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A NEW VARIATIONAL METHOD FOR THE $p(x)$-LAPLACIAN EQUATION

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Using a dual variational method we shall show the existence of solutions to the Dirichlet problem

$$\begin{aligned} &-\operatorname{div}\Big(\big|\nabla u(x)\big|^{p(x)-2}\nabla u(x)\Big) = F_u\big(x, u(x)\big),\ u \in W_0^{1,p(x)}(\Omega) \\ &x(y)|_{\partial\Omega} = 0. \end{aligned} \tag{0.1}$$

without assuming Palais-Smale condition.

1. INTRODUCTION

We shall show the existence of solutions to the Dirichlet problem

$$\begin{aligned} &-\operatorname{div}\Big(\big|\nabla u(x)\big|^{p(x)-2}\nabla u(x)\Big) = F_u\big(x, u(x)\big),\ u \in W_0^{1,p(x)}(\Omega) \\ &u(x)|_{\partial\Omega} = 0, \end{aligned} \tag{1.1}$$

by using a dual variational method. Here $\Omega \subset R^N$ is a bounded region, $p, q \in C(\overline{\Omega})$, and $1/\big(p(x)\big)+1/\big(q(x)\big) = 1$ for $x \in \Omega$, and $W_0^{1,p(x)}(\Omega)$ denotes the generalised Orlicz–Sobolev space, see [**5, 3**]. Let $p^- = \inf_{x\in\Omega} p(x)$. We assume that $p^- \geqslant 2$.

Variational problems with $(p,q)-$growth conditions have been studied in the last few years, see [**4, 6**] and references therein. Problems with (p,q)-growth conditions are applied in elastic mechanics and electrorheological fluid dynamics, see [**9, 10**] and references therein.

We construct a variational method which applies in the super-critical case without assuming a type of Palais–Smale conditions. Thus we think that our approach may shed new light on the study of Dirichlet problems with non-standart growth conditions. The case of subcritical growth is considered by using theory of monotonne operators in [**4**]. But in [**4**] a type of the Palais–Smale condition is assumed in order to obtain the existence of solutions in the supercritical case. The elliptic systems are considered in [**6**], where the ideas from [**4**] are applied. We shall make use of the duality theory for convex functions, see [**2**]. But since we need the convexity of F (the primitive of F_u with respect to the

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second variable) only on some interval, our approach seems to unite both super- and sub-critical cases. In the derivation of the dual variational method, we use some ideas developed in [8] for ordinary differential equations with super-critical growth.

In what follows C_S denotes the best Sobolev constant

$$\|u\|_{p(x)} \leqslant C_S \|\nabla u\|_{p(x)} \text{ for all } u \in W_0^{1,p(x)}(\Omega).$$

Since $W_0^{1,p(x)}(\Omega)$ is continuously embedded into $W_0^{1,p^-}(\Omega)$, ([3]), we denote by C_1 and C_2 the following constants

$$\|\nabla u\|_{p^-} \leqslant C_1 \|\nabla u\|_{p(x)} \tag{1.2}$$

$$\max_{x\in\Omega} |u(x)| \leqslant C_2 \|\nabla u\|_{p^-} \text{ for all } u \in W_0^{1,p(x)}(\Omega) \tag{1.3}$$

We make the following assumptions.

F1. $\operatorname{vol}(\Omega) \leqslant (1/p^- + 1/q^-)^{-1}$, and there exist a function $\overline{z} \in C_0^1(\Omega)$ such that $F_u(\cdot, \overline{z}(\cdot)) \in L^\infty(\Omega)$ and

$$C_S \operatorname*{ess\,sup}_{x\in\Omega} \left| F_u\Big(x, \sup_{s\in\Omega} |\overline{z}(s)|\Big) \right| \geqslant 1.$$

Let

$$I = \Big[-\sup_{x\in\Omega} |\overline{z}(x)|, \sup_{x\in\Omega} |\overline{z}(x)|\Big].$$

F2. $F : \Omega \times I$ and $F_u : \Omega \times I$ are Caratheodory functions, F is convex in u for almost all $x \in \Omega$, and moreover

$$C_1 C_2 C_S \operatorname*{ess\,sup}_{x\in\Omega} \left| F_u\Big(x, \sup_{s\in\Omega} |\overline{z}(s)|\Big) \right| \leqslant \sup_{s\in\Omega} |\overline{z}(s)| \tag{1.4}$$

F3. $F_u(x, 0) \neq 0$, for almost all $x \in \Omega$, and the functions $x \mapsto |F(x, 0)|$ and $x \mapsto |F^*(x, 0)|$ are integrable. Moreover for almost all $x \in \Omega$

$$\left| F_u\Big(x, -\sup_{s\in\Omega} |\overline{z}(s)|\Big) \right| \leqslant \left| F_u\Big(x, \sup_{s\in\Omega} |\overline{z}(s)|\Big) \right|, \tag{1.5}$$

These assumptions make $F : \Omega \times I \to R$ convex and lower semi-continuous, where F^* denotes the Fenchel–Young conjugate of F, see [2].

