HOMOGENEOUS OPERATORS AND ESSENTIAL COMPLEXES

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1. Introduction. The aim of this work is to present a new approach to the concept of essential Fredholm complex of Banach spaces ([**10**], [**2**]; see also [**11**], [**4**], [**6**], [**7**] etc. for further connections), by using non-linear homogeneous mappings. We obtain some generalized homotopic properties of the class of essential Fredholm complexes, in our sense, which are then applied to establish its relationship with similar concepts. We also prove the stability of this class under small perturbations with respect to the gap topology.

Throughout this paper we shall work with linear spaces over the field $\mathbb{K}$, which is either the real field $\mathbb{R}$ or the complex one $\mathbb{C}$. If X, Y are Banach spaces over $\mathbb{K}$, we denote by $\mathscr{L}(X, Y)$ the space of all linear and continuous operators from X into Y. The subspace of $\mathscr{L}(X, Y)$ consisting of all compact operators will be designated by $\mathscr{K}(X, Y)$. Let $\mathscr{H}(X, Y)$ be the space of all $\mathbb{K}$-homogeneous and continuous operators from X to Y, so that $T(\lambda x) = \lambda T(x)$ but possibly $T(x + y) \neq T(x) + T(y)$. Endowed with the usual operations and with the norm defined as in the case of linear operators, the space $\mathscr{H}(X, Y)$ becomes a Banach space which contains $\mathscr{L}(X, Y)$ as a closed subspace. Moreover, if $\mathscr{K}\mathscr{H}(X, Y)$ is the subspace of all compact operators from $\mathscr{H}(X, Y)$ (the definition of a compact $\mathbb{K}$-homogeneous operator is the same as for a linear operator), then $\mathscr{K}\mathscr{H}(X, Y)$ is a closed subspace of $\mathscr{H}(X, Y)$ and it obviously contains $\mathscr{K}(X, Y)$ (see [**12**], [**13**] or [**15**] for details).

Let $\mathrm{Ban}_{\mathbb{K}}$ be the category whose objects are Banach spaces over $\mathbb{K}$ and whose morphisms are bounded $\mathbb{K}$-linear operators. In the present work a *complex* in the category $\mathrm{Ban}_{\mathbb{K}}$ is a sequence $A = (A_p)_{p\geq 0}$, where $A_p \in \mathscr{L}(X_p, X_{p-1})$, $A_pA_{p+1} = 0$ and X_p is a Banach space over $\mathbb{K}$ for every integer $p \geq 0$, with $X_{-1} = \{0\}$, and therefore $A_0 = 0$. In addition, we assume that there exists an integer $n \geq 0$ (depending upon A) such that $X_p = \{0\}$ for all $p \geq n+1$ (and therefore $A_p = 0$ if $p \geq n+1$). In other words, we work only with complexes of *finite length*. A complex $A = (A_p)_{p\geq 0}$ can be represented in the usual way, that is, as a sequence of the form

$$0 \longrightarrow X_n \xrightarrow{A_n} X_{n-1} \xrightarrow{A_{n-1}} \dots \xrightarrow{A_1} X_0 \longrightarrow 0.$$

Nevertheless, since the space X_p is determined by the operator A_p (and it will be called in the sequel the *domain of definition* of A_p), we prefer the concentrated notation $A = (A_p)_{p\geq 0}$.

For a given complex $A = (A_p)_{p\geq 0}$, we shall denote by $H_p(A)$ the quotient $N(A_p)/R(A_{p+1})$, where $N(A_p)$ is the null-space of A_p and $R(A_{p+1})$ is the range of A_{p+1} (as a rule, we shall use the notation from [**9**]), that is, the *homology* of the complex A. When $H_p(A) = \{0\}$ for all integers $p \geq 0$, the complex A is said to be *exact*. More generally, if $\dim_{\mathbb{K}} H_p(A) < \infty$ for all $p \geq 0$, then the complex A is said to be *Fredholm*. In this case, one can define the *index* of A by the formula

$$\operatorname{ind}_{\mathbb{K}}(A)=\sum_{p\geqslant 0}(-1)^p \dim_{\mathbb{K}} H_p(A),$$

which has some stability properties (see **[4]**, **[14]**, **[1]**, **[2]** etc.).

An *essential complex* (**[10]**, **[2]**) in the category $\mathrm{Ban}_{\mathbb{K}}$ is a sequence $A=(A_p)_{p\geqslant 0}$, where $A_p\in\mathscr{L}(X_p,X_{p-1})$, $A_pA_{p+1}\in\mathscr{K}(X_{p+1},X_{p-1})$ and X_p is a Banach space over $\mathbb{K}$ for each integer $p\geqslant 0$, with $X_{-1}=\{0\}$. In addition, there exists an integer $n\geqslant 0$ (depending upon A) such that $X_p=0$ for all $p\geqslant n+1$. Obviously, every complex is an essential complex.

A standard procedure to study essential complexes is to transform them into ordinary ones by means of certain functors. We shall describe in the following three functors of this type.

(1) *The functor* λ_Z. Let Z be a fixed Banach space. For every Banach space X, we denote by $\lambda_Z(X)$ the quotient $\mathscr{L}(Z,X)/\mathscr{K}(Z,X)$. Note that every linear operator $S\in\mathscr{L}(X,Y)$ induces, by left multiplication, a linear and continuous operator $\lambda_Z(S)$ from $\lambda_Z(X)$ into $\lambda_Z(Y)$. It is easily seen that the assignment λ_Z defines a covariant functor from the category $\mathrm{Ban}_{\mathbb{K}}$ into itself. In addition, one has $\lambda_Z(S)=0$ for every Banach space Z iff $S\in\mathscr{K}(X,Y)$. This shows that if $A=(A_p)_{p\geqslant 0}$ is an essential complex, then $\lambda_Z(A)=(\lambda_Z(A_p))_{p\geqslant 0}$ is a complex.

The idea of studying classes of linear operators that are equivalent modulo compact operators goes back to Calkin. In connection with essential complexes, the functor λ_Z has been used in **[10]**.

