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# NOTE ON A PAPER OF J. HOFFSTEIN

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**Abstract.** The concept of a metaplectic form was introduced about 40 years ago by T. Kubota. He showed how Jacobi-Legendre symbols of arbitrary order give rise to characters of arithmetic groups. Metaplectic forms are the automorphic forms with these characters. Kubota also showed how higher analogues of the classical theta functions could be constructed using Selberg's theory of Eisenstein series. Unfortunately many aspects of these generalized theta series are still unknown, for example, their Fourier coefficients. The analogues in the case of function fields over finite fields can in principle be calculated explicitly and this was done first by J. Hoffstein in the case of a rational function field. Here we shall return to his calculations and clarify a number of aspects of them, some of which are important for recent developments.

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**1. Introduction.** Several years ago J. Hoffstein [**9**] published an investigation into the nature of those Dirichlet series whose coefficients are Gauss sums and whose analytic properties can be derived from the theory of metaplectic groups in the function field case. He restricted his considerations to rational function fields where the calculations are considerably easier than in the general case. The general theory [**10**, §II.3] shows that the functions in question are meromorphic with certain analytic properties. In the function field case this means that the functions are rational, with known denominators and with numerators of known degree. Their calculation reduces to a finite one. Moreover the residue at the (only) pole can also be given explicitly, even if the expression one obtains is not as illuminating as one would wish. J. Hoffstein uses the results to verify the function field analogue, for rational function fields, of a conjecture made by C. Eckhardt and the author [**5**] in the case of Gauss sums of order 4.

Unfortunately the notations used by J. Hoffstein and the present author are very different. The first object of this paper was to express the main results of [**9**] in the alternative language. This is not particularly difficult but it is hoped that it will prove useful. In his paper J. Hoffstein used one particular method to evaluate the coefficients of the Dirichlet series. He expresses them as double sums and by reversing the order of summation obtains a useful expression. Instead of this approach we shall here use one of the Davenport–Hasse theorems to express the Gauss sums in terms of a standard quantity. Consequently we express the coefficients as a single sum. This has two advantages. First of all there is much less cancellation between the terms of the sum. Secondly it shows that the theory of metaplectic forms allows one to investigate the following question and variants thereof. Let $\mathbb{F}_q$ be the usual finite field of $q$ elements. Let $n > 1$ be an integer so that $q \equiv 1 \pmod{n}$. For a monic polynomial $f(x) \in \mathbb{F}_q[x]$ let $D(f)$ denote the discriminant of $f$. The question is: of the $q^d - q^{d-1}$ monic polynomials without multiple factors of degree $d$ for how many is $D(f)$ an $n$th power? As far as the author knows, there have been no algebraic investigations into such questions.

To formulate the framework in which we shall be working we shall make some definitions. Let $p$ be the characteristic of $\mathbb{F}_q$. Let $\mu_n = \{x \in \mathbb{F}_q : x^n = 1\}$ and let $\chi : \mathbb{F}_q^\times \to \mu_n; x \mapsto x^{\frac{q-1}{n}}$. Let $R = \mathbb{F}_q[x]$, $k = \mathbb{F}_q(x)$ and $k_\infty = \mathbb{F}_q((x^{-1}))$ (the field of all Laurent series in $x^{-1}$, i.e. the completion of $k$ at the "infinite place"). Let deg denote the degree of an element of $R$. We shall write $\pi_\infty$ for $x^{-1}$ when we consider the latter as an element of $k_\infty$.

We define the Legendre-Jacobi symbol $\left(\frac{f}{g}\right)_n$ in $R$ in the usual way. It is easy to express this symbol in terms of the resultant (for example as in [**14**, §§28–29]) as follows:

$$\left(\frac{f}{g}\right)_n = \chi\left(b_0^{-m} R(g, f)\right)$$

where $b_0$ is the leading coefficient of $g$ and $m$ is the degree of $f$. This is [**14**, §29, Eqn. (2)] in the context of a finite field. Let now $a_0$ be the leading coefficient of $f$ and $n$ the degree of $g$. Then one has the symmetry property of the resultant:

$$R(f, g) = (-1)^{mn} R(g, f).$$

From these two equations one obtains the reciprocity law for the Legendre-Jacobi symbol:

$$\left(\frac{f}{g}\right)_n = \chi\left(b_0^{-m} a_0^n (-1)^{mn}\right) \left(\frac{g}{f}\right)_n .$$

We shall write

$$(f, g)_\infty = \chi\left(b_0^{-m} a_0^n (-1)^{mn}\right)$$

so that the reciprocity law takes on the form

$$\left(\frac{f}{g}\right)_n = (f, g)_\infty \left(\frac{g}{f}\right)_n .$$

These results go back to the thesis of F. K. Schmidt (see [**12**, §3.2.1 ] and [**13**]). They are often attributed to A. Weil (see [**2**, pp. 333–336]). Bass also attributes to Weil a lemma (*loc.cit.* Lemma 8.2) which allows one to extend the proof from rational to general function fields, but this is also due to F. K. Schmidt[1]. One should note that the very simple form of the reciprocity law means that one can compute the Legendre symbol by a straightforward Euclidean algorithm.

[1]The results of F. K. Schmidt appeared in his thesis with a summary in [**13**]. This is in a particularly obscure series and I have not been able to find it. There is a facsimile of the thesis in the library in Göttingen, presumably identical to the one referred to by Roquette in [**12**, *loc.cit.*].

