CYCLIC p-GROUPS OF SYMMETRIES OF SURFACES

by RAVI S. KULKARNI AND COLIN MACLACHLAN

(Received 31 January, 1990)

1. Introduction. Let Σ_g denote a compact orientable surface of genus $g \geqslant 2$. We consider finite groups G acting effectively on Σ_g and preserving the orientation—for short, G acts on Σ_g or G is a symmetry group of Σ_g. Each surface Σ_g admits only finitely many symmetry groups G and the orders of these groups are bounded by Wiman's bound of $84(g-1)$. This bound is attained for infinitely many values of g **[12]**, see also **[9]**, and all values of $g \leqslant 10^4$ for which it is attained are known **[4]**.

More detailed information on symmetry groups has been obtained by investigating this general set-up from different view points. First of all, the problem of determining all possible groups G acting on Σ_g for fixed g has been completed for $g \leqslant 5$ (e.g. **[17]**, **[13]**, **[10]**, **[11]**). Secondly the problem of determining the minimum genus g on which a given group G acts has been solved for various groups G ranging from cyclic to alternating (e.g. **[7]**, **[14]**, **[6]**, **[3]**). Also, if for a class of groups $\mathscr{C}$ one defines

$$M(\mathscr{C}, g) = \max\{o(G) \mid G \text{ acts on } \Sigma_g, G \in \mathscr{C}\}$$

upper and/or lower bounds for $M(\mathscr{C}, g)$ as linear functions of g have been obtained for various classes of groups ranging from all groups and cyclic groups through to solvable groups (e.g. **[1]**, **[15]**, **[7]**, **[8]**, **[18]**, **[2]**).

Recently Kulkarni **[8]** has shown that for any given group G there is an integer $n_0(G)$ such that if G acts on Σ_g then $g \equiv 1 \pmod{n_0(G)}$ and for all but a finite number of g such that $g \equiv 1 \pmod{n_0(G)}$, G acts on Σ_g. There is thus a minimum (reduced) stable g_0 (see §2) such that for all $g \geqslant g_0$ of the given form G acts on Σ_g and a gap sequence where G does not act, between the minimum (reduced) genus and the minimum (reduced) stable genus. Those groups G for which $n_0(G) = 1$ and so G acts on almost all Σ_g have been characterized **[8]** and include cyclic groups. For related problems on groups acting on non-closed surfaces see **[16]**.

In this paper, given a prime power p^e, we determine necessary and sufficient conditions on g, in terms of p-adic expansion, for Σ_g to admit a cyclic group of order p^e. From this one can deduce the minimum stable genus and in the case of prime order, a closed formula for the gap sequence, thereby determining all compact surfaces on which a given cyclic group of prime order acts. In the reverse direction, for a fixed genus g, we determine all primes p such that a cyclic group of order p acts on Σ_g. As will be seen the methods employed can be readily extended to other classes of p-groups and these problems will be taken up elsewhere. The computation of the gap sequence for small order cyclic groups and certain gaps for cyclic groups of prime order has recently played a role in determining "unstable" torsion in the cohomology of the mapping class group of genus g. **[5]**

2. Basic Definitions. Let G_p denote a finite p-group of exponent p^{e_p} and order p^{n_p}. The cyclic p-deficiency of G_p is defined to be $n_p - e_p$ and let

$$f_p = \begin{cases} n_p - e_p & \text{if } p \text{ is odd or } p = 2 \text{ and } n_2 = e_2 \\ n_2 - e_2 - 1 & \text{if } p = 2 \text{ and } n_2 > e_2. \end{cases}$$

In addition G_2 is said to be type I if either it is cyclic or the elements of order $<2^{e_2}$ do not form a subgroup of index 2, otherwise G_2 is of type II.

Then for any finite group G, define

$$n_0 = n_0(G) = \eta \prod_p p^{f_p}$$

where the product is over all primes p dividing $o(G)$, f_p being defined as above for any Sylow p-subgroup G_p of G, and $\eta = 1$ or 2 according as G_2 is of type I or II.