(2) *The functor* χ_Z. Let Z be again a fixed Banach space. For every Banach space X we denote by $\chi_Z(X)$ the quotient $\mathscr{K}(Z,X)/\mathscr{K}\mathscr{K}(Z,X)$. If $S\in\mathscr{L}(X,Y)$, then the left multiplication by S induces a linear and continuous operator from $\chi_Z(X)$ into $\chi_Z(Y)$. It is straightforward to see that χ_Z defines a covariant functor in the category $\mathrm{Ban}_{\mathbb{K}}$. Moreover, we have $\chi_Z(S)=0$ for all Banach spaces Z iff S is compact. Therefore the functor χ_Z also maps the class of essential complexes into the class of ordinary ones. This functor is seemingly new.

(3) *The functor* κ. For every Banach space X let $l^\infty(X)$ (resp. $\tau(X)$) be the Banach space of all bounded (resp. totally bounded) sequences consisting of elements of X, endowed with the natural linear structure and topology. Then we consider the quotient $\kappa(X)=l^\infty(X)/\tau(X)$. Every operator $S\in\mathscr{L}(X,Y)$ induces a linear and continuous operator $\kappa(S)$ from $\kappa(X)$ into $\kappa(Y)$ by its action on coordinates. We obtain again a covariant functor in the category $\mathrm{Ban}_{\mathbb{K}}$, such that $\kappa(S)=0$ iff S is compact.

The functor κ, which had been known for some time (see, for instance, **[3]**), was used to study essential complexes in **[6]**, **[7]**, **[2]** etc.

The above functors are needed to define various concepts of essential Fredholm complexes.

Definition 1.1. Let $A=(A_p)_{p\geqslant 0}$ be an essential complex in the category $\mathrm{Ban}_{\mathbb{K}}$. We say that A is

(1) *λ-Fredholm* if the complex $\lambda_Z(A) = (\lambda_Z(A_p))_{p\geq 0}$ is exact for every Banach space Z;

(2) *χ-Fredholm* if the complex $\chi_Z(A) = (\chi_Z(A_p))_{p\geq 0}$ is exact for every Banach space Z;

(3) *κ-Fredholm* if the complex $\kappa(A) = (\kappa(A_p))_{p\geq 0}$ is exact.

Let us mention that an essential λ-Fredholm complex is called in **[10]** simply *Fredholm*, and an essential κ-Fredholm complex is called in **[2]** *essentially Fredholm.*

The purpose of this paper is to introduce the class of essential χ-Fredholm complexes, to establish its relationship with the other classes of essential Fredholm complexes via some generalized homotopic properties, and to prove its stability under small perturbations with respect to the gap topology. Unlike the class of essential λ-Fredholm complexes, the class of essential χ-Fredholm complexes contains the family of all Fredholm complexes (see Theorem 2.7). Nevertheless, some of the properties of essential λ-Fredholm complexes can be restated in the context of essential χ-Fredholm complexes, as we shall see in the next section.

2. Generalized homotopic properties. Let $A = (A_p)_{p\geq 0}$ and $B = (B_p)_{p\geq 0}$ be two essential complexes in the category $\mathrm{Ban}_{\mathbb{K}}$. Let also X_p (resp. Y_p) be the domain of definition of A_p (resp. B_p) for every $p \geq 0$.

An *essential morphism* of A into B is a family of linear operators $F = (F_p)_{p\geq 0}$ such that $F_p \in \mathscr{L}(X_p, Y_p)$ and $B_{p+1}F_{p+1} - F_pA_{p+1} \in \mathscr{K}(X_{p+1}, Y_p)$ for all $p \geq 0$.

The families $1 = (1_p)_{p\geq 0}$ and $0 = (0_p)_{p\geq 0}$, where 1_p is the identity on X_p and 0_p is the zero map on X_p, are obviously essential morphisms of the essential complex $A = (A_p)_{p\geq 0}$ into itself.

REMARK 2.1. The essential morphism $F = (F_p)_{p\geq 0}$ of $A = (A_p)_{p\geq 0}$ into $B = (B_p)_{p\geq 0}$ induces a linear map from $H_p(\chi_Z(A))$ into $H_p(\chi_Z(B))$ for each $p \geq 0$ and every Banach space Z. Indeed, if $\sigma \in N(\chi_Z(A_p))$ and $\sigma_0 \in \mathscr{H}(Z, X_p)$ is in the equivalence class σ, then $A_p\sigma_0 \in \mathscr{K}\mathscr{H}(Z, X_{p-1})$. Therefore $B_pF_p\sigma_0 \in \mathscr{K}\mathscr{H}(Z, Y_{p-1})$, so that the coset of $F_p\sigma_0$ is in the null-space of $\chi_Z(B_p)$. If, in addition, $\sigma \in R(\chi_Z(A_{p+1}))$, a similar argument shows that the coset of $F_p\sigma_0$ is in $R(\chi_Z(B_{p+1}))$.

DEFINITION 2.2. Let $A = (A_p)_{p\geq 0}$ and $B = (B_p)_{p\geq 0}$ be two essential complexes and let X_p (resp. Y_p) be the domain of definition of A_p (resp. B_p). Let also $F = (F_p)_{p\geq 0}$ and $G = (G_p)_{p\geq 0}$ be two essential morphisms of A into B. We say that F and G are *χ-homotopic* if there exists a family of homogeneous operators $\theta = (\theta_p)_{p\geq 0}$, where $\theta_p \in \mathscr{H}(X_p, Y_{p+1})$, such that

$$F_p - G_p - B_{p+1}\theta_p - \theta_{p-1}A_p \in \mathscr{K}\mathscr{H}(X_p, Y_p), \qquad p \geq 0, \tag{2.1}$$

with $\theta_{-1} = 0$.

Note that if $\theta' = (\theta'_p)_{p\geq 0}$ and $\theta'' = (\theta''_p)_{p\geq 0}$ satisfy (2.1), then for every $t \in [0, 1]$ the family $\theta(t) = (t\theta'_p + (1-t)\theta''_p)_{p\geq 0}$ also satisfies (2.1).