Equation (1.1) is the Euler-Lagrange equation for the functional $J : W_0^{1,p(x)}(\Omega) \to \mathcal{R}$, ([4])

$$J(x) = \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)}\, dx - \int_\Omega F\big(x, u(x)\big)\, dx.$$

With the growth conditions given above J is not well defined on $W_0^{1,p(x)}(\Omega)$. We shall construct a subset of $W_0^{1,p(x)}$ on which the integral $\int_\Omega F\big(x, u(x)\big)\, dx$ is finite.

We invesitagate J, together with its dual functional $J_D : W \to \mathcal{R}$ given by

$$J_D(v) = \int_\Omega F^*\big(x, -\operatorname{div} v(x)\big)\,dx - \int_\Omega \frac{1}{q(x)}\big|v(x)\big|^{q(x)}\,dx,$$

here $W = \big\{v \in L^{q(x)}(\Omega) \mid \operatorname{div} v \in L^{q(x)}(\Omega)\big\}$.

The method which we use is based on the definition of the set X, see relation (2.2). In its abstract formulation it is used in topological methods, compare with [1] where a set similar to our set X is constructed but the existence result is obtained via a fixed point theorem. We investigate J on a set X and J_D on a set X^d dual to X in a certain sense. On these sets we look for critical values and critical points of both functionals. Having established the relationship between the relevant critical point we get the solution to (1.1).

2. Duality results

We shall seek solutions to (1.1) in the form of a pair $(x, v) \in W_0^{1,p(x)}(\Omega \times W)$ such that

$$\begin{aligned} \big|\nabla u(x)\big|^{p(x)-2}\nabla u(x) &= v(x), \\ -\operatorname{div} v(x) &= F_u\big(x, u(x)\big). \end{aligned} \tag{2.1}$$

The system (2.1) may be viewed as a Hamiltonian system, and will be obtained with the aid of a duality theory describing relations between critical points (that is, a variational principle) of a certain kind and critical values (that is, a duality principle) of the action and dual action functional.

In order to develop the duality theory for functionals J and J_D we shall construct certain nonlinear subsets of the spaces $W_0^{1,p(x)}(\Omega)$ and $\big(W_0^{1,p(x)}\big)^*$ on which we shall look for critical points.

We define $\overline{X}$ to be the set of functions $u \in W_0^{1,p(x)}(\Omega)$ such that

$$\|u\|_{L^{p(x)}(\Omega)} \leqslant C_S \operatorname*{ess\,sup}_{x\in\Omega} \left|F_u\Big(x, \sup_{s\in\Omega}\big|\overline{z}(s)\big|\Big)\right|, \quad \operatorname{div}\Big(\big|\nabla u(\cdot)\big|^{p(x)-2}\nabla u(\cdot)\Big) \in L^\infty(\Omega), \quad \text{and}$$
$$u(x) \in \Big[-\sup_{s\in\Omega}\big|\overline{z}(s)\big|, \sup_{s\in\Omega}\big|\overline{z}(s)\big|\Big] \text{ almost everywhere.}$$

We also define the set X such that for all $u \in X$ the relation

$$\operatorname{div}\Big(\big|\nabla\widetilde{u}(x)\big|^{p(x)-2}\nabla\widetilde{u}(x)\Big) = F_u\big(x, u(x)\big). \tag{2.2}$$

implies $\widetilde{u} \in X$.

Proposition 2.1. $X = \overline{X}$.

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PROOF: Take any $w \in \overline{X}$. The solution $x \in W_0^{1,p(x)}(\Omega)$ to

$$-\operatorname{div}\Big(\big|\nabla u(x)\big|^{p(x)-2}\nabla u(x)\Big) = F_u\big(x, w(x)\big),$$
$$u(x)|_{\partial\Omega} = 0,$$

exists by [**4**, Theorem 4.2]. Indeed, by F1, F2, and F3 we get for any $w \in \overline{X}$

$$F_u\Big(x, -\sup_{s\in\Omega}\big|\overline{z}(s)\big|\Big) \leqslant F_u\big(x, w(x)\big) \leqslant F_u\Big(x, \sup_{s\in\Omega}\big|\overline{z}(s)\big|\Big) \text{ for almost all } x \in \Omega.$$

Thus $F_u\big(\cdot, w(\cdot)\big) \in L^\infty(\Omega)$ and [**4**, Theorem 4.2] applies.

