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RIEMANN SURFACES AS ORBIT SPACES OF FUCHSIAN GROUPS

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1. Introduction. A *Fuchsian group* is a discrete subgroup of the hyperbolic group, L.F.(2, R), of linear fractional transformations

$$w = \frac{az + b}{cz + d} \quad (a, b, c, d \text{ real}, ad - bc = 1),$$

each such transformation mapping the complex upper half plane D into itself. If Γ is a Fuchsian group, the orbit space D/Γ has an analytic structure such that the projection map $p: D \rightarrow D/\Gamma$, given by $p(z) = \Gamma z$, is holomorphic and D/Γ is then a Riemann surface.

If N is a normal subgroup of a Fuchsian group Γ, then N is a Fuchsian group and $S = D/N$ is a Riemann surface. The factor group, $G = \Gamma/N$, acts as a group of automorphisms (biholomorphic self-transformations) of S for, if $\gamma \in \Gamma$ and $z \in D$, then $\gamma N \in G$, $Nz \in S$, and $(\gamma N)(Nz) = N\gamma z$. This is easily seen to be independent of the choice of γ in its N-coset and the choice of z in its N-orbit.

Conversely, if S is a compact Riemann surface, of genus at least two, then S can be identified with D/K, where K is a Fuchsian group acting without fixed points in D. D is the universal covering space of S and K is isomorphic to the fundamental group of S. We shall call a Fuchsian group which acts without fixed points in D, a *Fuchsian surface group*. If S admits a group of automorphisms G, then there is a Fuchsian group $\tilde{G}$, with K as a normal subgroup, such that G is isomorphic to $\tilde{G}/K$ and the action of $\tilde{G}/K$ on S, as described above, coincides with that of G.

In this paper we consider the situation of a Fuchsian group Γ with a normal subgroup N such that the orbit space $S = D/N$ is a compact Riemann surface of genus at least two. The subgroup N may have fixed points in D but there are Fuchsian groups $\tilde{G}$ and K, where K has no fixed points, such that S can be identified with D/K and $\tilde{G}/K$ is isomorphic to Γ/N. Furthermore, after identification the groups Γ/N and $\tilde{G}/K$ have the same action on S. We investigate the relationship between the pairs of Fuchsian groups Γ, N and $\tilde{G}$, K and show how $\tilde{G}$ and K can be determined when Γ and N are known.

2. Preliminaries. In this section we list some of the results that we shall require to determine the groups $\tilde{G}$ and K. For these and other facts about Fuchsian groups, see [**3**; **4**].

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If a Fuchsian group Γ has a compact orbit space, then it is known to have the following presentation:

$$\text{generators: } x_1, x_2, \ldots, x_r, a_1, b_1, a_2, b_2, \ldots, a_\gamma, b_\gamma,$$

(1) $$\text{relations: } \quad x_1^{m_1} = x_2^{m_2} = \ldots = x_r^{m_r} = \prod_{i=1}^{r} x_i \prod_{j=1}^{\gamma} a_j b_j a_j^{-1} b_j^{-1} = 1.$$

The only elements of Γ which have fixed points in D are the elements of finite order, namely the powers of the x_i and their conjugates.

Any Fuchsian group Γ, having a compact orbit space, has a compact fundamental region in D. (By a fundamental region for Γ we mean a closed set F, such that the images of F, under Γ, cover D while the images of the interior F° of F are disjoint and the measure of $F \backslash F^\circ$ is zero.) The elements of L.F.(2, R) map the set of circles and lines, orthogonal to the real axis, onto itself and any element of this set is called a *non-Euclidean line.* A fundamental region for Γ can be found which is bounded by finitely many non-Euclidean line segments.

We shall also use the following result, due to Armstrong [**2**].

Let G be a group of simplicial transformations of a simply connected space X, with H the normal subgroup of G generated by elements having a non-empty fixed point set. The fundamental group of the orbit space X/G is then isomorphic to the factor group G/H; i.e.,

(2) $$\pi_1(X/G) \cong G/H.$$

3. Main results. Suppose N is a normal subgroup of a Fuchsian group Γ, where N has a compact orbit space, $S = D/N$, of genus g, where $g \geqq 2$. Then N has a presentation:

$$\text{generators: } y_1, y_2, \ldots, y_t, c_1, d_1, c_2, d_2, \ldots, c_g, d_g,$$

(3) $$\text{relations: } \quad y_1^{\mu_1} = y_2^{\mu_2} = \ldots = y_t^{\mu_t} = \prod_{i=1}^{t} y_i \prod_{j=1}^{g} c_j d_j c_j^{-1} d_j^{-1} = 1.$$

Since D/Γ is conformally equivalent to $(D/N)/(\Gamma/N)$, it follows that Γ has a compact orbit space and we take Γ to have the presentation given by (1). For each i, $1 \leqq i \leqq r$, let n_i be the smallest positive integer such that $x_i^{k_i} \in N$. Then $x_i^\alpha \in N$ if and only if $n_i | \alpha$ and, in particular, $n_i | m_i$.