If in Definition 2.2 we can choose the operators $(\theta^p)_{p\geq 0}$ to be $\mathbb{K}$-linear, then we say that F and G are *λ-homotopic.*

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PROPOSITION 2.3. *Let A, B, F, G be as in Definition 2.2. If F and G are χ-homotopic, then they induce (by Remark 2.1) the same map from $H_p(\chi_Z(A))$ into $H_p(\chi_Z(B))$ for each $p \geq 0$ and every Banach space Z.*

Proof. Since F and G are χ-homotopic iff $F-G=(F_p-G_p)_{p\geq 0}$ and zero are χ-homotopic, we may assume with no loss of generality that F and $G=0$ are χ-homotopic. Then we shall show that the map induced by F from $H_p(\chi_Z(A))$ into $H_p(\chi_Z(B))$ is equal to zero. Indeed, if $\sigma \in N(\chi_Z(A_p))$ and $\sigma_0 \in \sigma$ (as in Remark 2.1), then, from (2.1), we infer that $F_p\sigma_0 - B_{p+1}\theta_p\sigma_0 \in \mathcal{K}\mathcal{H}(Z, Y_p)$ (since $\theta_{p-1}A_p\sigma_0$ is compact). This shows that the coset of $F_p\sigma_0$ equals the coset of $B_{p+1}\theta_p\sigma_0$, that is, it is in $R(\chi_Z(B_{p+1}))$. Hence the action induced by F from $H_p(\chi_Z(A))$ into $H_p(\chi_Z(B))$ must be null.

If $A=(A_p)_{p\geq 0}$ is an essential χ-Fredholm complex and X_p is the domain of definition of A_p, then there are linear operators $B_p \in \mathcal{L}(X_p, X_{p+1})$ and $C_p \in \mathcal{K}(X_p, X_p)$ such that $A_{p+1}B_p + B_{p-1}A_p = 1_p - C_p$ for all $p \geq 0$. The converse is also true (see [**10**, Proposition 2.3]). In other words, an essential complex is λ-Fredholm iff the identity and the zero morphism on it are λ-homotopic. A similar result holds for essential χ-Fredholm complexes.

THEOREM 2.4. *An essential complex $A=(A_p)_{p\geq 0}$ is χ-Fredholm if and only if the identity on A is χ-homotopic to zero.*

Proof. If the identity of A is χ-homotopic to zero, it follows from Proposition 2.3 that the identity on $H_p(\chi_Z(A))$ is zero for each integer $p \geq 0$ and every Banach space Z. This forces the spaces $H_p(\chi_Z(A))$ to be zero, that is, the essential complex A is χ-Fredholm.

Conversely, we shall use an inductive argument inspired by the proof of [**10**, Proposition 2.3]. We assume that the complex $\chi_Z(A)$ is exact for every Banach space Z. We shall show that there are operators $\theta_p \in \mathcal{K}(X_p, X_{p+1})$ and $v_p \in \mathcal{K}\mathcal{H}(X_p, X_p)$ such that

$$A_{p+1}\theta_p + \theta_{p-1}A_p = 1_p - v_p, \qquad p \geq 0, \tag{2.2}$$

with $\theta_{-1}=0$, where X_p is the domain of definition of A_p. We find the operators θ_p and v_p by induction with respect to p.

If $p=0$, then the exactness of the complex $\chi_Z(A)$ for $Z=X_0$ shows that we can choose an operator $\theta_0 \in \mathcal{H}(X_0, X_1)$ such that $A_1\theta_0 - 1_0 \in \mathcal{K}\mathcal{H}(X_0, X_0)$. Hence we may define $v_0 = 1_0 - A_1\theta_0$.

Assume now that we have found the mappings θ_q and v_q for all $q \leq p$. Note that

$$\begin{aligned} A_{p+1}(1_{p+1} - \theta_p A_{p+1}) &= A_{p+1} - (1_p - v_p - \theta_{p-1}A_p)A_{p+1} \\ &= v_p A_{p+1} + \theta_{p-1}A_pA_{p+1} \in \mathcal{K}\mathcal{H}(X_{p+1}, X_p), \end{aligned}$$

by (2.2). From the exactness of the complex $\chi_Z(A)$ with $Z=X_{p+1}$, we deduce the existence of an operator $\theta_{p+1} \in \mathcal{H}(X_{p+1}, X_{p+2})$ with $A_{p+2}\theta_{p+1} = 1_{p+1} - \theta_p A_{p+1} - v_{p+1}$, where $v_{p+1} \in \mathcal{K}\mathcal{H}(X_{p+1}, X_{p+1})$.

This shows that equations (2.2) have solutions for all $p \geq 0$, that is, the identity on A and zero are χ-homotopic.

COROLLARY 2.5. *Let $A = (A_p)_{p\geq 0}$ be an essential complex and let X_p be the domain of definition of A_p. Then A is χ-Fredholm if and only if the complex $\chi_Z(A)$ is exact for each Z in the family $(X_p)_{p\geq 0}$.*

Proof. It follows from the proof of Theorem 2.4 that equations (2.2) have solutions when $\chi_Z(A)$ is exact only for those Z in the family $(X_p)_{p\geq 0}$.

COROLLARY 2.6. *Let $A = (A_p)_{p\geq 0}$ be an essential complex.*
(1) *If A is λ-Fredholm, then A is χ-Fredholm.*
(2) *If A is χ-Fredholm, then A is κ-Fredholm.*

Proof. The assertion (1) follows from [**10**, Proposition 2.3] via Theorem 2.4.

Let us prove the assertion (2). Let us fix an integer $p \geq 0$, let X_p be the domain of definition of A_p and let $\{x_k\}_k$ be a sequence in $l^\infty(X_p)$ such that $\{A_p x_k\}_k \in \tau(X_{p-1})$. If we choose θ_p as in (2.2), we can see that $\{A_{p+1}y_k - x_k\}_k \in \tau(X_p)$, where $y_k = \theta_p(x_k)$, and hence $\{y_k\}_k \in l^\infty(X_{p+1})$. In other words, the complex $\kappa(A)$ is exact, that is, A is κ-Fredholm.

The class of essential χ-Fredholm complexes contains the class of Fredholm complexes, via Theorem 2.4. The next two results apply to complexes, not essential complexes.

