark. The Theorem 1 is true when Ω is the product of a finite number of half spaces (of different dimensions) with similar conditions on u and h.

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