THEOREM [8]. *If G acts on Σ_g, $g \geq 2$, then $g \in n_0\mathbb{N}$ and conversely, for all but a finite number of integers*

$$g \in 1 + n_0 N, \ G \text{ acts on } \Sigma_g.$$

If G acts on Σ_g with $g = 1 + n_0 g_0$, then we call g_0 a *reduced genus* of G. Also $\mu_0 = \mu_0(G)$ will denote the *minimum reduced genus* of G and $\sigma_0 = \sigma_0(G)$ the *minimum stable reduced genus of* G i.e. minimal with the property that all $g_0 \geq \sigma_0$ are reduced genera of G. In addition, the integers in the interval $[\mu_0, \sigma_0]$ which do not occur as reduced genera of G will constitute the *reduced gap-sequence* of G.

Those groups G for which $n_0 = 1$ are classified in [8] and these include all cyclic groups. Note that in that case, the actual values g such that G acts on Σ_g are related to the reduced genera g_0 by $g = 1 + g_0$.

By expressing an integer g_0 in its "p-adic form" we obtain necessary and sufficient conditions on this form so that g_0 should be a reduced genus for a cyclic p-group. As a consequence we obtain formulae for the invariant σ_0 for these cyclic p-groups (the invariant μ_0 is already known [7]) and for the gap-sequence for cyclic groups of prime order.

Now let $G = G_p$ be a finite p-group with $e = e_p$ and $n = n_p$. If G acts on Σ_g then $\Sigma_g/G \simeq \Sigma_h$ for some $h \geq 0$ and $\pi: \Sigma_g \to \Sigma_h$ is a branched covering with branch indices dividing p^e. If there are x_i branch points on Σ_h with branching index p^i, $i = 1, 2, \ldots, e$, then the Riemann-Hurwitz relation gives

$$2(g-1) = p^n\left[2(h-1) + \sum_{i=1}^{e} x_i(1 - p^{-i})\right]. \tag{1}$$

Since $g \geq 2$, the right hand side of (1) must be positive. Conversely, given integers $h \geq 0$, $x_i \geq 0$ such that the right hand side of (1) is positive, one can construct a (Fuchsian) group of signature

$$(h; p^{(x_1)}, p^{2(x_2)}, \ldots, p^{e(x_e)})$$

i.e. having a presentation of the form

$$\text{Generators: } a_1, b_1, \ldots, a_h, b_h, t_{11}, t_{12}, \ldots, t_{1x_1}, t_{21}, \ldots, t_{ex_e}$$

$$\text{Relations; } \quad t_{ij}^{p^i} = 1 \ (i = 1, 2, \ldots, e; \ j = 1, 2, \ldots, x_i)$$

$$\prod_k [a_k, b_k] \prod_{i,j} t_{ij} = 1.$$

Furthermore, if there exists a homomorphism of Γ onto G with torsion-free kernel such that (1) holds, then G acts on Σ_g. In that case, we say that the data $\{h, x_i\}$ corresponds to an action.

https://doi.org/10.1017/S0017089500008247 Published online by Cambridge University Press

3. Fundamental equation for p odd. Let p be an odd prime. From equation (1), it becomes necessary to examine in detail, for all p-groups, the solutions of the diophantine equation

$$N = p^e h + \sum_{i=1}^{e} x_i \tfrac{1}{2}(p^e - p^{e-i}). \tag{2$_e$}$$

Let $\Omega_e = \Omega_e(p)$ denote the set of solutions N of $(2)_e$ for which $h \geqslant 0$ and $x_i \geqslant 0$ all i. We will describe Ω_e in terms of a p-adic expansion of N. Given p and e, let

$$2N = a_0 + a_1 p + \ldots + a_e p^e$$

where $0 \leqslant a_i < p$ for $i = 0, 1, \ldots, e-1$ and $a_e \geqslant 0$. There is a unique expansion of this form. The results are expressed in terms of the sum of the coefficients of this expansion and so for brevity write

$$S_e(2N) = \sum_{k=0}^{e} a_k.$$

THEOREM 3.1. $\Omega_e = \{N \in \mathbb{N} \mid S_e(2N) \geqslant (e-i)(p-1)$ *where* a_i *is the first non-zero coefficient*$\}$.