From the relation

$$\int_\Omega -\operatorname{div}\Big(\big|\nabla u(x)\big|^{p(x)-2}\nabla u(x)\Big)u(x)\,dx = \int_\Omega F_u\big(x, w(x)\big)u(x)\,dx,$$

we obtain

$$\begin{aligned}\int_\Omega\big|\nabla u(x)\big|^{p(x)}\,dx &\leqslant \int_\Omega F_u\big(x, \overline{z}(x)\big)u(x)\,dx\\ &\leqslant \operatorname*{ess\,sup}_{x\in\Omega}\Big|F_u\Big(x, \sup_{s\in\Omega}\big|\overline{z}(s)\big|\Big)\Big|\int_\Omega\big|u(x)\big|\,dx\\ &\leqslant \Big(\frac{1}{p^-}+\frac{1}{q^-}\Big)\operatorname{vol}(\Omega)\operatorname*{ess\,sup}_{x\in\Omega}\Big|F_u\Big(x, \sup_{s\in\Omega}\big|\overline{z}(s)\big|\Big)\Big|\,\|u\|_{p(x)}.\end{aligned}$$

So

$$\int_\Omega\big|\nabla u(x)\big|^{p(x)}\,dx \leqslant C_S \operatorname*{ess\,sup}_{x\in\Omega}\Big|F_u\Big(x, \sup_{s\in\Omega}\big|\overline{z}(s)\big|\Big)\Big|\,\|\nabla u\|_{p(x)}.$$

Since

$$\|\nabla u\|_{p(x)} \leqslant 1 \Leftrightarrow \int_\Omega\big|\nabla u(x)\big|^{p(x)}\,dx \leqslant 1$$

we obtain by the assumptions that for $\|\nabla u\|_{p(x)} \leqslant 1$

$$\|\nabla u\|_{p(x)} \leqslant C_S \operatorname*{ess\,sup}_{x\in\Omega} F_u\Big(x, \sup_{s\in\Omega}\big|\overline{z}(s)\big|\Big).$$

If $\int_\Omega\big|\nabla u(x)\big|^{p(x)}\,dx \geqslant 1$ we get $\int_\Omega\big|\nabla u(x)\big|^{p(x)}\,dx \geqslant \|\nabla u\|_{p(x)}^{p^-}$ so

$$\|\nabla u\|_{p(x)} \leqslant \|\nabla u\|_{p(x)}^{p^- -1} \leqslant C_S \operatorname*{ess\,sup}_{x\in\Omega}\Big|F_u\Big(x, \sup_{s\in\Omega}\big|\overline{z}(s)\big|\Big)\Big|.$$

By direct calculations using inequalities (1.3) and (1.2) we obtain for all $x \in \Omega$

$$\begin{aligned}C_S \operatorname*{ess\,sup}_{x\in\Omega}\Big|F_u\Big(x, \sup_{s\in\Omega}\big|\overline{z}(s)\big|\Big)\Big| &\geqslant \|\nabla u\|_{p(x)} \geqslant \frac{1}{C_1}\|\nabla u\|_{p^-}\\ &\geqslant \frac{1}{C_1C_2}\max_{x\in\Omega}\big|u(x)\big| \geqslant \frac{1}{C_1C_2}\big|u(x)\big|\end{aligned}$$

So by the assumptions for almost everywhere $X \in \Omega$

$$|u(x)| \leqslant C_1 C_2 C_S \operatorname*{ess\,sup}_{X\in\Omega} \left|F_u\Big(X, \sup_{s\in\Omega}|\overline{z}(s)|\Big)\right| \leqslant \sup_{s\in\Omega}|\overline{z}(s)|$$

Thus $u \in \overline{X}$ and we may put $X = \overline{X}$. ∎

From now on we shall consider functional J on the set X. The dual functional J_D will be considered on a set X^d which *is the set of those $v \in W$ for which there exists $u \in X$ such that*

$$-\operatorname{div} v(x) = F_x\big(x, u(x)\big).$$

Moreover for any $v \in X^d$ there exists exactly one $u \in X$ and for any $u \in X$ there exists exactly one $v \in X^d$.

We have the following lemmas.