THEOREM 2.7. *Let $A = (A_p)_{p\geq 0}$ be a complex in the category* $\mathrm{Ban}_{\mathbb{K}}$. *Then A is Fredholm iff the identity of A and zero are χ-homotopic.*

Proof. Assume first that A is a Fredholm complex, that is, $\dim_{\mathbb{K}} H_p(A) < \infty$ for every $p \geq 0$. In particular, the space $R(A_{p+1})$ is closed and of finite codimension in $N(A_p)$. Let $\pi_{1,p}$ be a linear projection of $N(A_p)$ onto $R(A_{p+1})$. Let also $\pi_{2,p}$ be a homogeneous projection of X_p onto $N(A_p)$ (see, for instance, [**15**, Corollary 1.2]). Then π_p is a homogeneous projection of X_p onto $R(A_{p+1})$, where $\pi_p = \pi_{1,p}\pi_{2,p}$. Let $C_p : X_p \to X_p/N(A_p)$ be the canonical mapping and let $\rho_p : X_p/N(A_p) \to X_p$ be the homogeneous lifting associated with $\pi_{2,p}$ [**15**], that is, $\rho_p C_p = 1_p - \pi_{2,p}$. We define a mapping $\theta_p \in \mathscr{H}(X_p, X_{p+1})$ in the following way. Let

$$B_{p+1} : X_{p+1}/N(A_{p+1}) \to R(A_{p+1})$$

be the bijective operator induced by A_{p+1}. Then $(B_{p+1})^{-1}$ is a bounded operator, by the closed graph theorem. Set $\theta_p = \rho_{p+1}(B_{p+1})^{-1}\pi_p$, which is a continuous and homogeneous mapping from X_p into X_{p+1} for all $p \geq 0$. We shall show that the mappings θ_p satisfy (2.2).

Indeed, let $x \in X_p$, where $p \geq 0$ is fixed. Note that

$$\theta_{p-1}(A_p x) = (\rho_p(B_p)^{-1}\pi_{p-1}A_p)(x) = (\rho_p(B_p)^{-1})(A_p x) = \rho_p(C_p x) = x - \pi_{2,p}(x).$$

On the other hand, since $\pi_p(x) \in R(A_{p+1})$, we can write $\pi_p(x) = A_{p+1}v$ for some $v \in X_{p+1}$. Hence

$$\begin{aligned} A_{p+1}\theta_p(x) &= (A_{p+1}\rho_{p+1}(B_{p+1})^{-1}\pi_p)(x) \\ &= (A_{p+1}\rho_{p+1}(B_{p+1})^{-1})(A_{p+1}v) = (A_{p+1}\rho_{p+1}C_{p+1})(v) \\ &= A_{p+1}(v - \pi_{2,p+1}(v)) = A_{p+1}v = \pi_p(x). \end{aligned}$$

Consequently,

$$A_{p+1}\theta_p(x) + \theta_{p-1}(A_px) = x - v_p(x), \; x \in X_p,$$

and the homogeneous mapping

$$v_p = \pi_{2,p} - \pi_p = (1_p \mid N(A_p) - \pi_{1,p})\pi_{2,p}$$

is compact, since the linear operator $1_p \mid N(A_p) - \pi_{1,p}$ is a finite rank projection. Therefore the identity on the essential complex A is χ-homotopic to zero.

Conversely, let A be a complex such that its identity and zero are χ-homotopic. Then we can find operators $\theta_p \in \mathscr{H}(X_p, X_{p+1})$ and $v_p \in \mathscr{K}\mathscr{H}(X_p, X_p)$ such that (2.2) is fulfilled. By using (2.2), we shall derive that the complex A is Fredholm.

We first prove that $R(A_p)$ is a closed subspace of X_{p-1} for every $p \geqslant 0$. If this were not true for some p, then there would exist a sequence $\{y_k\}_k \subset R(A_p)$, $y_k = A_p(x_k)$, such that $y_k \to 0$ $(k \to \infty)$ but $\operatorname{dist}(x_k, N(A_p)) = 1$ for all k (where "dist" stands for distance). The sequence $\{x_k\}_k$ can obviously be assumed bounded. From the equality

$$A_{p+1}\theta_p(x_k) + \theta_{p-1}(y_k) = x_k - v_p(x_k),$$

which follows from (2.2), we infer that the sequence $\{x_k - A_{p+1}\theta_p(x_k)\}_k$ contains a convergent subsequence. Therefore, with no loss of generality, we may assume that

$$x_k - A_{p+1}\theta_p(x_k) \to v \in X_p \qquad (k \to \infty).$$

Then

$$A_p(x_k - A_{p+1}\theta_p(x_k)) = A_px_k = y_k \to 0 \qquad (k \to \infty),$$

so that $v \in N(A_p)$. Hence

$$\operatorname{dist}(x_k, N(A_p)) \leqslant \|x_k - A_{p+1}\theta_p(x_k) - v\| \to 0 \qquad (k \to \infty),$$

which contradicts the choice of the sequence $\{y_k\}_k$. Therefore $R(A_p)$ must be closed for all $p \geqslant 0$.

We now prove that $\dim_{\mathbb{K}} H_p(A) < \infty$ for each $p \geqslant 0$. If this were not true for a certain p, then in the Banach space $N(A_p)/R(A_{p+1})$ there would exist a bounded sequence $\{\xi_k\}_k$ which wouldn't contain any convergent subsequence. Let $x_k \in \xi_k$ be chosen such that the sequence $\{x_k\}_k$ is bounded in $N(A_p)$. Then we have

$$A_{p+1}\theta_p(x_k) - \theta_{p-1}(A_px_k) = A_{p+1}\theta_p(x_k) = x_k - v_p(x_k),$$

as a consequence of (2.2). Since v_p is compact, the sequence $\{v_p(x_k)\}_k$ contains a convergent subsequence. Therefore

$$\xi_k = x_k + R(A_{p+1}) = v_p(x_k) + R(A_{p+1})$$

contains a convergent subsequence, which contradicts the choice of the sequence $\{\xi_k\}_k$. Consequently the complex A is Fredholm.

We note that the "sufficiency" in Theorem 2.7 can also be obtained from the results of **[6]** or **[7]**.

Corollary 2.8. *Let $A = (A_p)_{p\geq 0}$ be a complex in the category $Ban_{\mathbb{K}}$. The following assertions are equivalent*:

(1) *A is Fredholm*;
(2) *A is χ-Fredholm*;
(3) *A is κ-Fredholm*.

Proof. The equivalence (1)$\Leftrightarrow$(2) follows from Theorems 2.4 and 2.7 whereas the equivalence (1)$\Leftrightarrow$(3) follows from [**6**] or [**7**].