Proof. If Ω_e' denotes the set described in the statement, we will prove by induction on e that $\Omega_e = \Omega_e'$.

Let $N \in \Omega_1$ so that $2N = 2ph + x_1(p-1)$. Furthermore let $x_1 = c_0 + c_1 p$ with $0 \leqslant c_0 < p$. Then

$$2N = \begin{cases} [2h + c_1(p-1) + c_0 - 1]p + (p - c_0) & \text{if} \quad c_0 > 0 \\ [2h + c_1(p-1)]p & \text{if} \quad c_0 = 0 \end{cases}$$

$$S_1(2N) = \begin{cases} 2h + (c_1+1)(p-1) & \text{if} \quad a_0 \neq 0 \\ 2h + c_1(p-1) & \text{if} \quad a_0 = 0. \end{cases}$$

Thus $N \in \Omega_1'$.

Now suppose $N \in \Omega_1'$. Hence

$$S_1(2N) \geqslant \begin{cases} p-1 & \text{if} \quad a_0 \neq 0, \\ 0 & \text{if} \quad a_0 = 0. \end{cases}$$

In the second case, clearly $N \in \Omega_1$. In the first case, set

$$S_1(2N) = (p-1) + 2k.$$

Thus $N \in \Omega_1$ with $h = k$ and $x_1 = p - a_0$.

Now assume that $\Omega_{e-1} = \Omega_{e-1}'$, and let

$$2N = a_0 + a_1 p + \ldots + a_e p^e$$

with a_i the first non-zero coefficient. Define $N' \in \mathbb{N}$ by

$$N' = \frac{1}{p}\left[N - m(p - a_0)\left(\frac{p^e - 1}{2}\right)\right]$$

where $m = \min\{a_0, 1\}$.

Let $N \in \Omega_e$ and so from equation $(2)_e$ we deduce that $x_e = (p - a_0) + p x_e'$ with $x_e' \geqslant 0$.

Then we obtain that $N' \in \Omega_{e-1} = \Omega'_{e-1}$. Now let

$$2N' = b_0 + b_1 p + \ldots + b_{e-1} p^{e-1}$$

so that $S_{e-1}(2N') \geqslant (e-1-i)(p-1)$ where b_i is the first non-zero coefficient. But then

$$\begin{aligned} 2N = a_0 + (p-1)p + (p-1)p^2 + \ldots + (p-1)p^i + (b_i - 1)p^{i+1} \\ + b_{i+1}p^{i+2} + \ldots + b_{e-2}p^{e-1} + (b_{e-1} + (p - a_0))p^e \end{aligned}$$

$$S_e(2N) = a_0 + i(p-1) + S_{e-1}(2N') - 1 + p - a_0$$

$$S_e(2N) \geqslant (i+1)(p-1) + (e-1-i)(p-1) = e(p-1).$$

Now suppose that $N \in \Omega'_e$ so that $S_e(2N) \geqslant e(p-1)$.

$$2N' = (a_1 + 1) + a_2 p + \ldots + a_{e-1}p^{e-2} + (a_0 + a_e - p)p^{e-1}.$$

Since each of the coefficients $a_1, a_2, \ldots, a_{e-1} \leqslant p-1$ it follows that $a_0 + a_e \geqslant p-1$ with equality only in the case $a_1 = \ldots = a_{e-1} = p-1$. In that exceptional case, $2N' = (a_0 + a_e + 1 - p)p^{e-1}$ so $N' \in \Omega'_{n-1}$. Let k be the first integer so that $k \geqslant 1$ and $a_k \neq p-1$. Then we obtain the p-adic expansion

$$2N' = (a_k + 1)p^{k-1} + a_{k+1}p^k + \ldots + a_{e-1}p^{e-2} + (a_0 + a_e - p)p^{e-1}$$

$$\begin{aligned} S_{e-1}(2N') &= \sum_{j=k}^{e} a_j + 1 + a_0 - p = S_e(2N) + 1 - p - (k-1)(p-1) \\ &\geqslant [(e-1) - (k-1)](p-1). \end{aligned}$$