It is also worth noting here that if $f$ is a polynomial as above with leading term $a_0$ as above then the discriminant $D(f)$ of $f$ is given by [**14**, §29, Eqn. (2)]

$$D(f) = a_0^{-1} R(f, f').$$

Next we shall construct an additive character on $k_\infty$. To do this we shall first let $e_o$ be the additive character on $\mathbb{F}_p$ given by $e_o(j \pmod p) = \exp(2\pi i j/p)$. We use this to define an additive character $e_\star$ on $\mathbb{F}_q$ by $e_\star(x) = e_o(\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x))$. Now let $\omega$ be a global differential on the rational curve. Let now $e_\omega(f) = e_\star(\mathrm{Res}_\infty(\omega f))$ for $f$ in $k_\infty$ where $\mathrm{Res}_\infty$ denotes the residue of the differential at infinity. If $v \neq \infty$ is a place of $k$ then it corresponds to an irreducible polynomial $\phi \in R$. We shall let $\mathrm{Res}_v(\omega)$ be the sum of the residues of the differential $\omega$ over the zeroes of $\phi$. This then lies in $\mathbb{F}_q$ as the sum is over a set of conjugates. Abel's theorem on the residues affirms that $\mathrm{Res}_\infty(\omega) + \sum_{v \text{ finite}} \mathrm{Res}_v(\omega) = 0$ for any global differential $\omega$. It follows that if $\omega$ is regular outside $\infty$ then $e_\omega | R = 1$. If $\omega = \mathrm{d}x/x^2$ then one has that $\{y \in k : e_\omega | yR = 1\} = R$. For this reason we shall denote by $e$ the additive character on $k_\infty$ defined by $\omega = \mathrm{d}x/x^2$.

Next let $\varepsilon : \mu_n \to \mathbb{C}^\times$ be a homomorphism. For most of our purposes we shall assume that $\varepsilon$ is injective – but not always. Then we define, for $r, c \in R$, the Gauss sum

$$g(r, \varepsilon, c) = \sum_{y \pmod c} \varepsilon\left(\left(\frac{y}{c}\right)_n\right) e\left(\frac{ry}{c}\right)$$

where we sum over those $y$ that are coprime to $c$. Our main interest will be in the behaviour of $g(r, \varepsilon, c)$ as a function of $c$. To this end we shall introduce a family of functions. For $a, b \in k_\infty^\times$ we write $a \sim b$ if $a/b \in k_\infty^{\times n}$. Then we define for $\eta \in k_\infty^\times$

$$\psi(r, \varepsilon, \eta, s) = (1 - q^{n-ns})^{-1} \sum_{c \in R, c \sim \eta} g(r, \varepsilon, c) q^{-\deg(c)s}.$$

Since for $u \in \mathbb{F}_q^\times$ one has $\psi(r, \varepsilon, u\eta, s) = \varepsilon((u, \eta)_\infty)\psi(r, \varepsilon, \eta, s)$ it suffices to consider $\eta$ of the form $\pi_\infty^i$. Note that for $u$ as above one also has $\psi(ur, \varepsilon, \eta, s) = \varepsilon\big((u, \eta)_\infty^{-1}\big)\psi(r, \varepsilon, \eta, s)$ so that it involves no loss of generality to consider only monic $r$.

In the notation of [**9**] we have, if $n$ is odd,

$$\frac{q-1}{n} Z_{m'}(t, i) = \psi(m', \mathbf{j}^{-1}, \pi_\infty^{-i}, s_1)$$

and, if $n$ is even,

$$\frac{q-1}{n} Z_{m'}(t, i) = \psi(m', \mathbf{j}^{-1}, \pi_\infty^{-i}, s_1) + \psi\big(m', \mathbf{j}^{-1}, \pi_\infty^{-i+n/2}, s_1\big)$$

where $t = q^{-s_1}$ and the $s$ of [**9**] is $s_1/2$. The numerical factor $\frac{q-1}{n}$ arises as Hoffstein considers monic polynomials $c$ whereas we consider those whose leading coefficient is an $n$th power.

The general theory, [**10**, §II.3], also recalled in [**9**], shows that $\psi(r, \varepsilon, \pi_\infty^{-i}, s)$ is a rational function in $q^{-s}$ with denominator $1 - q^{n+1-ns}$. The definition shows that if we choose $i$ to satisfy $0 \le i < n$ then the numerator of $\psi(r, \varepsilon, \pi_\infty^{-i}, s)$ is of the form $q^{-is}$ times a polynomial in $q^{-ns}$. This fact means that we can separate the two components of Hoffstein's $Z_{m'}(t, i)$ in the case that $n$ is even; this allows us to

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take over his results without any further arguments. In particular we can compute the degree of the numerator. What we find on using [**9**, Proposition 2.1] is that $\psi(r, \varepsilon, \pi_\infty^{-i}, s) = q^{-is}(1 - q^{n+1-ns})^{-1}\Psi(r, \varepsilon, i, q^{-ns})$ where the degree of the polynomial $\Psi(r, \varepsilon, i, T)$ in $T$ is at most $[(1 + \deg(r) - i)/n]$. This follows from the proposition in question by considering the behaviour on both sides as $t \to \infty$; we shall consider the functional equation in detail in Section 3. Note that the bound will be negative when $\deg(r) \leq i - 2$; in this case $\Psi(r, \varepsilon, i, T) = 0$.

We let

$$C(r, \varepsilon, i) = \sum_{\substack{\deg(c)=i \\ c \text{ monic}}} g(r, \varepsilon, c). \tag{1}$$

Then for $i$ with $0 \leq i < n$ we have

$$(1 - q^{n-ns})\psi(r, \varepsilon, \pi_\infty^{-i}, s) = \frac{q-1}{n} \sum_{\substack{i' \geq i \\ i' \equiv i \pmod{n}}} C(r, \varepsilon, i')q^{-i's}.$$

This means that

$$\Psi(r, \varepsilon, i, T) = \frac{1 - q^{n+1}T}{1 - q^n T}\frac{q-1}{n}\sum_{j \geq 0} C(r, \varepsilon, i + nj)T^j. \tag{2}$$

We have thus reduced the calculation of the $\Psi(r, \varepsilon, i, T)$, and thereby of the $\psi(r, \varepsilon, \pi_\infty^{-i}, s)$, to that of the $C(r, \varepsilon, i)$ with $i \leq \deg(r) + 1$. For the $j \geq [(1 + \deg(r) - i)/n]$ and $0 \leq i < N$ we have recurrence relation $C(r, \varepsilon, i + (j + 1)n) = q^{n+1}C(r, \varepsilon, i + jn)$.

