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CONTINUOUS ASSOCIATION SCHEMES AND HYPERGROUPS

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Abstract

Classical finite association schemes lead to finite-dimensional algebras which are generated by finitely many stochastic matrices. Moreover, there exist associated finite hypergroups. The notion of classical discrete association schemes can be easily extended to the possibly infinite case. Moreover, this notion can be relaxed slightly by using suitably deformed families of stochastic matrices by skipping the integrality conditions. This leads to a larger class of examples which are again associated with discrete hypergroups. In this paper we propose a topological generalization of association schemes by using a locally compact basis space X and a family of Markov-kernels on X indexed by some locally compact space D where the supports of the associated probability measures satisfy some partition property. These objects, called continuous association schemes, will be related to hypergroup structures on D. We study some basic results for this notion and present several classes of examples. It turns out that, for a given commutative hypergroup, the existence of a related continuous association scheme implies that the hypergroup has many features of a double coset hypergroup. We, in particular, show that commutative hypergroups, which are associated with commutative continuous association schemes, carry dual positive product formulas for the characters. On the other hand, we prove some rigidity results in particular in the compact case which say that for given spaces X, D there are only a few continuous association schemes.

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1. Introduction

In this paper we study a topological generalization of the notion of classical finite association schemes by using the notion of hypergroups in the sense of Dunkl, Jewett, and Spector. To explain this, let us start with the notion of a finite association scheme which is common in algebraic combinatorics; see, for example, the monographs [4, 5, 48].

DEFINITION 1.1. Let X, D be finite nonempty sets and $(R_i)_{i\in D}$ a disjoint partition of $X \times X$ with $R_i \neq \emptyset$ ($i \in D$) and with the following properties.

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(1) There exists $e \in D$ with $R_e = \{(x,x) : x \in X\}$.
(2) There exists an involution $i \mapsto \bar{i}$ on D such that for $i \in D$, $R_{\bar{i}} = \{(y,x) : (x,y) \in R_i\}$.
(3) For all $i, j, k \in D$ and $(x,y) \in R_k$, the number

$$p_{i,j}^k := |\{z \in X : (x,z) \in R_i \quad \text{and} \quad (z,y) \in R_j\}|$$

is independent of $(x,y) \in R_k$.

Then $\Lambda := (X, D, (R_i)_{i\in D})$ is called a finite association scheme with intersection numbers $(p_{i,j}^k)_{i,j,k\in D}$ and identity e.

Now let $\Lambda := (X, D, (R_i)_{i\in D})$ be a finite association scheme. Form the adjacency matrices $A_i \in \mathbb{R}^{X\times X}$ $(i \in D)$ with

$$(A_i)_{x,y} := \begin{cases} 1 & \text{if} \quad (x,y) \in R_i \\ 0 & \text{otherwise} \end{cases} \qquad (i \in D,\ x,y \in X).$$

Then A_e is the identity matrix I_X, and the transposed matrices satisfy $A_i^T = A_{\bar{i}}$ for $i \in D$. Moreover, for $i, j \in D$, we have for the usual matrix product $A_iA_j = \sum_{k\in D} p_{i,j}^k A_k$.

Define the valencies

$$\omega_i := p_{i,\bar{i}}^e = |\{z \in X : (x,z) \in R_i\}| \in \mathbb{N}$$

of R_i (or $i \in D$), where these numbers are independent of $x \in X$. Then the renormalized adjacency matrices $S_i := (1/\omega_i)A_i \in \mathbb{R}^{X\times X}$ are stochastic, that is, all row sums are equal to 1. Moreover, the products

$$S_iS_j = \sum_{k\in D} \frac{\omega_k}{\omega_i\omega_j} p_{i,j}^k S_k \quad (i, j \in D)$$

are convex combinations of the S_i, and the linear span of the S_i is a finite-dimensional algebra. This algebra is isomorphic with the algebra of measures of some finite hypergroup structure on D in the sense of Dunkl, Jewett, and Spector, where the S_i are identified with the point measures δ_i of $i \in D$. For this we recapitulate the definition of a finite hypergroup; see [8, 15, 22], and in the finite case, [34, 45, 46]. We point out that we do not use another definition of hypergroups where products of sets are considered, and which runs under the subject classification 20N20.