Thus $N' \in \Omega'_{e-1} = \Omega_{e-1}$ and it then follows easily that $N \in \Omega_e$.

Definition. Let $\sigma_e(p)$ denote the smallest stable solution of $(2)_c$ i.e. $\sigma_e(p)$ is minimal with the property that all $N \geqslant \sigma_e(p)$ lie in $\Omega_e(p)$.

Corollary 3.2. $\sigma_e(p) = \frac{1}{2}[e(p-1)p^e - 3(p^e - 1)]$.

Proof. From the above description of Ω_e it is clear that the largest integer which fails to lie in Ω_e occurs when $S_e(2N) = e(p-1) - 2$ and $a_0 \neq 0$. Taking $a_0 = 1$ and $a_e = e(p-1) - 3$ gives the result.

4. The Fundamental equation for $p = 2$. The presence of the factors 2 in equation (1) leads to a slightly different diophantine equation in the case of $p = 2$.

$$N = 2^e h + \sum_{i=1}^{e-1} x_i(2^{e-1} - 2^{e-1-i}) \tag{3$_c$}$$

Let $\Omega_e(2)$ denote the set of solutions N of $(3)_c$ for which $h, x_i \geqslant 0$. This is only defined for $e \geqslant 2$ and for $e = 2$, $\Omega_2 = \mathbb{N}$. Thus assume throughout that $e \geqslant 3$. For fixed e, consider the 2-adic expansion

$$N = a_0 + a_1 2 + \ldots + a_{e-1}2^{e-1}$$

where $a_i = 0, 1$ for $i = 0, 1, \ldots, e-2$ and $a_{e-1} \geqslant 0$. Let $S_e(N)$ denote the sum of the coefficients in this expansion. In a manner analogous to Theorem 3.1 we obtain

Theorem 4.1. $\Omega_e = \{N \in \mathbb{N} \mid S_e(N) \geqslant e - 1 - i$ *where* a_i *is the first non-zero coefficient*$\}$.

COROLLARY 4.2. *The least stable solution of* $(3)_e$, $\sigma_e(2)$ *is given by*

$$\sigma_e(2) = (e-3)2^{e-1} + 2.$$

5. Cyclic p-groups, p odd. In this section, we obtain necessary and sufficient conditions on an integer g_0 in terms of its p-adic expansion in order that it should be a reduced genus for a Z_pe action, p an odd prime. From this one readily deduces $\sigma_0(Z_pe)$ the minimum stable reduced genus and a formula for the gap sequence for Z_p actions.

One readily sees that every $g_0 \geqslant 1$ is a reduced genus for Z_3. Thus throughout this section we assume that $p^e > 3$.

Let integers $h \geqslant 0$, $x_i \geqslant 0$, $i = 1, 2, \ldots, e$ be given. For this data $\{h, x_i\}$ define $M = \min\{n : x_j = 0 \text{ for each } j > n\}$. By the criterion given at the end of §2, we deduce in this case that the data $\{h, x_i\}$ corresponds to an action of Z_pe with reduced genus $g_0 = g - 1 \geqslant 1$ if and only if g satisfies equation (1) and the data satisfies at least one of the conditions in the table below

	h	M	x_M
1	$\geqslant 2$	0	
2	$\geqslant 1$	$\geqslant 1$	$\geqslant 2$
3	$\geqslant 0$	e	$\geqslant 2$ and r.h.s. of (1) positive.