Lemma 2.2. *There exists a constant γ such that for all $u \in X$*

$$\left|\int_\Omega F\big(x, u(x)\big)\, dx\right| \leqslant \gamma.$$

Proof: By the convexity of F we get for almost all $x \in \Omega$

$$F\big(x, u(x)\big) \leqslant F(x, 0) + F_u\big(x, u(x)\big)u(x).$$

Thus by the assumptions

$$\begin{aligned}\left|\int_\Omega F\big(x, u(x)\big)\, dx\right| &\leqslant \int_\Omega |F(x,0)|\, dx + \int_\Omega \left|F_u\big(x, u(x)\big)u(x)\right| dx \\ &\leqslant \int_\Omega |F(x,0)|\, dx + \sup_{s\in\Omega}|\overline{z}(s)| \int_\Omega \left|F_u\Big(x, \sup_{s\in\Omega}|\overline{z}(s)|\Big)\right| dx = \gamma\end{aligned}$$

∎

Lemma 2.3. *There exists a constant η such that for all $v \in X^d$*

$$\left|\int_\Omega F^*\big(x, -\operatorname{div} v(x)\big)\, dx\right| \leqslant \eta.$$

Proof: We need only show that the integral

$$\int_\Omega F^*\big(x, -\operatorname{div} v(x)\big)\, dx$$

is finite for any $v \in X^d$. By the convexity of F we get for almost all $x \in \Omega$

$$F^*\big(x, -\operatorname{div} v(x)\big) \leqslant F^*(x, 0) + \xi(x)\big(-\operatorname{div} v(x)\big),$$

where $\xi(x) \in \partial F^*\big(x, -\operatorname{div} v(x)\big)$. Since $v \in X^d$, it follows that there exists $u \in X$ such that

$$-\operatorname{div} v(x) = F_u\big(x, u(x)\big).$$

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Thus by convexity $u(x) \in \partial F^*(x, -\operatorname{div} v(x))$. By the assumptions

$$\left|\int_\Omega F^*(x, -\operatorname{div} v(x))\,dx\right| \leqslant \int_\Omega |F^*(x,0)|\,dx + \int_\Omega |u(x)| \cdot |-\operatorname{div} v(x)|\,dx$$
$$\leqslant \int_\Omega |F^*(x,0)|\,dx + \sup_{s\in\Omega}|\overline{z}(s)| \int_\Omega \left|F_u\left(x, \sup_{s\in\Omega}|\overline{z}(s)|\right)\right| dx = \eta. \qquad \square$$

From the above lemmas it follows that J and J_D are well defined on X and X^d. We consider a functional $J^\# : X \times X^d \to R$ given by the formula

$$J^\#(u,v) = \int_\Omega F^*(x, -\operatorname{div} v(x))\,dx + \int_\Omega \frac{1}{p(x)}|\nabla u(x)|^{p(x)}\,dx - \int_\Omega \nabla u(x) v(x)\,dx.$$

In the proof of the duality principle we shall make use of the following lemmas.

Lemma 2.4. *For any $v \in X^d$*

$$\inf_{u\in X} J^\#(u,v) = J_D(v).$$

Proof: Fix $v \in X^d$. We obtain by a Fenchel–Young inequality

$$\sup_{u\in X}\left\{\int_\Omega \nabla u(x)v(x)\,dx - \int_\Omega \frac{1}{p(x)}|\nabla u(x)|^{p(x)}\,dx\right\} \leqslant \sup_{u\in X}\int_\Omega \frac{1}{q(x)}|v(x)|^{q(x)}\,dx$$
$$= \int_\Omega \frac{1}{q(x)}|v(x)|^{q(x)}\,dx \tag{2.3}$$

By the definition of X^d for a given $v \in X^d$ there exists $u_v \in X$ satisfying

$$|\nabla u_v(x)|^{p(x)-2}\nabla u_v(x) = v(x). \tag{2.4}$$

Indeed, by definition for a given $v \in X^d$ there exists $u \in X$ such that

$$-\operatorname{div} v(x) = F_u(x, u(x)).$$

By relation (2.2) in turn for this u there exists u_v such that

$$-\operatorname{div}\left(|\nabla u_v(x)|^{p(x)-2}\nabla u_v(x)\right) = F_u(x, u(x)).$$

Thus we can define u_v by (2.4). Relation (2.4) and the convexity relations give

$$\int_\Omega \frac{1}{p(x)}|\nabla u_v(x)|^{p(x)}\,dx + \int_\Omega \frac{1}{q(x)}|v(x)|^{q(x)}\,dx = \int_\Omega \nabla u_v(x)v(x)\,dx.$$

By the above

$$-J^\#(u_v,v) = \int_\Omega \nabla u_v(x)v(x)\,dx - \int_\Omega \frac{1}{p(x)}|\nabla u_v(x)|^{p(x)}\,dx - \int_\Omega F^*(x, -\operatorname{div} v(x))\,dx$$
$$= \int_\Omega \frac{1}{q(x)}|v(x)|^{q(x)}\,dx - \int_\Omega F^*(x, -\operatorname{div} v(x))\,dx. \tag{2.5}$$

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As a consequence of (2.3) and (2.5) we have

$$-J_D(v) = -J^{\#}(u_v, v) \leqslant \sup_{u\in X} -J^{\#}(u,v) \leqslant -J_D(v).$$