Let Γ_1 be the group given by the presentation:

$$\text{generators: } \quad x_1', x_2', \ldots, x_r', a_1', b_1', a_2', b_2', \ldots, a_\gamma', b_\gamma',$$

$$\text{relations: } \quad (x_1')^{n_1} = (x_2')^{n_2} = \ldots = (x_r')^{n_r} = \prod_{i=1}^{r} x_i' \prod_{j=1}^{\gamma} a_j' b_j' a_j'^{-1} b_j'^{-1} = 1.$$

Define a homomorphism ψ: $\Gamma \to \Gamma_1$ by taking

$$\psi(x_i) = x_i' \qquad (i = 1, 2, \ldots, r),$$
$$\psi(a_j) = a_j', \psi(b_j) = b_j' \qquad (j = 1, 2, \ldots, \gamma).$$

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We take M to be the kernel of ψ, so that Γ_1 is isomorphic to the factor group Γ/M. Since M is generated by the $x_i^{n_i}$ and their conjugates, all of which lie in N, M is a normal subgroup of N and the factor group N/M is isomorphic to the subgroup N_1 of Γ_1, where $N_1 = \psi(N)$. By the second isomorphism theorem,

$$\Gamma/N \cong (\Gamma/M)/(N/M) \cong \Gamma_1/N_1$$

and we now show that, up to isomorphism, Γ_1 and N_1 are the groups $\tilde{G}$ and K, mentioned above.

LEMMA. *M is generated by the elements of finite order in N.*

Proof. If y has finite order, $y \in N \subset \Gamma$, then $y = tx_i^{\alpha}t^{-1}$. Since N is normal in Γ, $x_i^{\alpha} \in N$ so $n_i|\alpha$ and $y \in M$.

Remark. Since being of finite order and having a fixed point are equivalent in a Fuchsian group, we can rephrase the lemma so as to say that M is generated by those elements of N which have a fixed point in D.

If we write $c_j' = \psi(c_j)$ and $d_j' = \psi(d_j)$, $j = 1, 2, \ldots, g$, then we have the following presentation for N_1.

$$\text{(4)} \qquad \begin{aligned} &\text{generators: } c_1', d_1', c_2', d_2', \ldots, c_g', d_g', \\ &\text{relation: } \prod_{j=1}^{g} c_j'd_j'c_j'^{-1}d_j'^{-1} = 1, \end{aligned}$$

which follows from (3).

Writing D_1 for the orbit space D/M, D_1 is a Riemann surface and, identifying N_1 with N/M, N_1 is a group of automorphisms of D_1. We also identify Γ_1 with Γ/M.

THEOREM 1. *D_1 is a covering space of the orbit space D_1/N_1.*

Proof. Let $z_1 \in D_1$ and suppose $z_1 = Mz$ for $z \in D$. Since N is a Fuchsian group, there is an open neighbourhood V of z such that, if $V \cap yV \neq \emptyset$ for $y \in N$, then $y(z) = z$. Hence, if such a y exists, by the lemma, $y \in M$. Let $p_1 \colon D \to D_1$ be the projection mapping and let $V_1 = p_1(V)$, so that V_1 is an open neighbourhood of $p_1(z) = Mz = z_1$. If $y' \in N_1$ and $V_1 \cap y'V_1 \neq \emptyset$, then $MV \cap yMV \neq \emptyset$ for $y' = yM$. Then, for some $m_1, m_2 \in M$, $m_1V \cap ym_2V \neq \emptyset$ and hence $m_1^{-1}ym_2 \in M$ so that $y \in M$.

Hence, for $z_1 \in D_1$, there is an open neighbourhood V_1 of z_1 such that, for $y' \in N_1$, $y' \neq 1$,

$$\text{(5)} \qquad V_1 \cap y'V_1 = \emptyset.$$

Thus N_1 has a strong type of discontinuous action on D_1 and this is easily verified to be sufficient to ensure that D_1 is a covering space of D_1/N_1.