From equation (1) it follows that $g_0 = p^{e-M} g_0'$ and

$$2g_0' = 2p^M(h-1) + \sum_{i=1}^{M} x_i(p^M - p^{M-i}).$$

Condition *1* is equivalent to $g_0' \geqslant 1$.

Condition *2* is equivalent to $g_0' - (p^M - 1) \in \Omega_M$

Condition *3* is equivalent to $g_0 + 1 \in \Omega_e \backslash \{0\}$. Note that $g_0 + 1 - p^e \in \Omega_e$ implies that $g_0 + 1 \in \Omega_e$. Thus

THEOREM 5.1. *Let* $g_0 \geqslant 1$. *Then* g_0 *is a reduced genus for* Z_pe *if and only if*

(i) $g_0 + 1 \in \Omega_e$,

or (ii) *for some* $1 \leqslant M < e$, $g_0 = p^{e-M} g_0'$ *and* $g_0' + 1 - p^M \in \Omega_M$,

or (iii) $g_0 = p^e g_0'$.

THEOREM 5.2. *Let* $g_0 \geqslant 1$. *Let* $2g_0 = a_0 + a_1 p + \ldots + a_e p^e$ *where* $0 \leqslant a_i < p$ *for* $i = 0, 1, \ldots, e-1$ *and* $a_e \geqslant 0$. *Let* i *be the first integer so that* $a_i \neq 0$ *and let* j *be the first integer* $\geqslant i$ *such that* $a_j \neq p-1$ *or* $j = e$. *Then* g_0 *is a reduced genus for* Z_pe *if and only if*

A) $i = 0$ *and* $S_e(2g_0) \geqslant (e+j)(p-1) - 2$

B) $0 < i = j \leqslant e$ *and* $S_e(2g_0) \geqslant (e-i)(p-1)$

C) $1 < i < j \leqslant e$ *and* $S_e(2g_0) \geqslant (e-i+1)(p-1)$

D) $1 = i < j \leqslant e$ *and* $S_e(2g_0) \geqslant e(p-1) - 2$.

Proof. A) $i = 0$ so that $g_0 \not\equiv 0 \pmod p$. Now

$$2(g_0 + 1) = (a_0 + 2) + a_1 p + \ldots + a_e p^e.$$

https://doi.org/10.1017/S0017089500008247 Published online by Cambridge University Press

If $a_0 \neq p-2, p-1$, so that $j=0$, the inequality follows immediately from 5.1 and 3.1. If $a_0 = p-2$, so that $j=0$, let $m \geq 1$ be the first integer so that $a_m \neq p-1$ or $m=e$. Thus

$$2(g_0+1) = (a_m+1)p^m + a_{m+1}p^{m+1} + \ldots + a_e p^e.$$

It thus follows from 3.1 that

$$S_e(2g_0) \geq (e-m)(p-1) - 1 + (p-2) + (m-1)(p-1) = e(p-1) - 2.$$

If $a_0 = p-1$ so that $j>0$, from 3.1 it follows that

$$S_e(2g_0) \geq (e+j)(p-1) - 2.$$

B) $0 < i = j \leq e$. So we have

$$2g_0 = a_i p^i + \ldots + a_e p^e$$

with $a_i \neq p-1$. Note if $i=e$ all such g_0 (≥ 1) are reduced genera and the inequality agrees.

For all M such that $e-M \leq i$, $g_0 \equiv 0 \pmod{p^{e-M}}$. Thus g_0 is a reduced genus if and only if either (i) $g_0+1 \in \Omega_e$ or (ii) for some such M, $g_0/p^{e-M}+1-p^M \in \Omega_M$ i.e. if and only if one of the following four conditions holds:

(α) $S_e(2g_0) \geq e(p-1)-2$
(β) $e-M<i$ and $S_e(2g_0) \geq M(p-1)$
(γ) $e-M=i$, $a_i \neq p-2$ and $S_e(2g_0) \geq M(p-1)$
(δ) $e-M=i$, $a_i = p-2$ and
$S_e(2g_0) \geq (M-m)(p-1) - 1 + (p-2) + (m-1)(p-1) + 2 = M(p-1)$

where m is the first integer $>i$ such that $a_m \neq p-1$ or $m=e$. (The conditions and inequalities in cases β), γ), δ) all imply that $a_e \geq 2$). Thus g_0 is a reduced genus if and only if either $S_e(2g_0) \geq e(p-1)-2$ or $S_e(2g_0) \geq (e-i)(p-1)$. Since $i \geq 1$, the first inequality implies the second and the result follows.