∎

Lemma 2.5. *For any $u \in X$*

$$\inf_{v\in X^d} J^{\#}(u,v) = J(u).$$

Proof: Fix $u \in X$. We obtain by the Fenchel–Young inequality

$$\begin{aligned}(2.6)\qquad &\sup_{v\in\left(W_0^{1,p(x)}\right)^*}\left\{\int_\Omega \big(-\operatorname{div} v(x)\big)u(x)\,dx - \int_\Omega F^*\big(x,-\operatorname{div} v(x)\big)\,dx\right\}\\ &\qquad\leqslant \sup_{f\in L^{q(x)}(\Omega)}\left\{\int_\Omega u(x)f(x) - \int_\Omega F^*\big(x,f(x)\big)\,dx\right\}\\ &\qquad\leqslant \sup_{f\in L^{q(x)}(\Omega)}\int_\Omega F\big(x,u(x)\big)\,dx = \int_\Omega F\big(x,u(x)\big)\,dx.\end{aligned}$$

By definition of X^d we get there exists v_u such that

$$-\operatorname{div} v_u(x) = F_u\big(x,u(x)\big).$$

By (2.7) and using the convexity of F we get

$$\begin{aligned}-J^{\#}(u,v_u) &= \int_\Omega\big(-\operatorname{div} v_u(x)\big)u(x)\,dx - \int_\Omega F^*\big(x,-\operatorname{div} v_u(x)\big)\,dx - \int_\Omega \frac{1}{p(x)}\left|\nabla u(x)\right|^{p(x)}dx\\ (2.7)\qquad &= \int_\Omega F\big(x,u(x)\big)\,dx - \int_\Omega \frac{1}{p(x)}\left|\nabla u(x)\right|^{p(x)}dx = -J(u).\end{aligned}$$

As a consequence of the above relation we obtain by (2.6) and (2.7)

$$\begin{aligned}-J(u) &= \int_\Omega \nabla u(x)v_u(x)\,dx - \int_\Omega \frac{1}{p(x)}\left|\nabla u(x)\right|^{p(x)}dx - \int_\Omega F^*\big(x,-\operatorname{div} v_u(x)\big)\,dx\\ &\leqslant \sup_{v\in W}\left\{\int_\Omega \nabla u(x)v(x)\,dx - \int_\Omega F^*\Big(x,-\operatorname{div} v(x)\Big)\,dx\right\}\\ &\qquad\qquad - \int_\Omega \frac{1}{p(x)}\left|\nabla u(x)\right|^{p(x)}dx \leqslant -J(u).\end{aligned}$$

∎

Theorem 2.6. (Duality Principle.)

$$\inf_{u\in X} J(u) = \inf_{v\in X^d} J_D(v).$$

Proof: By Lemmas 2.4 and 2.5 we obtain

$$\inf_{u\in X} J(u) = \inf_{u\in X}\inf_{v\in X^d} J^{\#}(u,v) = \inf_{v\in X^d}\inf_{u\in X} J^{\#}(u,v) = \inf_{v\in X^d} J_D(v).$$

∎

3. Variational principles

We shall use the duality results to derive necessary conditions for the existence of solutions to (1.1).

Theorem 3.1. (Variational Principle.) *Suppose that there exists $\overline{u} \in X$ such that*

$$-\infty < J(\overline{u}) = \inf_{u \in X} J(u) < \infty.$$

Then there exists $\overline{v} \in X^d$ such that

$$-\operatorname{div} \overline{v}(x) = F_u\big(x, \overline{u}(x)\big), \tag{3.1}$$

$$\left|\nabla \overline{u}(x)\right|^{p(x)-2} \nabla \overline{u}(x) = \overline{v}(x). \tag{3.2}$$

Moreover

$$\inf_{v \in X^d} J_D(v) = J_D(\overline{v}) = J(\overline{u}) = \inf_{u \in X} J(u). \tag{3.3}$$

Proof: Since $\overline{u} \in X$, we may take $\overline{v} \in X^d$ such that

$$-\operatorname{div} \overline{v}(x) = F_x\big(x, \overline{u}(x)\big).$$

Thus (3.1) holds. By (3.1) and by the Fenchel-Young inequality we have

$$\begin{aligned}
J(\overline{u}) &= \int_\Omega \frac{1}{p(x)} \left|\nabla \overline{u}(x)\right|^{p(x)} dx - \int_\Omega F\big(x, \overline{u}(x)\big)\, dx \\
&= -\int_\Omega \big(-\operatorname{div} \overline{v}(x)\big) \nabla \overline{u}(x)\, dx + \int_\Omega F^*\big(x, -\operatorname{div} \overline{v}(x)\big)\, dx + \int_\Omega \frac{1}{p(x)} \left|\nabla \overline{u}(x)\right|^{p(x)} dx \\
&= -\int_\Omega \overline{v}(x) \nabla \overline{u}(x)\, dx + \int_\Omega \frac{1}{p(x)} \left|\nabla \overline{u}(x)\right|^{p(x)} dx + \int_\Omega F^*\big(x, -\operatorname{div} \overline{v}(x)\big)\, dx \\
&\geqslant -\int_\Omega \frac{1}{q(x)} \left|\overline{v}(x)\right|^{q(x)} dx + \int_\Omega F^*\big(x, -\operatorname{div} \overline{v}(x)\big)\, dx = J_D(\overline{v}).
\end{aligned}$$