Remark 2.9. The class of Fredholm complexes is strictly larger than the class of those complexes that are λ-Fredholm. Indeed, let X_1 be a Banach space which contains a closed linear subspace X_2 that is not complemented in X_1. Let also $X_0 = X_1/X_2$. If $A_2 : X_2 \to X_1$ is the inclusion and $A_1 : X_1 \to X_0$ is the canonical mapping, then the complex $A = (0, A_2, A_1, 0)$ is Fredholm but not λ-Fredholm, in virtue of [**6**, Theorem 2.7]. Hence the complex $\lambda_Z(A)$ fails to be exact for at least one $Z \in \{X_0, X_1, X_2\}$ by [**10**, Proposition 2.3].

It is known that for every essential λ-Fredholm complex one can define an index that is stable under compact and small perturbations, and coincides with the natural one in the case of Fredholm complexes (see [**10**] for details). Therefore, a natural question arises in our context: is it possible to assign an index to every essential χ-Fredholm complex? Of course, such an index should be stable under small and compact perturbations and coincide with the usual one in the case of Fredholm complexes. This problem, whose answer is not known to the author of this text, is intimately connected with the Problem 5.1 from [**1**] as well as with the problem raised in [**2**, Remark 4.8(3)], which have received so far only partial answers (see, for instance, [**8**]).

Finally, let us remark that Theorem 2.7 leads to a non-linear version of a well-known characterization of Fredholm operators.

Corollary 2.10. *Let X_1, X_0 be Banach spaces. A linear operator $A_1 \in \mathscr{L}(X_1, X_0)$ is Fredholm if and only if there exists a homogeneous operator $\theta_1 \in \mathscr{H}(X_0, X_1)$ such that $A_1\theta_1 - 1_0 \in \mathscr{K}\mathscr{H}(X_0, X_0)$ and $\theta_1 A_1 - 1_1 \in \mathscr{K}\mathscr{H}(X_1, X_1)$.*

3. Stability under small perturbations. In this section we shall prove that the class of essential χ-Fredholm complexes is stable under perturbations that are small with respect to the gap topology (see [**6**], [**7**], [**10**], [**2**] etc. for similar results concerning the other types of essential Fredholm complexes). We shall rely heavily upon some perturbation results from [**2**].

Let $\mathscr{X}$ be a fixed Banach space over the field $\mathbb{K}$. We denote by $\mathscr{G}(\mathscr{X})$ the family of all closed linear subspaces of $\mathscr{X}$. The space $\mathscr{G}(\mathscr{X})$ is endowed with the gap topology, which is defined by means of the functions $\delta(X, Y) = \sup\{\mathrm{dist}(x, Y); x \in X, \|x\| = 1\}$ and $\hat{\delta}(X, Y) = \max\{\delta(X, Y), \delta(Y, X)\}$, where $X, Y \in \mathscr{G}(\mathscr{X})$ (see [**9**, IV.2.1]).

Lemma 3.1. *Let $X, Y \in \mathscr{G}(\mathscr{X})$ and let $\delta > \delta(X, Y)$. Then there exists an operator $\tau \in \mathscr{H}(X, Y)$ such that $\|x - \tau(x)\| \leq \delta \, \|x\|$ for each $x \in X$.*

Proof. Let $S(X)$ be the unit sphere of X and let $x \in S(X)$. Then there exists a vector $y_x \in Y$ such that $\|x - y_x\| < \delta$. Let V_x be an open neighbourhood of x in $S(X)$ such that $\|v - y_x\| < \delta$ for all $v \in V_x$. The family $\{V_x\}_{x \in S(X)}$ is an open cover of the metric (and therefore paracompact) space $S(X)$, so that we can choose a partition of unity subordinated to this cover. In other words, there exists a family of continuous and non-negative functions $\{f_\alpha\}_{\alpha \in A}$ on $S(X)$ such that $\mathrm{supp}(f_\alpha) \subset V_{x_\alpha}$ for every $\alpha \in A$, the family of closed sets $\{\mathrm{supp}(f_\alpha)\}_{\alpha \in A}$ is locally finite and $\sum_{\alpha \in A} f_\alpha(x) = 1$ for all $x \in S(X)$. We define a continuous Y-valued function by the formula

$$\tau_0(x) = \sum_{\alpha \in A} f_\alpha(x) y_{x_\alpha}, \qquad x \in S(X),$$

where y_{x_α} corresponds to the vector $x_\alpha \in V_{x_\alpha}$, as above. Obviously, $\|x - \tau_0(x)\| \leq \delta$ for all $x \in S(X)$. Since $\|y_{x_\alpha}\| \leq \delta + \|x_\alpha\| \leq \delta + 1$, the function τ_0 is bounded.

We shall obtain from τ_0 a homogeneous operator by using a procedure inspired from [**16**, Theorem 1.1]. We first extend the function τ_0 to the whole space X by setting $\tau_1(x) = \|x\|\, \tau_0(x/\|x\|)$ if $x \neq 0$ and $\tau_1(0) = 0$. Since τ_0 is bounded, it is easily seen that τ_1 is continuous on X. Moreover, τ_1 is positive-homogeneous. In addition, if $x \neq 0$,

$$\|x - \tau_1(x)\| = \|x\|\, \|x/\|x\| - \tau_0(x/\|x\|)\| \leq \delta\, \|x\|,$$

and the final estimate also holds for $x = 0$.

We shall discuss two cases. If $\mathbb{K} = \mathbb{R}$, then we define

$$\tau(x) = 2^{-1}(\tau_1(x) - \tau_1(-x)).$$

The function τ is $\mathbb{R}$-homogeneous, Y-valued and continuous on X. Moreover,

$$\|x - \tau(x)\| \leq 2^{-1}\, \|x - \tau_1(x)\| + 2^{-1}\, \|x + \tau_1(-x)\| \leq \delta\, \|x\|$$

for all $x \in X$.

If $\mathbb{K} = \mathbb{C}$, then we define

$$\tau(x) = (2\pi)^{-1} \int_0^{2\pi} e^{-it} \tau_1(e^{it}x)\, dt, \qquad x \in X.$$

It is easy to check that τ is continuous, Y-valued and $\mathbb{C}$-homogeneous. Furthermore,

$$\|x - \tau(x)\| = (2\pi)^{-1} \left\| \int_0^{2\pi} e^{-it}(e^{it}x - \tau_1(e^{it}x))\, dt \right\| \leq (2\pi)^{-1} \int_0^{2\pi} \|e^{it}x - \tau_1(e^{it}x)\|\, dt \leq \delta\, \|x\|,$$

and the proof of the lemma is complete.