The quantity which will interest us most is

$$\begin{aligned} \rho(r, \varepsilon, i) &= \Psi(r, \varepsilon, i, q^{-n-1}) \\ &= \lim_{s \to 1 + \frac{1}{n}} (1 - q^{n+1-ns})q^{is}\psi(r, \varepsilon, \pi_\infty^{-i}, s) \end{aligned} \tag{3}$$

which is also the Fourier coefficient of a generalized theta series. It follows from the discussion above that

$$\rho(r, \varepsilon, i) = \frac{q-1}{n}.\frac{C(r, \varepsilon, i')}{q^{\frac{n+1}{n}(i'-i)}}$$

where $i' \equiv i \pmod{n}$ and $i' > \deg(r)$. In view of (1) this is an explicit formula for $\rho(r, \varepsilon, i)$. We note here that $\rho(r, \varepsilon, i + nN) = q^{(n+1)N}\rho(r, \varepsilon, i)$ for $N \in \mathbb{Z}$.

In [**9**] the $C(r, \varepsilon, i)$ are denoted by $S_i(r)$ (generally with $m'$ in place of $r$). Also, corresponding to $\rho(r, \varepsilon, i)$ Hoffstein uses $\frac{q-1}{n}r_{m'}(i)$ with $m' = r$.

We shall discuss the evaluation of the $C(r, \varepsilon, i)$ and the consequences for the understanding of the $\rho(r, \varepsilon, i)$, and also of the $\psi(r, \varepsilon, \pi_\infty^{-i}, s)$, or, what is effectively the same, the $\Psi(r, \varepsilon, i, T)$.

**2. The Davenport-Hasse theorem.** The version of the Davenport–Hasse theorem which we need here is the following. Let $\chi$, $\varepsilon$, $e_\star$ be as as above. Let $m > 1$ be an integer.

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Let $q' = q^m$. Let $\tau(\varepsilon)$ be the Gauss sum

$$\tau(\varepsilon) = \sum_{y \in \mathbb{F}_q^\times} \varepsilon(\chi(y)) e_\star(y).$$

Then (see [**4**, §3,I])

$$\sum_{y \in \mathbb{F}_{q'}^\times} \varepsilon\big(\chi\left(\mathrm{Norm}_{\mathbb{F}_{q'}/\mathbb{F}_q}(y)\right)\big) e_\star\big(\mathrm{Tr}_{\mathbb{F}_{q'}/\mathbb{F}_q}(y)\big) = -(-\tau(\chi))^m.$$

Let $\mu$ denote the Möbius function in $R$. Then we shall use the Davenport–Hasse theorem to prove the following theorem.

THEOREM 2.1. *Let $r, c \in R$ be coprime. Then we have*

$$g(r, \varepsilon, c) = \mu(c) \varepsilon \left( \left( \frac{-r}{c} \right)_n^{-1} \left( \frac{c'}{c} \right)_n \right) (-\tau(\varepsilon))^{\deg(c)}$$

*where, as usual, $c'$ denotes the derivative of $c$ with respect to $x$.*

*Proof.* First of all we recall that if $c$ is not square–free and if $r$ is coprime to $c$ then $g(r, \varepsilon, c) = 0$. Thus in this case both sides of the formula are zero. Next, under the assumption that $r$ and $c$ are coprime it is also elementary that $g(r, \varepsilon, c) = \varepsilon((\frac{r}{c})_n^{-1}) g(1, \varepsilon, c)$ and so we can assume that $r = 1$. It is a further elementary property of Gauss sums that if $c_1, c_2 \in R$ are coprime then

$$g(r, \varepsilon, c_1 c_2) = \varepsilon \left( \left( \frac{c_1}{c_2} \right)_n \left( \frac{c_2}{c_1} \right)_n \right) g(r, \varepsilon, c_1) g(r, \varepsilon, c_2).$$

One verifies also easily from the chain rule that if $c_1$ and $c_2$ are coprime in $R$ then

$$\left( \frac{(c_1 c_2)'}{c_1 c_2} \right)_n = \left( \frac{c_1}{c_2} \right)_n \left( \frac{c_2}{c_1} \right)_n \left( \frac{c_1'}{c_1} \right)_n \left( \frac{c_2'}{c_2} \right)_n.$$

It follows that we are thus reduced to proving the theorem in the case where $c$ is irreducible and $r = 1$. In this case we have that $g(r, \varepsilon, c)$ is equal to

$$\sum_{y \pmod{c}} \varepsilon \left( \left( \frac{y}{c} \right)_n \right) e_\star \left( \mathrm{Res}_\infty \left( \frac{y \, \mathrm{d}x}{c} \right) \right)$$

which we rewrite as

$$\sum_{y \pmod{c}} \varepsilon \left( \left( \frac{y}{c} \right)_n \right) e_\star \left( \mathrm{Res}_\infty \left( \frac{y \, \mathrm{d}c}{c' c} \right) \right).$$

As before we can extract the $c'$ as an external factor $(\frac{c'}{c} n)$ since $c'$ is coprime to $c$ (as $c$ is irreducible). Let $m = \deg(c)$. Then we can identify $R/(c)$ with $\mathbb{F}_{q^m}$. It remains to show that $\mathrm{Res}_\infty(\frac{y \, \mathrm{d}c}{c}) = -\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(y)$ and $(\frac{y}{c})_n = \mathrm{Norm}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(y)$. The first of these follows from first using Abel's theorem to convert the residue to the sum of the residues over the zeros of $c$ (over a suitable extension field) and we obtain precisely the result claimed.