Hence $J(\overline{u}) \geqslant J_D(\overline{v})$. By Theorem 2.6 it follows that $J(\overline{u}) \leqslant \inf_{v \in X^d} J_D(v) \leqslant J_D(\overline{v})$. Hence $J(\overline{u}) = J_D(\overline{v})$ and

$$\begin{aligned}
&\int_\Omega \frac{1}{p(x)} \left|\nabla \overline{u}(x)\right|^{p(x)} dx - \int_\Omega F\big(x, \overline{u}(x)\big)\, dx \\
&\qquad = -\int_\Omega \frac{1}{q(x)} \left|\nabla \overline{v}(x)\right|^{q(x)} dx + \int_\Omega F^*\big(x, -\operatorname{div} \overline{v}(x)\big)\, dx.
\end{aligned} \tag{3.4}$$

By (3.1) and (3.4) it follows that

$$\frac{1}{p(x)} \left|\nabla \overline{u}(x)\right|^{p(x)} + \frac{1}{q(x)} \left|\nabla \overline{v}(x)\right|^{q(x)} = \overline{v}(x) \nabla \overline{u}(x).$$

Hence (3.2) holds. Assertion (3.3) follows by Duality Principle and since $J(\overline{u}) = J_D(\overline{v})$. ∎

A similar result may be derived for minimising sequences. Theorem 3.2 may be viewed as an ε-variational principle and it will be used in the proof of the existence theorem. It differs from Theorem 3.1 in relation (3.2), which is now presented in a ε-subdifferential form.

THEOREM 3.2. *Let $\{u_j\}$, $u_j \in X$, $j \in \mathcal{N}$ be a minimising sequence for J and let $-\infty < \inf_{j\in\mathcal{N}} J(u_j) < \infty$. If for $v_j \in X^d$, $j \in \mathcal{N}$ we have*

$$-\operatorname{div} v(v_j) = F_u\big(x, u_j(x)\big), \tag{3.5}$$

then $\{v_j\}$ is a minimising sequence for J_D. For any $\varepsilon > 0$ there exists j_0 such that for $j \geqslant j_0$

$$0 \leqslant \int_\Omega \frac{1}{p(x)} |\nabla u_j(x)|^{p(x)}\, dx + \int_\Omega \frac{1}{q(x)} |v_j(x)|^{q(x)}\, dx - \int_\Omega v_j(x) \nabla u_j(x)\, dx \leqslant \varepsilon. \tag{3.6}$$

Moreover

$$\inf_{v\in X^d} J_D(v) = \inf_{j\in\mathcal{N}} J_D(v_j) = \inf_{u\in X} J(u) = \inf_{j\in\mathcal{N}} J(u_j). \tag{3.7}$$

PROOF: Since $u_j \in X$ for $j \in \mathcal{N}$ we obtain that there exists $v_j \in X^d$ such that (3.5) is satisfied. We shall show that $\{v_j\}$ is a minimising sequence for J_D. By the above and Fenchel-Young inequality we obtain reasoning as in the proof of Theorem 3.1 that for any $j \in \mathcal{N}$

$$J(u_j) \geqslant J_D(v_j). \tag{3.8}$$

By the Duality Principle and (3.8) we get

$$\inf_{j\in\mathcal{N}} J_D(v_j) \geqslant \inf_{v\in X^d} J_D(v) = \inf_{u\in X} J(u) = \inf_{j\in\mathcal{N}} J(u_j) \geqslant \inf_{j\in\mathcal{N}} J_D(v_j).$$

Thus we get (3.7) and it follows that $\{v_j\}$ is a minimising sequence for J_D.