Definition 1.2. A finite hypergroup $(D, *)$ is a finite nonempty set D with an associative, bilinear, probability-preserving multiplication $*$ (called convolution) on the vector space $M_b(D)$ of all complex measures on D with the following properties.

(1) There exists a neutral element $e \in D$ with $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$ for $x \in D$.
(2) There exists an involution $x \mapsto \bar{x}$ on D such that for all $x, y \in D$, $e \in \operatorname{supp}(\delta_x * \delta_y)$ if and only if $y = \bar{x}$.
(3) If for $\mu \in M_b(D)$, μ^- is the image of μ under the involution, then $(\delta_x * \delta_y)^- = \delta_{\bar{y}} * \delta_{\bar{x}}$ for all $x, y \in D$.

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Now let $\Lambda := (X, D, (R_i)_{i\in D})$ be a finite association scheme as above. It is easy to see that then the unique bilinear extension of the convolution

$$\delta_i * \delta_j := \sum_{k\in D} \frac{\omega_k}{\omega_i\omega_j} \cdot p_{i,j}^k \delta_k \quad (i, j \in D)$$

of point measures leads to a finite hypergroup $(D, *)$, the so-called hypergroup associated with Λ.

Classical examples of finite association schemes and hypergroups appear from groups.

Example 1.3. Let H be a subgroup of a finite group G with identity e. Consider the set $X := G/H := \{gH : g \in G\}$ of cosets as well as the set $D := G//H := \{HgH : g \in G\}$ of double cosets. It can be easily checked and is well known that the partition

$$R_{HgH} := \{(xH, yH) \in X \times X : Hx^{-1}yH = HgH\} \quad (HgH \in D)$$

of $X \times X$ leads to a finite association scheme with identity HeH and involution $HgH \mapsto Hg^{-1}H$. The associated hypergroup is the so-called double coset hypergroup $(D, *)$ with the double coset convolution

$$\delta_{HxH} * \delta_{HyH} = \frac{1}{|H|} \sum_{h\in H} \delta_{HxhyH} \quad (x, y \in G).$$

The associated convolution algebra $M_b(D)$ is often also called a Hecke-algebra.

Typical commutative examples for 1.3 appear, if one considers so-called distance-transitive graphs X on which the group G of all graph automorphisms acts where H is the fix group of some vertex. We then have $G/H \equiv X$ in a canonical way, and $G//H$ can be identified with $\{0, 1, \ldots, N\}$ with the diameter N of X. We do not give further details here and refer to [5]. The set of distance-transitive graphs is a proper subset of the set of distance-regular graphs where again canonical-related commutative association schemes and commutative hypergroups exist, and where the construction of 1.3 via groups usually is no longer available. Here, we skip details of the theory of distance-regular graphs and refer to the monographs [5, 10] and the recent survey [12].

Let us now consider generalizations of association schemes. We first skip the condition that X and D are finite, where we keep a finiteness condition for the partition; see Definition 3.1 below. It turns out that then most statements for finite association schemes remain valid. In particular, there exist associated discrete hypergroups $(D, *)$ as above. Typical examples appear when we consider totally disconnected, locally compact groups G with a compact, open subgroup H. Then the spaces $X := G/H$ and $D := G//H$ as above are discrete with respect to the quotient topology, and X, D and a partition as in 1.3 lead to a possibly infinite association scheme. The associated hypergroup $(D, *)$ is then just the double coset hypergroup studied in hypergroup theory; see [22]. The details are worked out in [43]. Typical examples for this are the infinite association schemes related to homogeneous trees and, slightly more

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general, to infinite distance-transitive graphs; see [43]. We also mention that there exists examples of higher rank with X as sets of vertices of affine buildings; see, for example, [1] and references therein.