LEMMA 3.2. *Let $X, Y \in \mathcal{G}(\mathcal{X})$. Then for every Banach space Z we have the estimates*

$$\delta(\mathcal{H}(Z, X), \mathcal{H}(Z, Y)) \leq \delta(X, Y),$$

$$\delta(\mathcal{KH}(Z, X), \mathcal{KH}(Z, Y)) \leq \delta(X, Y).$$

Proof. Let $\delta > \delta(X, Y)$. By the previous lemma, we can choose $\tau \in \mathcal{H}(X, Y)$ such that $\|x - \tau(x)\| \leq \delta\, \|x\|$ for all $x \in X$. Let $\sigma \in \mathcal{H}(Z, X)$ be arbitrary and let $\theta = \tau\sigma \in$

$\mathcal{H}(Z, Y)$. Then

$$\|\sigma - \theta\| = \sup_{\|x\| \le 1} \|\sigma(x) - \tau(\sigma(x))\| \le \delta \sup_{\|x\| \le 1} \|\sigma(x)\| \le \delta \, \|\sigma\|.$$

Therefore $\delta(\mathcal{H}(Z, X), \mathcal{H}(Z, Y)) \le \delta$. Since $\delta > \delta(X, Y)$ is arbitrary, we infer easily the first estimate from the statement.

To obtain the second estimate from the statement, we only note that if $\sigma \in \mathcal{KH}(Z, X)$, then $\theta = \tau\sigma \in \mathcal{KH}(Z, Y)$. A similar argument then leads to the desired conclusion.

LEMMA 3.3. *Let $X_1, Y_1, X_2, Y_2 \in \mathcal{G}(\mathcal{X})$, let $S_1 \in \mathcal{L}(X_1, Y_1)$ and let $S_2 \in \mathcal{L}(X_2, Y_2)$. Then*

$$\delta(X_1, X_2) \le (1 + \|S_1\|)\delta(S_1, S_2).$$

In addition, if $(1 + \|S_1\|)\delta(S_2, S_1) < 1$, then

$$\|S_2\| \le (1 - (1 + \|S_1\|)\delta(S_2, S_1))^{-1}((1 + \|S_1\|)\delta(S_2, S_1) + \|S_1\|).$$

This result is contained in [**2**, Lemma 4.6].

LEMMA 3.4. *Let $X_1, Y_1, X_2, Y_2 \in \mathcal{G}(\mathcal{X})$, and let $S_j \in \mathcal{L}(X_j, Y_j)$ $(j = 1,2)$. We define the operators $M_j : \mathcal{H}(Z, X_j) \to \mathcal{H}(Z, Y_j)$ by the equality $M_j\sigma = S_j \circ \sigma$ $(\sigma \in \mathcal{H}(Z, X_j))$, where Z is an arbitrary Banach space. Then we have the estimate $\delta(M_1, M_2) \le 2^{1/2}\delta(S_1, S_2)$.*

Proof. We have, by definition (see [**9**]),

$$\delta(S_1, S_2) = \delta(G(S_1), G(S_2)),$$

where $G(S_j) \in \mathcal{G}(\mathcal{X} \times \mathcal{X})$ is the graph of the operator S_j $(j = 1,2)$.

Let $\delta > \delta(S_1, S_2)$. By virtue of Lemma 3.1, there exists an operator $\tau \in \mathcal{H}(G(S_1), G(S_2))$ such that

$$\|(x, S_1x) - \tau(x, S_1x)\| \le \delta \, \|(x, S_1x)\|, \quad x \in X_1. \tag{3.1}$$

We fix an element $\sigma_1 \in \mathcal{H}(Z, X_1)$. Then the map

$$Z \ni z \to \tau(\sigma_1(z), S_1\sigma_1(z)) \in G(S_2)$$

is continuous and $\mathbb{K}$-homogeneous. Therefore it must be of the form

$$\tau(\sigma_1(z), S_1\sigma_1(z)) = (\sigma_2(z), S_2\sigma_2(z)), \qquad z \in Z,$$

where $\sigma_2 \in \mathcal{H}(Z, X_2)$. If we assume that the norm of $\mathcal{X} \times \mathcal{X}$ is given by $\|(x_1, x_2)\|^2 = \|x_1\|^2 + \|x_2\|^2$, from the estimate (3.1) we infer that

$$\begin{aligned}&\|\sigma_1(z) - \sigma_2(z)\|^2 + \|S_1\sigma_1(z) - S_2\sigma_2(z)\|^2 \\ &\qquad \le \delta^2(\|\sigma_1(z)\|^2 + \|S_1\sigma_1(z)\|^2) \le \delta^2(\|\sigma_1\|^2 + \|S_1\sigma_1\|^2) \, \|z\|^2,\end{aligned}$$

so that

$$\|\sigma_1 - \sigma_2\|^2 \le \delta^2(\|\sigma_1\|^2 + \|S_1\sigma_1\|^2),$$
$$\|S_1\sigma_1 - S_2\sigma_2\|^2 \le \delta^2(\|\sigma_1\|^2 + \|S_1\sigma_1\|^2).$$

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Hence

$$\|(\sigma_1, M_1\sigma_1) - (\sigma_2, M_2\sigma_2)\|^2 \leqslant 2\delta^2(\|\sigma_1\|^2 + \|M_1\sigma_1\|^2),$$

which leads to the desired estimate.

To state the next lemma, we need a special notation from [**2**]. Let X, X_0, Y, $Y_0 \in \mathcal{G}(\mathcal{X})$ be such that $X_0 \subset X$ and $Y_0 \subset Y$, and let $S \in \mathcal{L}(X/X_0, Y/Y_0)$. Then we set

$$G_0(S) = \{(x, y) \in \mathcal{X} \times \mathcal{X} : S(x + X_0) = y + Y_0\}.$$

If $\tilde{X}$, $\tilde{X}_0$, $\tilde{Y}$, $\tilde{Y}_0 \in \mathcal{G}(\mathcal{X})$ have similar properties and $\tilde{S} \in \mathcal{L}(\tilde{X}/\tilde{X}_0, \tilde{Y}/\tilde{Y}_0)$, then we define

$$\begin{aligned} \delta_0(S, \tilde{S}) &= \delta(G_0(S), G_0(\tilde{S})), \\ \hat{\delta}_0(S, \tilde{S}) &= \max\{\delta_0(S, \tilde{S}), \delta_0(\tilde{S}, S)\}, \end{aligned} \tag{3.2}$$

which are computed in $\mathcal{G}(\mathcal{X} \times \mathcal{X})$.