Let us take arbitrary $\varepsilon > 0$. It follows that there exists j_0 such that for $j \geqslant j_0$ we have

$$J(u_j) < \inf_{j\in\mathcal{N}} J(u_j) + \varepsilon.$$

By (3.8) it now follows that for $j \geqslant j_0$ we have $J_D(v_j) < \inf_{j\in\mathcal{N}} J(u_j) + \varepsilon$. So

$$J_D(v_j) \leqslant J(u_j) < \inf_{j\in\mathcal{N}} J(u_j) + \varepsilon \leqslant J_D(v_j).$$

Thus we get (3.6) using the definitions of J and J_D. ∎

4. The existence of solutions

We shall show that there exists an element $\overline{u} \in W_0^{1,p(x)}(\Omega)$ which together with a corresponding $\overline{v} \in W$ satisfies system (2.1). We shall make use of the ε-variational principle for minimising sequences and the construction of sets X and X^d. It is not the existence of the minimising sequences that is a really difficult to establish, but their convergence to the pair satisfying system (2.1). Here the duality theory plays again an important part.

Theorem 4.1. *There exists a pair $(\overline{u}, \overline{v}) \in W_0^{1,p(x)}(\Omega) \times W$ satisfying the system* (2.1), *that is,*

$$|\nabla \overline{u}(x)|^{p(x)-2} \nabla \overline{u}(x) = \overline{v}(x), \tag{4.1}$$

$$-\operatorname{div} \overline{v}(x) = F_u(x, \overline{u}(x)), \tag{4.2}$$

$$\inf_{v \in X^d} J_D(v) = J_D(\overline{v}) = J(\overline{u}) = \inf_{u \in X} J(u). \tag{4.3}$$

Proof: We first show that J is bounded from below on X. From Lemma 2.2 it follows that

$$J(x) = \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)}\, dx - \int_\Omega F(x, u(x))\, dx \geqslant -\gamma.$$

Now we put $b = J(x_0)$ for a fixed $x_0 \in X$ and consider the Lebesgue set

$$S_b = \{x \in X : J(x) \leqslant b\}.$$

By a direct calculation we get for all $x \in S_b$

$$\int_\Omega \frac{1}{p^+} |\nabla u(x)|^{p(x)}\, dx \leqslant b + \gamma.$$

It follows that S_b is relatively weakly compact in $W_0^{1,p(x)}(\Omega)$. Thus we may choose in S_b a weakly convergent minimising sequence $\{u_j\}$ for a functional J. This sequence is up to a subsequence strongly convergent in $L^{p^-}(\Omega)$ and thus convergent almost everywhere. We denote its limit by $\overline{u}$. We may now observe that J is weakly lower semicontinuous on S_b. Indeed,

$$W_0^{1,p(x)}(\Omega) \ni x \to \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)}\, dx \in R$$

being convex and lower semicontinuous is weakly lower semicontinuous (**[2]**). Since the limit

$$\lim_{j \to \infty} \int_\Omega F_u(x, u_j(x))\, dx = \int_\Omega F_u(x, \overline{u}(x))\, dx$$

exists we get

$$\liminf_{j \to \infty} J(u_j) \geqslant J(\overline{u}). \tag{4.4}$$

Thus $J(\overline{u}) = \inf_{u \in X} J(u)$.

We now choose the sequence $\{v_j\}$ in such a way that $v_j \in X^d$ for $j \in \mathcal{N}$ and

$$- \operatorname{div} v_j(x) = F_u\big(x, u_j(x)\big). \tag{4.5}$$

We investigate the convergence of $\{v_j\}$ and $\{- \operatorname{div} v_j\}$.

From relation (4.5) we obtain that $\{- \operatorname{div} v_j\}$ is convergent almost everywhere to a certain function $w(x) = F_u\big(x, \overline{u}(x)\big)$. We observe that $w \in L^\infty(\Omega)$. From Theorem 3.2 it follows that $\{v_j\}$ is a minimising sequence for J_D and we get for j sufficiently large and a fixed $\varepsilon > 0$,

$$\int_\Omega \frac{1}{p(x)} \left|\nabla u_j(x)\right|^{p(x)} dx - \int_\Omega F\big(x, u_j(x)\big)\, dx \\ - \int_\Omega F^*\big(x, - \operatorname{div} v_j(x)\big)\, dx + \int_\Omega \frac{1}{q(x)} \left|v_j(x)\right|^{q(x)} dx \leqslant \varepsilon.$$

By the above, by definition of $\{u_j\}$ and by Lemma 2.3 it follows that $\{v_j\}$ is weakly convergent in $L^{q(x)}(\Omega)$. We denote its limit by $\overline{v}$. We show that $\operatorname{div} \overline{v} = \overline{w}$. Since $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(x)}(\Omega)$, ([5]), we may proceed as follows. We take any $f \in C_0^\infty(\Omega)$ and calculate

$$\begin{aligned} \int_\Omega \nabla f(x) \overline{v}(x)\, dx &= \lim_{j \to \infty} \int_\Omega \nabla f(x) v_j(x)\, dx \\ &= - \lim_{j \to \infty} \int_\Omega f(x) \operatorname{div} v_j(x)\, dx \\ &= - \int_\Omega f(x) \overline{w}(x)\, dx \end{aligned}$$

Thus by the Euler–Lagrange lemma for mutliple integrals, ([7]), we get $\operatorname{div} \overline{v} = \overline{w}$ and so relation (4.2) holds .