For the second we recall that $\mathrm{Norm}_{\mathbb{F}_{q'}/\mathbb{F}_q}(y) = y^{1+q+q^2+\cdots+q^{m-1}} = y^{\frac{q^m-1}{q-1}}$. When we apply $\chi$ to this we obtain

$$\chi\left(\mathrm{Norm}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(y)\right) = y^{\frac{q^m-1}{q-1}\frac{q-1}{n}} = \left(\frac{y}{c}\right)_n$$

in $R/(c)$. The result in the case where $c$ is irreducible follows now directly from the Davenport–Hasse theorem. This completes the proof of the theorem. □

One consequence of this theorem is that for $c$ coprime to $r$ one has $g(r, \varepsilon, c)^n = \mu(c)^n((-\tau(\varepsilon))^n)^{\deg(c)}$. The second factor on the right–hand side is a Größencharakter and the formula is simply the Eisenstein–Weil theorem in this context.

We shall use this to give a more useful expression for the $C(r, \varepsilon, i)$. We let

$$C^*(r, \varepsilon, i) = \sum_{\substack{\deg(c)=i \\ c \text{ monic}, gcd(r,c)=1}} g(r, \varepsilon, c). \tag{4}$$

If we write $c = r^* c_1$ in (1) where $r^*$ is monic and divides some power of $r$ and $c_1$ is coprime to $r$ then we obtain

$$C(r, \varepsilon, i) = \sum_{r^*} g(r, \varepsilon, r^*)\varepsilon(\chi(-1))^{(i-1)\deg(r^*)} C^*\left(rr^{*(n-2)}, \varepsilon, i - \deg(r^*)\right). \tag{5}$$

The set of $r^*$ is finite and easily described. On the other hand our theorem shows that

$$C^*\left(rr^{*(n-2)}, \varepsilon, i\right) = (-\tau(\varepsilon))^i \sum_{\substack{c_1 \text{ monic} \\ \deg(c_1)=i,\ \gcd(c_1,r)=1}} \mu(c_1)\varepsilon\left(\left(\frac{r}{c_1}\right)_n^{-1}\left(\frac{r^*}{c_1}\right)_n^2\left(\frac{c_1'}{c_1}\right)_n\right). \tag{6}$$

This method of calculation has the advantage that, it reduces the number of Gauss sums that have to be computed to a finite set depending only on $r$; as one of our questions concerns the behaviour of $\rho(r_o\pi^j, \varepsilon, i)$ as a function of $j$ for field $r_o$ (not divisible by $\pi$). For most purposes $\pi$ will be irreducible. It is also to be noted that the degree of cancellation expected is not excessive since one knows that at least for large $i$ the function $C^*(r, \varepsilon, i)$ grows like $q^{i(1+\frac{1}{n})}$. The inner sum grows then like $q^{i(\frac{1}{2}+\frac{1}{n})}$ and has of the order $q^i$ terms.

J. Hoffstein's method for computing $C(r, \varepsilon, i)$ was to start from

$$C(r, \varepsilon, i) = \sum_{\substack{\deg(c)=i \\ c \text{ monic}}} g(r, \varepsilon, c).$$

and to expand it to

$$C(r, \varepsilon, i) = \sum_{\substack{\deg(c)=i \\ c \text{ monic}}} \sum_{y \pmod{c}} \varepsilon\left(\left(\frac{y}{c}\right)_n\right) e(y/c).$$

One can take as a set of residues $\pmod{c}$ the set of $y$ of degree $< i$ which is independent of $c$. Thus the sum on the right-hand side can be regarded as a double sum where the outer sum is over $j < i$ and the inner sum is over $c, y$ where the $\deg(y) = j$. For $i$ small this can be used effectively but we shall not make any use of it here. Generally it has the disadvantage that there is a much larger amount of cancellation which takes place here; the number of terms is of the order $q^{2i}$ but the sum is of the order $q^{i(1+\frac{1}{n})}$. Hoffstein uses it only when the degrees are small.

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**3. Hecke theory and the functional equation.** Let $S$ be a set of prime elements of $R$ and let $\psi_S(r, \varepsilon, \eta, s)$ be defined in the same way as $\psi(r, \varepsilon, \eta, s)$ but with the restriction that the $c$ are coprime to all elements of $S$. One can verify that if $\pi$ is a prime not is $S$ and if $r$ is coprime to $\pi$ then

$$\psi_S(r\pi^j, \varepsilon, \eta, s) = \frac{1-\mathrm{N}(\pi)^{n-1-ns}-(1-\mathrm{N}(\pi)^{-1})\mathrm{N}(\pi)^{(j+1)(n-ns)}}{1-\mathrm{N}(\pi)^{n-ns}}\psi_{S\cup\{\pi\}}(r\pi^j, \varepsilon, \eta, s)$$
$$+ g(-r, \varepsilon^{j+1}, \pi)\varepsilon(\eta, \pi^{j+1})_\infty \mathrm{N}(\pi)^{j-(j+1)s}\psi_{S\cup\{\pi\}}(r\pi^{-j-2}, \varepsilon, \eta\pi^{-j-1}, s)$$

where for any $c \in R$, $c \neq 0$ we write $\mathrm{N}(c) = q^{\deg(c)}$. Also $J = [j/n]$ and the exponent $-j-2$ appearing in the arguments of the last function is to be understood $\pmod n$. This formula can be considered as a sort of "Hecke theory" for these functions. Note that the function $\psi_{S\cup\{\pi\}}(r\pi^j, \varepsilon, \eta, s)$ depends on $j$ only $\pmod n$ whereas $\psi_S(r\pi^j, \varepsilon, \eta, s)$ has a more complicated behaviour in $j$ – described by the formula. If $j < n-1$ then the formula simplifies to

$$\psi_S(r\pi^j, \varepsilon, \eta, s) = \psi_{S\cup\{\pi\}}(r\pi^j, \varepsilon, \eta, s)$$
$$+ g(-r, \varepsilon^{j+1}, \pi)\varepsilon(\eta, \pi^{j+1})_\infty \mathrm{N}(\pi)^{j-(j+1)s}\psi_{S\cup\{\pi\}}(r\pi^{-j-2}, \varepsilon, \eta\pi^{-j-1}, s)$$

and for $j = n-1$ one has

$$\psi_S(r\pi^j, \varepsilon, \eta, s) = \psi_{S\cup\{\pi\}}(r\pi^j, \varepsilon, \eta, s)(1-\mathrm{N}(\pi)^{n-1-ns}).$$