C) and D) follow by a similar analysis.

COROLLARY 5.3. $\sigma_0(Z_pe) = \sigma_e(p) - 1 = \frac{1}{2}[(e(p-1)-3)p^e+1]$.

Proof. By Theorem 5.1, $\sigma_0(Z_pe) \leq \sigma_e(p)-1$. Now let $g_0 = \sigma_e(p)-2$ so that

$$2g_0 = (p-1) + (p-1)p + \ldots + (p-1)p^{e-1} + [e(p-1)-4]p^e$$

so that, in the notation of Theorem 5.2, $i=0$ and $j=e$. But the inequality given under A) then fails.

Now consider the particular case of Z_p. For $p=3$, Z_3 acts on all surfaces of genus $g \geq 2$. Otherwise from Theorem 5.2 we can read off the reduced minimum genus $\mu_0 = \frac{1}{2}(p-3)$, (but this is already known for all cyclic groups [**7**]) and the reduced minimum stable genus $\sigma_0 = \sigma_1(p) - 1 = \frac{1}{2}[p(p-4)+1]$.

COROLLARY 5.4. *The gap sequence for* Z_p *is*

$$\left\{\tfrac{1}{2}(p-1) + \varepsilon\left(\frac{p+3}{2}\right) + i + kp \mid \varepsilon = 0, 1,\ i,k \geq 0,\ 0 \leq i+k \leq \left(\frac{p-5}{2}\right) - 2\varepsilon\right\}.$$

Proof. If $2g_0 = a_0 + a_1 p$, then C) and D) of Theorem 5.2 do not apply in this case and the other cases reduce to (i) $a_0 = 0$ all a_1 (ii) $a_0 \neq 0$, $a_0 = p-1$, $a_0 + a_1 \geqslant 2(p-2)$ (iii) $a_0 \neq 0 a_0 \neq p - 1 a_0 + a_1 \geqslant p - 3$.

Let $g_0 = \frac{1}{2}(p-1) + b_0 + b_1 p$ where $0 \leqslant b_0 < p$ and $b_1 \geqslant 0$.

A) $b_0 = (p+1)/2$. All such g_0 are reduced genera.

B) $b_0 = 0$. Then $b_1 \geqslant (p-3)/2$.

C) (i) $1 \leqslant b_0 \leqslant (p-1)/2$. Then $2g_0 = (2b_0 - 1) + (2b_1 + 1)p$ so that $b_0 + b_1 \geqslant (p-3)/2$

(ii) $(p+3)/2 \leqslant b_0 \leqslant p - 1$. Then $2g_0 = (2b_0 - 1 - p) + (2b_1 + 2)p$ so that $b_0 + b_1 \geqslant p - 2$.

6. Cyclic 2-groups. The results and methods are similar to those in the preceding section and so only an outline is given.

Using the same notation as in §5, the data $\{h, x_i\}$ corresponds to an action of Z_2e if and only if at least one of the conditions set out in the table below holds.

	h	M	x_M
1	$\geqslant 2$	0	
2	$\geqslant 1$	$\geqslant 1$	Even
3	$\geqslant 0$	e	Even and r.h.s. of (1) positive.

THEOREM 6.1. *All $g_0 \geqslant 1$ are reduced genera for Z_2 and Z_4. Let $g_0 \geqslant 1$ and $e \geqslant 3$. Then g_0 is a reduced genus for Z_2e if and only if*

(i) $g_0 + 1 \in \Omega_e$

or (ii) *for some $2 \leqslant M < e$, $g_0 = 2^{e-M} g_0'$ and $g_0' + 1 - 2^M \in \Omega_M$*

or (iii) $g_0 = 2^{e-1} g_0'$.