Lemma 3.5. *Let X_1, Y_1, X_2, $Y_2 \in \mathcal{G}(\mathcal{X})$ and let $S_j \in \mathcal{L}(X_j, Y_j)$ $(j = 1,2)$. Then for every Banach space Z we have the estimate*

$$\delta_0(\chi_Z(S_1), \chi_Z(S_2)) \leqslant 3(1 + \|S_1\|)\max\{\delta(Y_1, Y_2), \delta(S_1, S_2)\}.$$

Proof. Let $\delta_0 > \delta(Y_1, Y_2)$, let $\delta > \delta(S_1, S_2)$ and let $(\sigma_1, \theta_1) \in G_0(\chi_Z(S_1))$. Then $\theta_1 - S_1\sigma_1 \in \mathcal{KH}(Z, Y_1)$. By Lemma 3.2, we can choose $\nu_2 \in \mathcal{KH}(Z, Y_2)$ such that $\|\theta_1 - S_1\sigma_1 - \nu_2\| < \delta_0 \|\theta_1 - S_1\sigma_1\|$.

With the notation of Lemma 3.4, $(\sigma_1, S_1\sigma_1) \in G(M_1)$. Then, by this lemma, we can find an element $(\sigma_2, S_2\sigma_2) \in G(M_2)$ such that

$$\|\sigma_1 - \sigma_2\|^2 + \|S_1\sigma_1 - S_2\sigma_2\|^2 \leqslant 2\delta^2(\|\sigma_1\|^2 + \|S_1\sigma_1\|^2).$$

If we set $\theta_2 = S_2\sigma_2 + \nu_2 \in \mathcal{H}(Z, Y_2)$, then $(\sigma_2, \theta_2) \in G_0(\chi_Z(S_2))$. In addition,

$$\begin{aligned} \|\sigma_1 - \sigma_2\|^2 + \|\theta_1 - \theta_2\|^2 &\leqslant \|\sigma_1 - \sigma_2\|^2 + (\|\sigma_1 - S_1\sigma_1 - \nu_2\| + \|S_1\sigma_1 - S_2\sigma_2\|)^2 \\ &\leqslant 2\delta^2(\|\sigma_1\|^2 + \|S_1\sigma_1\|^2) + 2\delta_0^2 \|\theta_1 - S_1\sigma_1\|^2 + 4\delta^2(\|\sigma_1\|^2 + \|S_1\sigma_1\|^2) \\ &< 9(1 + \|S_1\|^2)\max\{\delta_0^2, \delta^2\}(\|\sigma_1\|^2 + \|\theta_1\|^2), \end{aligned}$$

whence we obtain the claimed estimate.

Let $C = (C_p)_{p \geqslant 0}$ be a complex such that the domain of definition of C_p is of the form X_p/Y_p, where X_p, $Y_p \in \mathcal{G}(\mathcal{X})$, $Y_p \subset X_p$ for all $p \geqslant 0$. Let also $\tilde{C} = (\tilde{C}_p)_{p \geqslant 0}$ be a complex of the same type, that is, $\tilde{C}_p$ is defined on $\tilde{X}_p/\tilde{Y}_p$, with $\tilde{X}_p$, $\tilde{Y}_p \in \mathcal{G}(\mathcal{X})$, $\tilde{Y}_p \subset \tilde{X}_p$. Then [**2**] one can define the quantity

$$\hat{\delta}_0(C, \tilde{C}) = \sup_{p \geqslant 0} \hat{\delta}_0(C_p, \tilde{C}_p), \tag{3.3}$$

with $\hat{\delta}_0(C_p, \tilde{C}_p)$ given by (3.2). We also set

$$\gamma(C) = \inf_{p \geqslant 0} \gamma(C_p), \tag{3.4}$$

where $\gamma(C_p)=\inf\{\|C_px\|;\operatorname{dist}(x,N(C_p))=1\}$ is the reduced minimum modulus of C_p ([**9**]). If C is Fredholm, in particular if C is exact, then obviously $\gamma(C)>0$.

PROPOSITION 3.6. *Let C and $\tilde{C}$ be as above. If C is exact and $3\hat{\delta}_0(C,\tilde{C})(1+\gamma(C)^{-2})^{1/2}<1$, then $\tilde{C}$ is also exact.*

Proof. The assertion is a consequence of [**2**, Corollary 2.12]. Indeed, let us define the numbers

$$r=\gamma(C)^{-1}\geqslant\max\{\gamma(C_p)^{-1},\gamma(C_{p+1})^{-1}\},$$
$$\delta_0=\hat{\delta}_0(C,\tilde{C})\geqslant\max\{\delta_0(\tilde{C}_p,C_p),\delta_0(C_{p+1},\tilde{C}_{p+1})\},$$
$$\begin{aligned}\delta&=(1+\gamma(C)^{-2})^{1/2}\hat{\delta}_0(C,\tilde{C})\\&\geqslant\max\{(1+\gamma(C_p)^{-2})^{1/2}\delta_0(\tilde{C}_p,C_p),(1+\gamma(C_{p+1})^{-2})^{1/2}\delta_0(C_{p+1},\tilde{C}_{p+1})\}.\end{aligned}$$

Then we have

$$\delta+\delta_0(1+\delta)(1+r^2)^{1/2}\leqslant\delta+2\delta_0(1+r^2)^{1/2}=3\hat{\delta}_0(C,\tilde{C})(1+\gamma(C)^{-2})^{1/2}<1,$$

which insures, by [**2**, Corollary 2.12], that $R(\tilde{C}_{p+1})=N(\tilde{C}_p)$ for all $p\geqslant0$, that is, the exactness of the complex $\tilde{C}$.