These equations can be inverted to show that

$$\psi_{S\cup\{\pi\}}(r\pi^j, \varepsilon, \eta, s)(1-\mathrm{N}(\pi)^{n-1-ns})$$

is equal to

$$\psi_S(r\pi^j, \varepsilon, \eta, s) - g(-r, \varepsilon^{j+1}, \pi)\varepsilon(\eta, \pi^{-(j+1)})_\infty \mathrm{N}(\pi)^{j-(j+1)s}\psi_S(r\pi^{n-j-2}, \varepsilon, \eta\pi^{-j-1}, s)$$

for $0 \leq j \leq n-2$; if $j = n-1$ the appropriate formula has already be given above. This formula can be extended to a set of primes $\{\pi_1, \pi_2, \ldots, \pi_t\}$. Let, for $1 \leq i \leq t$, $j_i$ be such that $1 \leq j_i \leq n-2$. Let $r$ be coprime to all the $\pi_j$, $(1 \leq j \leq t)$ Then we find that

$$\psi_{S\cup\{\pi_1,\pi_2,\ldots,\pi_t\}}\left(r\pi_1^{j_1}\pi_2^{j_2}\cdots\pi_t^{j_t}, \varepsilon, \eta, s\right)\prod_{1\leq i\leq t}(1-\mathrm{N}(\pi_t)^{n-1-ns})$$

is equal to a sum over all the subsets of $\{1, 2, \ldots, t\}$ where the term corresponding to $T \subset \{1, 2, \ldots, t\}$ is

$$(-1)^{\mathrm{Card}(T)}\prod_{i\in T}\left(g(-r, \varepsilon^{j_i+1}, \pi_i)\prod_{\substack{\ell\in T\\ \ell\neq i}}\left(\frac{\pi_\ell}{\pi_i}\right)_n^{j_i+1}\varepsilon\left(\eta, \pi_i^{-j_i-1}\right)_\infty \mathrm{N}(\pi_i)^{j_i-(j_i+1)s}\right)$$
$$\psi_S\left(r\prod_{\ell\notin T}\pi_\ell^{j_\ell}\prod_{i\in T}\pi_i^{n-j_i-2}, \varepsilon, \eta\prod_{i\in T}\pi_i^{-j_i-1}, s\right).$$

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The main application of this which we shall need is the observation that if $r$ is a unit and $0 \le i < n$ then

$$\psi_{\{\pi_1,\pi_2,\dots,\pi_t\}}\big(r\pi_1^{j_1}\pi_2^{j_2}\cdots\pi_t^{j_t},\varepsilon,\pi_\infty^{-i},s\big)\prod_{1\le i\le t}(1-\mathrm{N}(\pi_t)^{n-1-ns})$$

is of the form $q^{-is}\Psi_{\{\pi_1,\pi_2,\dots,\pi_t\}}(r\pi_1^{j_1}\pi_2^{j_2}\cdots\pi_t^{j_t},\varepsilon,i,q^{-ns})/(1-q^{n+1-ns})$ where $\Psi_{\{\pi_1,\pi_2,\dots,\pi_t\}}\big(r\pi_1^{j_1}\pi_2^{j_2}\cdots\pi_t^{j_t},\varepsilon,i,X\big)$ is a polynomial of degree $\le \sum_{1\le j\le t}\deg(\pi_j)$. This is not an optimal estimate but it is nevertheless useful.

Next, the Periodicity Theorem [**10**, p. 134] implies that for $\pi \notin S$

$$\lim_{s\to 1+\frac{1}{n}} q^{is}(1-q^{n+1-ns})\psi_{S\cup\{\pi\}}(r,\varepsilon,\pi_\infty^{-i},s)(1+\mathrm{N}(\pi)^{-1})$$
$$=\lim_{s\to 1+\frac{1}{n}} q^{is}(1-q^{n+1-ns})\psi_S(r,\varepsilon,\pi_\infty^{-i},s).$$

From this one concludes that

$$\rho\big(r\pi_1^{j_1}\pi_2^{j_2}\cdots\pi_t^{j_t},\varepsilon,i\big)=\prod_{1\le i\le t}(1-\mathrm{N}(\pi_i)^{-1})^{-1}\Psi_{\{\pi_1,\pi_2,\dots,\pi_t\}}\big(r\pi_1^{j_1}\pi_2^{j_2}\cdots\pi_t^{j_t},\varepsilon,\pi_\infty^{-i},q^{-n-1}\big).$$

On the other hand we have

$$\Psi_{\{\pi_1,\pi_2,\dots,\pi_t\}}\big(r\pi_1^{j_1}\pi_2^{j_2}\cdots\pi_t^{j_t},\varepsilon,\pi_\infty^{-i},q^{ns}\big)(1-q^{n+1-ns})$$

is equal to

$$\prod_{1\le j\le t}(1-\mathrm{N}(\pi_j)^{n-1-ns})\sum_{\ell\ge 0}C^*\big(\pi_1^{j_1}\pi_2^{j_2}\cdots\pi_t^{j_t},\varepsilon,i+n\ell\big)q^{-n\ell s}.$$