THEOREM 6.2. *Let $g_0 \geqslant 1$ and $e \geqslant 3$. Let*

$$g_0 = a_0 + a_1 2 + \ldots + a_{e-1} 2^{e-1}$$

where $a_i \in \{0, 1\}$ for $i = 0, 1, \ldots, e-2$ and $a_{e-1} \geqslant 0$. Let i be the first integer so that $a_i \neq 0$ and let j be the first integer $\geqslant i$ such that $a_j = 0$ or $j = e - 1$. Then g_0 is a reduced genus for Z_2e if and only if

A) *$i = 0$ and $S_e(g_0) \geqslant e - 2$*

B) *$1 < j \leqslant j \leqslant e - 1$ and $S_e(g_0) \geqslant e - i$*

C) *$1 = i < j \leqslant e - 1$ and $S_e(g_0) \geqslant e - 2$.*

COROLLARY 6.3. *$\sigma_0(Z_8) = 1$ and for $e \geqslant 4$*

$$\sigma_0(Z_2e) = 5 + (e-4)2^{e-1}.$$

7. Primes dividing the order of a symmetry group. In the preceding sections, starting with a cyclic group of order p^e we have determined conditions on g such that the group acts on Σ_g. In this section, we consider the opposite problem of: given g, determine the primes which divide $o(G)$ where G is a symmetry group of Σ_g. Note that the earlier results were in terms of the reduced genus g_0, while here the results are in terms of the actual genus g $(=g_0 + 1)$.

DEFINITION. Let $\pi_g = \{\text{primes } p : Z_p \text{ acts on } \Sigma_g\}$.

The cyclic groups of order 2 and 3 act on surfaces of all genera, so that $2, 3 \in \pi_g$ for all g. The minimum genus of a surface on which a cyclic group of prime order p acts is $\frac{1}{2}(p-1)$ (e.g. **[7]**) so that $p \leqslant 2g+1$. In addition, for every genus g, there are surfaces of genus g which admit cyclic symmetry groups of orders $2(2g+1)$ **[7]**, $2(g+1)$ **[1][15]** and $4g$. [This last arising from a triangle group of signature $(0; 2, 4g, 4g)$]. Similarly one easily obtains that, if $g \equiv 0 \pmod 2$ ($g \equiv 0 \pmod 3$) then Σ_g admits cyclic groups of order $g+2$ ($2g+3$ resp), the corresponding group having signature $(0; \frac{1}{2}(g+2), \frac{1}{2}(g+2), g+2, g+2)$ ($(0; \frac{1}{3}(2g+3), 2g+3, 2g+3)$ resp). From theorem 5.2B) we see that if $p \mid g-1$ then $p \in \pi_g$. From 5.3, $p \in \pi_g$ whenever $g \geqslant \frac{1}{2}[p(p-4)+3]$ so that π_g includes all primes $p \leqslant \sqrt{2g+1}+2$. Let S_g be the set of primes p that satisfy at least one of the following

$$p \leqslant \sqrt{2g+1}+2 \tag{1}$$

$$p \mid g(g^2-1)(2g+1) \tag{2}$$

$$p \mid g+2(2g+3) \text{ if } g \equiv 0 \pmod 2 \ (g \equiv 0 \pmod 3). \tag{3}$$

THEOREM 7.1. *With notation as above*, $\pi_g = S_g \cup T_g$ *where*

$$T_g = \left\{ \text{primes } p \mid p < \frac{g}{2} \text{ and } 2g-2 = a_0 + a_1 p \right.$$
$$\left. \text{with } 0 < a_0 < p-1 \text{ and } a_0 + a_1 \geqslant p-3 \right\}.$$

Proof. If $a_0 = p-1$, and $p \in \pi_g$, by 5.2A), $a_1 \geqslant p-3$. This implies that $p \leqslant \sqrt{2}g+1$ and so $p \in S_g$.