We can now establish the exactness of the class of χ-Fredholm complexes under small perturbations. We shall denote by $\partial_e(\mathscr{X})$ the family of those essential complexes $A=(A_p)_{p\geqslant0}$ such that $X_p\in\mathscr{G}(\mathscr{X})$ for all $p\geqslant0$, where X_p is the domain of definition of A_p.

If $A=(A_p)_{p\geqslant0}$, $B=(B_p)_{p\geqslant0}$ are members of $\partial_e(\mathscr{X})$, we set

$$\hat{\delta}(A,B)=\sup_{p\geqslant0}\hat{\delta}(A_p,B_p),\tag{3.5}$$

where $\hat{\delta}(A_p,B_p)=\max\{\delta(A_p,B_p),\delta(B_p,A_p)\}$ ([**9**]; see also Lemma 3.4).

Let also

$$\|A\|=\sup_{p\geqslant0}\|A_p\|,\qquad A=(A_p)_{p\geqslant0}\in\partial_e(\mathscr{X}).\tag{3.6}$$

THEOREM 3.7. *Let $A=(A_p)_{p\geqslant0}\in\partial_e(\mathscr{X})$ be χ-Fredholm. Then there exists a positive number δ_A such that if $B=(B_p)_{p\geqslant0}\in\partial_e(\mathscr{X})$ and $\hat{\delta}(A,B)<\delta_A$, then B is also χ-Fredholm.*

Proof. Let X_p be the domain of definition of A_p and let $(\theta_p)_{p\geqslant0}$ be a fixed family of homogeneous operators such that $\theta_p\in\mathscr{H}(X_p,X_{p+1})$ and $A_{p+1}\theta_p+\theta_{p-1}A_p-1_p\in\mathscr{K}\mathscr{H}(X_p,X_p)$ for all $p\geqslant0$ (which exists by Theorem 2.4). With no loss of generality we may assume that $\hat{\delta}(A,B)$ is small enough so that

$$(1+\|A\|)\hat{\delta}(A,B)\leqslant2^{-1}.\tag{3.7}$$

In particular, $(1+\|A_p\|)\delta(B_p,A_p)\leqslant2^{-1}$, and hence

$$\begin{aligned}\|B_p\|&\leqslant(1-(1+\|A_p\|)\delta(B_p,A_p))^{-1}((1+\|A_p\|)\delta(B_p,A_p)+\|A_p\|)\\&\leqslant2(2^{-1}+\|A_p\|)=1+2\,\|A_p\|,\end{aligned}$$

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by Lemma 3.3. Therefore

$$\|B\| = \sup_{p\geq 0} \|B_p\| \leq 1 + 2\,\|A\|. \tag{3.8}$$

Let Z be a fixed Banach space. With $(\theta_p)_{p\geq 0}$ as above (note that $\theta_p = 0$ for all but a finite family of indices), we have

$$\gamma(\chi_Z(A))^{-1} \leq r < \infty, \tag{3.9}$$

where $r = \max\{\|\theta_p\| : p \geq 0\}$. Indeed, let $\varepsilon > 0$ be given, let $\sigma \in N(\chi_Z(A_p))$ for some p and let $\sigma_0 \in \sigma$ be such that $\|\sigma_0\| \leq (1+\varepsilon)\,\|\sigma\|$. Then, from (2.2),

$$\chi_Z(A_{p+1})(\theta_p\sigma_0 + \mathscr{K}\mathscr{H}(Z, X_{p+1})) = \sigma.$$

Moreover,

$$\|\theta_p\sigma_0 + \mathscr{K}\mathscr{H}(Z, X_{p+1})\| \leq \|\theta_p\sigma_0\| \leq (1+\varepsilon)\,\|\theta_p\|\,\|\sigma\|.$$

Using the definition of $\gamma(\chi_Z(A_{p+1}))$, we deduce that

$$\gamma(\chi_Z(A_{p+1}))^{-1} \leq (1+\varepsilon)\,\|\theta_p\|.$$

Since $\varepsilon > 0$ is arbitrary, from the last estimate we derive easily (3.9).

Next we prove that

$$\hat{\delta}_0(\chi_Z(A), \chi_Z(B)) \leq 12(1+\|A\|)^2\hat{\delta}(A, B). \tag{3.10}$$

Indeed, by Lemma 3.3,

$$\begin{aligned}
\hat{\delta}(X_p, Y_p) &\leq (1 + \max\{\|A_p\|, \|B_p\|\})\hat{\delta}(A_p, B_p)\\
&\leq (1 + \max\{\|A\|, \|B\|\})\hat{\delta}(A, B)\\
&\leq 2(1+\|A\|)\hat{\delta}(A, B),
\end{aligned}$$

where we have used (3.8). Therefore, Lemma 3.5 and the above calculation show that

$$\begin{aligned}
\hat{\delta}_0(\chi_Z(A_p), \chi_Z(B_p)) &\leq 3(1 + \max\{\|A_p\|, \|B_p\|\}) \times \max\{\hat{\delta}(X_{p-1}, Y_{p-1}), \hat{\delta}(A_p, B_p)\}\\
&\leq 12(1+\|A\|)^2\hat{\delta}(A, B),
\end{aligned}$$

where Y_p is the domain of definition of B_p, which obviously leads to (3.10).

To obtain the final conclusion, we intend to apply Proposition 3.6. Note that

$$3\hat{\delta}_0(\chi_Z(A), \chi_Z(B))(1 + \gamma(\chi_Z(A))^{-2})^{1/2} \leq 36(1+\|A\|)^2(1+r^2)^{1/2}\hat{\delta}(A, B), \tag{3.11}$$

by (3.9) and (3.10). Hence, if

$$36(1+\|A\|)^2\left(1 + \max_p \|\theta_p\|^2\right)^{1/2}\hat{\delta}(A, B) < 1, \tag{3.12}$$

then, by Proposition 3.6, the complex $\chi_Z(B)$ is also exact. Since the coefficient of $\hat{\delta}(A, B)$ in (3.12) does not depend on Z, the fulfillment of (3.12) (provided (3.7) is also fulfilled), implies the exactness of $\chi_Z(B)$ for every Banach space Z, that is, the essential complex B is χ-Fredholm.

COROLLARY 3.8. *The family of χ-Fredholm complexes is open in $\partial_e(\mathscr{X})$.*

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