We shall write

$$\sum_{\ell\ge 0}C^{**}\big(\pi_1^{j_1}\pi_2^{j_2}\cdots\pi_t^{j_t},\varepsilon,i+n\ell\big)X^\ell$$

for

$$\prod_{1\le j\le t}(1-q^{\deg(\pi_i)(n-1)}X^{\deg(\pi_i)})\sum_{\ell\ge 0}C^*\big(\pi_1^{j_1}\pi_2^{j_2}\cdots\pi_t^{j_t},\varepsilon,i+n\ell\big)X^\ell$$

and observe that the coefficients $C^{**}(r^*,\varepsilon,i)$ can be computed by a difference scheme from the $C^{**}(r^*,\varepsilon,i)$. Finally we remark that if

$$(1-q^{n+1}X)\sum_\ell d_\ell X^\ell$$

is a polynomial of degree $N$ in $X$, say $\sum_{0\le\ell\le N}D_\ell X^\ell$ then $D_\ell = d_\ell - q^{n+1}d_{\ell-1}$ if $\ell>0$ and $D_0=d_0$. This means that $\sum_{0\le\ell\le N}D_\ell q^{-(n+1)\ell}$ telescopes to $d_N q^{-N(n+1)}$ and consequently that

$$\rho\big(r\pi_1^{j_1}\pi_2^{j_2}\cdots\pi_t^{j_t},\varepsilon,i\big)=\prod_{1\le i\le t}(1-\mathrm{N}(\pi_i)^{-1})^{-1}C^{**}\big(r\pi_1^{j_1}\pi_2^{j_2}\cdots\pi_t^{j_t},\varepsilon,i+Mn\big)q^{-M(n+1)}$$

for $M>\sum_{1\le j\le t}\deg(\pi_j)$.

The functional equation takes on the following form (this is [**9**, Proposition 2.1] with the necessary modifications) for $i$ with $0 \leq i < n$ and $r$ monic:

$$\begin{aligned} q^{is}\psi\left(r, \varepsilon, \pi_\infty^{-i}, s\right) = {} & q^{n(s-1)E} q^{(2-s)i}\psi(r, \varepsilon, \pi_\infty^{-i}, 2-s)\tfrac{1-q^{-1}}{1-q^{ns-n-1}} \\ & + \varepsilon\chi(-1)^{i(1+\deg(r))}\tau(\varepsilon^{2i-\deg(r)-1})q^{n(2-s)([(1+\deg(r)-i)/n]-2)}q^{2n-\deg(r)+2i-2} \\ & \times q^{(2-s)(1+\deg(r)-i)_n}\psi(r, \varepsilon, \pi_\infty^{i-1-\deg(r)}, 2-s)\tfrac{1-q^{n-ns}}{1-q^{ns-n-1}} \end{aligned}$$

where

$$E = 2 + \left[\frac{2i-2-\deg(r)}{n}\right] = 1 - \left[\frac{\deg(r)+1-2i}{n}\right]$$

and $(x)_n = x - n[x/n]$.

The restriction that $r$ be monic represents no loss of generality as $\psi(ur, \varepsilon, i, s) = \chi(u)^{-i}\psi(r, \varepsilon, i, s)$ for a unit $u$.

We can also express the functional equations as relations between the appropriate $\Psi(r, \varepsilon, i, T)$ and $\Psi(r, \varepsilon, i', q^{-2}T^{-1})$. This becomes a set of relations between five $C(r, \varepsilon, i, j)$; we shall not make any use of them here.

There are some important relations between the $\rho(r, \varepsilon, i)$. The first, the Periodicity Theorem, (see [**10**, p. 134]) asserts that this function depends only on $r$ modulo $n$th powers. One can deduce easily from this and the Hecke relations above that for $0 \leq j < n-1$ one has

$$\begin{aligned} \rho(r_o\pi^j, \varepsilon, i) = {} & \mathrm{N}(\pi)^{-\frac{j+1}{n}} g(-r_o, \varepsilon^{j+1}, \pi)\varepsilon\chi(-1)^{i(j+1)\deg(\pi)} \\ & \times \rho(r_o\pi^{n-2-j}, \varepsilon, i-(j+1)\deg(\pi)) \end{aligned} \tag{7}$$

and

$$\rho(r_o\pi^{n-1}, \varepsilon, i) = 0 \tag{8}$$

where $\pi$ is irreducible, monic and coprime to $r_o$. There is a form of this relation for the "infinite prime"; this has been given in [**9**, Proposition 2.2]. It asserts

$$\rho(r, \varepsilon, i) = \tau\left(\varepsilon^{2i-1-\deg(r)}\right)\varepsilon\chi(-1)^{i(1+\deg(r))}\rho(r, \varepsilon, (1+\deg(r)-i)_n)q^{2i+E-3-\deg(r)} \tag{9}$$

and

$$\rho(r, \varepsilon, i) = 0 \tag{10}$$

if $2i \equiv 1 + \deg(r) \pmod{n}$. Note that in [**9**] the last assertion is not asserted but it follows because the Gauss sum is $-1$.

In the case $n$ odd then the number of undetermined $\rho(r_o\pi^j, \varepsilon, i)$ for fixed $r_o$ drops from $n^2$ to $((n-1)/2)^2$. The situation when $n$ is even is rather more difficult to analyze and depends more sensitively on the context, especially on the nature of $n$ and $\deg(r_o)$ but nevertheless the number of independent coefficients is reduced by something like a factor of 4.

**4. Some examples.** We have already seen that the degree of $\Psi(r, \varepsilon, i, T)$ is bounded by $[(\deg(r)+1-i)/n]$. Thus if $\deg(r) < i-1$ then $\Psi(r, \varepsilon, i, T) = 0$. This applies only

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to $r$ with $\deg(r) \le n-2$ as $0 \le i < n$. Let $c_o = \frac{q-1}{n}$. If now $0 \le \deg(r) + 1 - i < n$, i.e. $i - 1 \le \deg(r) < i - 1 + n$ then $\Psi(r, \varepsilon, i, T) = C(r, \varepsilon, i)$. In these cases $\rho(r, \varepsilon, i) = c_o C(r, \varepsilon, i)$. When we use this in combination with the relations between different $\rho(r, \varepsilon, i)$ described in the previous section then we can compute, for small $n$, the $\rho(r, \varepsilon, i)$ for a useful number of $r$ (or $\deg(r)$) and $i$.