By Theorem 3.2 and relation (4.5) it follows that

$$\liminf_{j \to \infty} \left(\int_\Omega \frac{1}{p(x)} \left|\nabla u_j(x)\right|^{p(x)} dx + \int_\Omega \frac{1}{q(x)} \left|v_j(x)\right|^{q(x)} dx - \int_\Omega v_j(x) \nabla u_j(x)\, dx \right) = 0.$$

Thus

$$\begin{aligned} 0 &\geqslant \liminf_{j \to \infty} \left(\int_\Omega \frac{1}{p(x)} \left|\nabla u_j(x)\right|^{p(x)} dx + \int_\Omega \frac{1}{q(x)} \left|v_j(x)\right|^{q(x)} dx - \int_\Omega v_j(x) \nabla u_j(x)\, dx \right) \\ &\geqslant \liminf_{j \to \infty} \int_\Omega \frac{1}{p(x)} \left|\nabla u_j(x)\right|^{p(x)} dx + \liminf_{k \to \infty} \int_\Omega \frac{1}{q(x)} \left|v_j(x)\right|^{q(x)} dx \\ &\qquad + \lim_{j \to \infty} \int_\Omega \operatorname{div} v_j(x) u_j(x)\, dx \\ &\geqslant \int_\Omega \frac{1}{p(x)} \left|\nabla \overline{u}(x)\right|^{p(x)} dx + \int_\Omega \frac{1}{q(x)} \left|\overline{v}(x)\right|^{q(x)} dx + \int_\Omega \operatorname{div} \overline{v}(x) \overline{u}(x)\, dx \geqslant 0. \end{aligned}$$

The last relation follows by the Fenchel–Young inequality. Hence

$$\int_\Omega \frac{1}{p(x)}\left|\nabla\overline{u}(x)\right|^{p(x)}dx - \int_\Omega \overline{v}(x)\nabla\overline{u}(x)\,dx + \int_\Omega \frac{1}{q(x)}\left|\overline{v}(x)\right|^{q(x)}dx = 0.$$

So (4.2) is satisfied. Relation (4.3) follows by Duality Principle, (4.4) and since $J_D(\overline{v}) = J(\overline{u})$. ∎

Corollary 4.2. *There exists $\overline{u} \in X$ such that*

$$-\operatorname{div}\Big(\left|\nabla\overline{u}(x)\right|^{p(x)-2}\nabla\overline{u}(x)\Big) = F_u\big(x,\overline{u}(x)\big),$$
$$J(\overline{u}) = \inf_{u\in X} J(u).$$

Proof: It suffices to prove that $\lim_{j\to\infty} u_j = \overline{u} \in X$ in the proof of Theorem 4.1. Indeed, we must show that

$$\|\overline{u}\|_{L^{p(x)}(\Omega)} \leqslant C_S \operatorname*{ess\,sup}_{x\in\Omega}\left|F_u\Big(x,\sup_{s\in\Omega}\left|\overline{z}(s)\right|\Big)\right|, \tag{4.6}$$

$$\overline{u}(x) \in \Big[-\sup_{s\in\Omega}\left|\overline{z}(s)\right|, \sup_{s\in\Omega}\left|\overline{z}(s)\right|\Big] \text{ almost everywhere} \tag{4.7}$$

and

$$-\operatorname{div}\Big(\left|\nabla\overline{u}(\cdot)\right|^{p(x)-2}\nabla\overline{u}(\cdot)\Big) \in L^\infty(\Omega). \tag{4.8}$$

We infer that

$$\|u_j\|_{L^{p(x)}(\Omega)} \leqslant C_S \operatorname*{ess\,sup}_{x\in\Omega}\left|F_u\Big(x,\sup_{s\in\Omega}\left|\overline{z}(s)\right|\Big)\right|$$

for all j and

$$\liminf_{j\to\infty}\|u_j\|_{L^{p(x)}(\Omega)} \geqslant \|\overline{u}\|_{L^{p(x)}(\Omega)}.$$

Thus (4.6) holds. By definiton of the sequence $\{u_j\}$ we also get

$$\left|u_j(x)\right| \leqslant \sup_{s\in\Omega}\left|\overline{z}(s)\right|.$$

Since $\{u_j\}$ is convergent almost everywhere, we get (4.7). To prove (4.8) we observe that

$$F_u\big(x,\overline{u}(s)\big) \leqslant F_u\Big(x,\sup_{s\in\Omega}\left|\overline{z}(s)\right|\Big)$$

for almost all x. ∎

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