If $p \geqslant g/2$, then $2g-2 \leqslant 4p-2$ so that $a_1 \leqslant 3$. If $a_1 = 3$, then $a_0 \geqslant p-6$ and $2g-2 = 4p-6$, $4p-4$, $4p-2$. But that implies that $p \in S_g$, these cases being covered in (2) and (3). Otherwise if $a_1 = 2$, then $2g-2 = 3p-5$, $3p-3$ again both cases giving $p \in S_g$. Similarly we reject $a_1 = 1,0$.

REFERENCES

1. R. D. M. Accola, On the number of automorphisms of a closed Riemann surface. *Trans. Amer. Math. Soc.* **131** (1968), 398–408.

2. B. P. Chetiya, On genuses of compact Riemann surfaces admitting solvable automorphism groups. *Indian J. Pure appl. Math.* **12** (1981), 1312–1318.

3. M. D. E. Conder, Generators for alternating and symmetric groups. *J. London Math. Soc.* **22** (1980), 75–86.

4. M. D. E. Conder, The genus of Compact Riemann surfaces with Maximal Automorphism Group. *J. of Algebra* **108** (1987), 204–247.

5. H. Glover and G. Mislin, Torsion in the mapping class group and its cohomology. *J. of Pure & Appl. Algebra* **44** (1987), 177–189.

6. H. Glover and D. Sjerve, The genus of $PSL_2(q)$ *J. Reine und Angew Math.* **380** (1987), 59–86.

7. W. J. Harvey, Cyclic groups of automorphisms of a compact Riemann surface. *Quart. J. Math.* **17** (1966), 86–97.

8. R. S. Kulkarni, Symmetries of surfaces. *Topology* **26** (1987), 195–203.

9. R. S. Kulkarni, Normal subgroups of fuchsian groups. *Quart. J. Math. Oxford* **36** (1985), 325–344.

10. A. Kuribayashi and H. Kimura, On Automorphism Groups of Compact Riemann surfaces of Genus 5. *Proc. Japan Academy* **63** Ser A (1987), 126–130.

11. I. Kuribayashi and A. Kuribayashi, On Automorphism groups of a Compact Riemann Surface of genus 4 as subgroups in GL(4, $\mathbb{C}$). *Bull. Facul. Sc. Eng. Chuo Univ.* **28** (1985), 11–28.

12. A. M. Macbeath, On a Theorem of Hurwitz *Proc. Glasgow Math. Assoc.* **5** (1961), 90–96.

13. C. Maclachlan, *Groups of Automorphisms of compact Riemann surfaces* Ph.D. Thesis Birmingham Univ. 1966.

14. C. Maclachlan, Abelian groups of automorphisms of compact Riemann surfaces. *Proc. London Math. Soc.* **15** (1965), 699–712.

15. C. Maclachlan, A bound for the number of automorphisms of a compact Riemann surface. *J. London Math. Soc.* **44** (1969), 265–272.

16. D. McCullough, A. Miller and B. Zimmermann, Group actions on non-closed 2-manifolds (*to appear*).

17. A. Wiman, Uber die hyperelliptischen Kurven und diejenigen vom Geschlechte $p = 3$, welche eindeutigen Transformationen in sich Zulassen *Bihang. Till Kongl. Svenska Veienskaps—Akademiems Hadlingar* (*Stockholm* 1895-6), **21,** 1–23.

18. R. Zomorrodian, Nilpotent automorphism groups of Riemann surfaces, *Trans. Amer. Math. Soc.* **288** (1985), 241–255.

INST. MITTAG-LEFFLER
AURAVÄGEN 17
5-182 62 DJURSHOLM
SWEDEN

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF ABERDEEN
ABERDEEN
SCOTLAND

https://doi.org/10.1017/S0017089500008247 Published online by Cambridge University Press