The cases $n = 2$ and $n = 3$ are not that relevant to us here because the relations (7) and (8) suffice to determine the $\rho(r, \varepsilon, i)$ completely once we have treated the cases with $\deg(r) = 0$. If $n = 2$ we have only the cases $i = 0, 1$ and the considerations above show that $\rho(1, \varepsilon, 0) = c_o$. The relation (9) shows that $\rho(1, \varepsilon, 1) = c_o \tau(\varepsilon) q$. If $n = 3$ then again $\rho(1, \varepsilon, 0) = c_o$ and $\rho(1, \varepsilon, 1) = c_o \tau(\varepsilon) q$. Finally (10) shows that $\rho(1, \varepsilon, 2) = 0$.

If $n = 4$ then we have again $\rho(1, \varepsilon, 0) = c_o$ and $\rho(1, \varepsilon, 1) = c_o \tau(\varepsilon) q$. Further $\rho(1, \varepsilon, i) = 0$ for $i = 2, 3$. If $\deg(r) = 1$ and $r$ is monic then we have $\rho(r, \varepsilon, 0) = c_o$, $\rho(r, \varepsilon, 2) = c_o \tau(\varepsilon^2) q^2$ and $\rho(r, \varepsilon, i) = 0$ for $i = 1, 3$. If $\deg(r) = 2$ then we have $\rho(r, \varepsilon, i) = c_o C(r, \varepsilon, i)$ for all $i$. This case is interesting because the relations (9) apply. These show that in this case

$$C(r, \varepsilon, 2) = \tau(\varepsilon) q C(r, \varepsilon, 1) \varepsilon \chi(-1)$$

and, as $C(r, \varepsilon, 0) = 1$,

$$C(r, \varepsilon, 3) = \tau(\varepsilon^3) q^3.$$

These equations are not self-evident. When $\deg(r) = 3$ one has by (10) that $\rho(r, \varepsilon, i) = 0$ for $i = 0, 3$ and $\rho(r, \varepsilon, i) = c_o C(r, \varepsilon, i)$ for $i = 1, 3$. Here one has

$$C(r, \varepsilon, 3) = \tau(\varepsilon^2) q^2 C(r, \varepsilon, 1).$$

If $\deg(r) = 4$ and $i = 0, 1$ then the degree of $\Psi(r, \varepsilon, i, T)$ is at most 1 and so this polynomial depends on $C(r, \varepsilon, i)$ and $C(r, \varepsilon, i+4)$ for these values of $i$. On the other hand for $i = 3, 4$ we have $\rho(r, \varepsilon, i) = c_o C(r, \varepsilon, i)$ and

$$C(r, \varepsilon, 3) = \tau(\varepsilon) q^2 C(r, \varepsilon, 2).$$

For larger values of $\deg(r)$ we could give expressions but they are more complicated and we shall not discuss them here.

For larger values of $n$ the same methods allow us to compute $\rho(r, \varepsilon, i)$ for all $i$ and $\deg(r) \le n - 1$. The case $\deg(r) = n - 1$ and $i = 0$ is special since we have to use (10) from which we obtain $\rho(r, \varepsilon, 0) = 0$. For $\deg(r) < n - 1$ one has $\rho(r, \varepsilon, 0) = c_o$. For other values of $\deg(r)$ and $i$ we have $\rho(r, \varepsilon, i) = 0$ if $\deg(r) < i - 1$ and $\rho(r, \varepsilon, i) = c_o C(r, \varepsilon, i)$ otherwise. Again the equations (9) lead to non-trivial relations between the $C(r, \varepsilon, i)$.

We next turn to the evaluation of the $C(r, \varepsilon, i)$ when $\deg(r) \le n - 1$. This we do using (5) and (6). In the former the set of $r^*$ satisfies $\deg(r^*) \le i < n$ and so no $r^*$ has a factor which is a non-trivial $n$th power. It follows from this that there is only one $r^*$ and it is such that $r/r^*$ is $r^o$, the product over the irreducible polynomials which divide $r$. If $\deg(r^o) > i$ then $C(r, \varepsilon, i) = 0$, otherwise $C(r, \varepsilon, i) = g(r, \varepsilon, r/r^o) C^*(r, \varepsilon, i - \deg(r/r^o))$. The factor $C^*(r, \varepsilon, i')$ has been evaluated by (6). There are some special cases which one can deal with more completely. We have $C^*(r,\varepsilon,0)=1$. Next

$$C^*(r,\varepsilon,1)=(-\tau(\varepsilon))\sum_{\substack{a\in\mathbb{F}_q\\ r(a)\neq 0}}\varepsilon(\chi(r(a)))^{-1}$$

which is a character sum of classical type. If $r$ is of degree 0 then this sum is $(q-1)\varepsilon(\chi(r(0)))^{-1}$. If $r$ is of degree 1 then it is 0. If $r$ is if degree 2 then it is essentially a Jacobi sum; the cases where $r$ is irreducible and where it is reducible are a little different – see, for example, [**9**, Remark following Prop. 3.4].

For $r$ of degree greater than 2 the nature of the sum is less clear and one can verify that the prime decomposition is no longer made up of divisors of $q$. This corresponds to the fact that the analogue of the $\sum_{\substack{a\in\mathbb{F}_q\\ r(a)\neq 0}}\varepsilon(\chi(r(a)))^{-1}$ in complex analysis is a beta integral if $\deg(r)=2$ but a hypergeometric function if $\deg(r)=3$.