Denote the covering map from D_1 to D_1/N_1 by p_2. Then D_1/N_1 can be given an analytic structure such that it is a Riemann surface and such that p_2 is holomorphic. The map p_2 is given by $p_2(z_1) = N_1 z_1$ and, since $z_1 = Mz$ for some $z \in D$ and $N_1 = N/M$, this can be written $p_2(Mz) = (N/M)Mz$.

If $Nz = Nz'$ for $z, z' \in D$, then there is a $y \in N$ with $y(z) = z'$. Let $x \in M$; then for $y_0 = x^{-1}y$, $y_0 \in N$ and $Mz' = My_0(z) = (y_0 M)Mz$ so that $(N/M)Mz = (N/M)Mz'$. Thus the mapping $h: D/N \to D_1/N_1$ given by $h(Nz) = (N/M)Mz$ is well defined.

THEOREM 2. *h is a conformal homeomorphism between D/N and D_1/N_1.*

Proof. From its definition, h is onto and, if $(N/M)Mz = (N/M)Mz'$, then $Mz' = My(z)$ for some $y \in N$ so that $z' = xy(z)$ for some $x \in M$. Since $xy \in N$, $Nz = Nz'$ and h is one-to-one. In terms of the projection maps $p: D \to D/N$, $p_1: D \to D_1$, and $p_2: D_1 \to D_1/N_1$, the definition of h is $h \circ p = p_2 \circ p_1$ and so $h = p_2 \circ p_1 \circ p^{-1}$. Since each projection is open and continuous, h and h^{-1} are also and h is a homeomorphism.

p_2 is a covering map, while p and p_1 are covering maps save at those points of D which are fixed points of N. Since the projection maps are analytic, h is analytic except, possibly, at projections of fixed points. Since these points are isolated, h is analytic everywhere. Then h^{-1} is also analytic and D/N and D_1/N_1 are conformally equivalent.

We use h to identify D_1/N_1 with D/N and may refer to either as S. We have now got two possibly distinct actions on S of the isomorphic groups $G = \Gamma/N$ and Γ_1/N_1 but one can readily verify that these actions do coincide.

THEOREM 3. *D_1 is conformally equivalent to the upper half plane.*

Proof. We first triangulate the upper half plane D. Let F be a convex fundamental region for N, bounded by a finite number of non-Euclidean line segments and let z be an interior point of F. Join z to each vertex of F. F is then triangulated and the images under N form a triangulation of D. As a subgroup of N, M is a group of simplicial transformations of D.

We now apply (2), with D as the space X and M as the group G. Since M is generated by the elements having a non-empty fixed point set, $H = M$ and (2) gives

$$\pi_1(D/M) \cong M/M = \{1\}.$$

Thus D_1 is a simply connected Riemann surface and so is conformally equivalent to one of the complex plane, the disc, or the Riemann sphere [**1**]. However, D_1 is the universal covering space of the Riemann surface S and so cannot be conformally equivalent to the plane or the sphere. Since the disc is conformally equivalent to the upper half plane, the theorem is proved.

Identifying D_1 with the upper half plane, N_1 is a group of automorphisms of the upper half plane. Thus N_1 is a subgroup of L.F.$(2, R)$ and it follows from (5) that N_1 is discrete. Thus N_1 is a Fuchsian group; in fact, it is a Fuchsian surface group since it has no fixed points in D_1. From its presentation (4), N_1 is not a cyclic group, so its normalizer in L.F.$(2, R)$ is also a Fuchsian group.

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Since Γ_1 is a group of automorphisms of D_1, contained in the normalizer of N_1, it, too, will be a Fuchsian group. Since D_1 is the universal covering space of S, with $D_1/N_1 = S$, N_1 is the surface group denoted by K earlier. Also, Γ_1/N_1 is isomorphic to G and has the same action on S so that Γ_1 is the group $\tilde{G}$, the group lying over G.

We summarize our results in the following theorem.

THEOREM 4. *Let N be a normal subgroup of a Fuchsian group Γ, such that the orbit space $S = D/N$ is a compact Riemann surface of genus $g \geqq 2$. Let G denote the group Γ/N acting as a group of automorphisms of S. If M is the normal subgroup of N generated by elements having a non-empty fixed point set, then:*

(i) *the orbit space $D_1 = D/M$ is the universal covering space of S;*

(ii) *the surface group for S is the factor group $N_1 = N/M$, with $S = D_1/N_1$;*

(iii) *the covering group $\tilde{G}$ is the factor group $\Gamma_1 = \Gamma/M$.*

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