In (6) we can split the sum over $c_1$ into two sums. We consider semi-simple commutative algebras $A$ over $\mathbb{F}_q$ of rank $i$. For each $c_1$ the algebra $\mathbb{F}_q[x]/(c_1)$ is such an $A$. We can then regard the sum in (6) as an outer sum over all such $A$ and then an inner sum over elements $a\in A$ for which the discriminant does not vanish. These sums are in general not easy to analyze. It is worth noting that they are essentially known for $\deg(r)\leq 2$ and, when $\deg(r)=2$, $r$ is reducible. The sums are then analogues of Selberg's integral and the evaluation was effected by Anderson and Evans, see [**8**]. For the convenience of the reader we shall quote the more general version given in [**15**] in language closer to that of this paper. Suppose that $q$ is odd. Let $\varepsilon_1,\varepsilon_2,\varepsilon_3$ be homomorphisms $\mathbb{F}_q^\times\to\mathbb{C}^\times$ with $\varepsilon_3$ non-trivial. Let $d$ be the order of $\varepsilon_3$. Let $\phi$ be the unique character of $\mathbb{F}_q^\times$ of order 2. We shall assume that $\varepsilon_1$ and $\varepsilon_2$ are powers of $\varepsilon_3$. We define $j,k$ by $\varepsilon_1\varepsilon_3^j=1$, $\varepsilon_2\varepsilon_3^k=1$, and $0\leq j<d$, $0\leq k<d$. Let $i$ be an integer with $i>j$ and $i>k$. Let

$$C(\varepsilon_1,\varepsilon_2,\varepsilon_3,i)=\sum_c\varepsilon_1((-1)^ic(0))\varepsilon_2(c(1))\varepsilon_3(D(c))\phi(D(c))$$

where the sum is over all polynomials of degree $i$ without multiple factors. Let

$$D(\varepsilon_1,\varepsilon_2,\varepsilon_3,i)=\prod_{0\leq\ell<i}\frac{\tau\left(\varepsilon_1\varepsilon_3^\ell\right)\tau\left(\varepsilon_2\varepsilon_3^\ell\right)\tau\left(\varepsilon_3\varepsilon_3^\ell\right)\bar{\tau}\left(\varepsilon_1\varepsilon_1\varepsilon_3^{n-1+\ell}\right)}{q\tau(\varepsilon_3)}$$

Then $C(\varepsilon_1,\varepsilon_2,\varepsilon_3,i)/D(\varepsilon_1,\varepsilon_2,\varepsilon_3,i)$ is a rational integer which is given by

$$q^{[i/d]}\left(-i+\sum_{0\leq s\leq 2[i/d]}(2s+1)q^{2[i/d]-s}\right)$$

if $(n)_d\leq j$ and $(j+k+1-n)_d>k$ , by

$$q^{[i/d]+1}\left(-i+\sum_{0\leq s\leq 2[i/d]+1}(2s+1)q^{2[i/d]+1-s}\right)$$

if $(n)_d>k$ and $(j+k+1-n)_d\leq j$ and by

$$1+(q-1)i$$

otherwise.

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The proof of this beautiful formula is based on an idea of G.W.Anderson [**1**]. It is based on studying the $L$-functions associated with characters of the form $\varepsilon((\frac{V}{*})_{q-1})$ and interpreting the Jacobi-Legendre symbol, as above, in terms of the resultant. J. Denef and F. Loeser [**6**] gave a variant of the proof and were led in [**7**] to a more general related result.

The formulæ (2),(3), (5) and (6) show that

$$\rho(r, \varepsilon, i) \in \tau(\varepsilon^i) \times \frac{q-1}{n} q^{-(n+1)[(1+\deg(r)-i)/n]}\mathbb{Z}[1^{1/n}]$$

where, as before $\rho(r, \varepsilon, i) = 0$ if $1 + \deg(r) < i$. In simple cases the element of $\mathbb{Z}[1^{1/n}]$ is a product of Jacobi sums but this will not be so in general. The cases which we have discussed above show that the prime decomposition can be somewhat unpredictable. If, for example, the polynomial $r$ is of degree 3 then the sum defining $C$ is an analogue of the hypergeometric function and so one would not expect that its behaviour as a function of $r$ will be simple.

There is every reason to expect that much the same holds in characteristic 0, at least up to a transcendental factor – see [**9**, Conjecture 4.10]. In this context "transcendental" means that it is a special value of a function built out of standard transcendental functions, not that it is transcendental in the stricter use of this word. Nevertheless it probably is transcendental in this sense, but not provably so. Also there is no obvious "denominator" corresponding to the factor $q^{-(n+1)[(1+\deg(r)-i)/n]}$.

When $n = 3$ and $k = \mathbb{Q}(\sqrt{-3})$ one has a closed formula [**11**, Theorem 9.1] in which the transcendental factor is given explicitly.

When $n = 4$ this conjecture is compatible with the evidence of [**5**]. Here no interpretation of the transcendental factor was proffered ([**5**, p. 240]).

When $n = 6$ G.Wellhausen has computed $\rho(r, \varepsilon, \eta)$ for 44 values of $r$. He was able to give an interpretation of the transcendental factor. He found in the range he considered a number of 'sporadic' factors – and these seem to be the analogues of those we have seen above. Consequently it seems to be the case that one has to give up the idea that there is a "closed formula" for the $\rho(r, \varepsilon, \eta)$. One has to consider them as a a new class of arithmetic functions (in $r$). They have certain characteristics which are analogous to Gauss sums. If we regard Gauss sums as, essentially, $n$th roots of Größenchakaktere (by the Eisenstein-Weil theorem) then the $\rho(r, \varepsilon, \eta)$ are a further class of the same type. The relationship between them will essentially be given by the "inner products", that is, the asymptotic behaviour of $r \mapsto g(r', \varepsilon, r)\rho(rr'', \varepsilon, \eta)$. This leads us to the investigations on "Weyl multiple Dirichlet series" (see, for example, [**3**] and its sequels). This theory is far from complete but it offers considerable promise in understanding the arithmetic role of the coefficients of generalised theta series on general linear groups as considered in [**10**, §II.3].

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