We next consider a further discrete extension from [43]. Fix some (possibly infinite) association scheme with an associated partition and the associated stochastic, renormalized adjacency matrices S_i; assume that we in addition have a further algebra of matrices generated by 'deformed' stochastic matrices $\tilde{S}_i \in \mathbb{R}^{X\times X}$ for $i \in D$ where any entry of any S_i is positive if and only if so is the corresponding entry of $\tilde{S}_i$. We add some further technical axioms like $\tilde{S}_e = S_e$ and that there is a measure on X which replaces the counting measure of an association scheme and which satisfies some adjoint relation; see Definition 3.5. It turns out that these so-called generalized association schemes with the matrices $\tilde{S}_i$ instead of the S_i also admit associated hypergroups $(D, *)$ as above.

In this paper we use this notion of generalized association schemes from [43] and present a topological extension in Definition 4.2 by using families of Markov-kernels on locally compact spaces X which are indexed by some locally compact space D instead of stochastic matrices as before. We require that the supports of the measures associated with these kernels admit partition properties similar to those of association schemes, and we require that the kernels generate an algebra such that again the product linearizations of the kernels fit to some hypergroup structure $(D, *)$. In addition, some natural topological conditions are added. We point out that here we require from the beginning that there exists an associated hypergroup structure $(D, *)$ (different from the discrete case). We have done this as we otherwise would run into technical topological problems (which we want to avoid in this paper), and as for all known examples this hypergroup property is available from the beginning. This is in particular the case for standard classes of examples of such continuous association schemes (CAS for short). Here is a short incomplete list of examples of CAS.

(1) If H is a compact subgroup of a locally compact group G, then $X := G/H$ and $D := G//H$ lead to canonical CAS associated with groups analog to the finite case or the case where $H \subset G$ is compact and open (in which case G/H and $G//H$ are discrete).

(2) All (unimodular) association schemes and all generalized association schemes as above are CAS.

(3) If a noncompact commutative CAS is given, then it often can be deformed via so-called pairs (α, φ) of positive multiplicative functions on D and X; see Section 8 for details. This construction often leads to plenty of interesting families of deformed CAS with deformed Markov-kernels, where the spaces X, D remain unchanged. On the other hand, in the compact and in particular finite case, the situation is much more rigid. It turns out that for given compact spaces X, D, there is at most one associated CAS structure; see Corollary 5.14. Moreover, each finite CAS is automatically an association scheme, that is, there is in fact no freedom in the choice of the stochastic matrices $\tilde{S}_i$ of a generalized association

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scheme in the finite case. This difference between the compact and noncompact case is remarkable.

(4) Besides the examples indicated above we point out that there are several further standard constructions to get new CAS from given ones; see Section 11.

This paper is organized as follows. In Section 2 we recapitulate some facts about hypergroups in the sense of Dunkl, Jewett, and Spector with a focus on the commutative case; the main references are the monograph of Bloom and Heyer [8] and Jewett [22]. Some technical details of Section 2 may be skipped at a first reading. In Section 3 we recapitulate some notations and facts on possibly infinite classical association schemes and their discrete generalizations mentioned above. This discrete generalization motivates the definition of continuous association schemes (CAS for short) on the basis of Markov-kernels and associated transition operators in Section 4. The central Section 4 contains the discussion of basic properties and some natural classes of examples. In Section 5 we add some further axioms to the basic definition, called translation properties (T1) and (T2), which are needed to get stronger interrelation between the analysis on X and D. It turns out that all compact CAS as well as all CAS associated with groups and all classical discrete association schemes have these properties. As a byproduct we obtain some rigidity result, for example, that all finite CAS are in fact association schemes.

Section 6 is then devoted to positive definite functions on D and X for commutative CAS. We in particular obtain that each commutative hypergroup $(D, *)$ which is associated with some CAS with property (T2) admits a dual positive convolution on the support of the Plancherel measure of $(D, *)$; see Theorem 6.9. This central result will be improved in Section 7 where we consider two possibly different commutative CAS structures with the same basic spaces X, D where we assume that one of them has property (T2) and where the schemes are related in some way. The central positive definiteness result in Theorem 7.1 will also lead to further rigidity results.

Sections 8 to 11 are mainly devoted to examples of CAS and construction principles of examples beyond the group cases and discrete association schemes. We start in Section 8 with nontrivial functions φ on X which are eigenfunctions under all transition operators of the given commutative CAS. It turns out that these φ are always related to multiplicative functions α of the hypergroup $(D, *)$. The interrelations between φ and α will lead to further results regarding the properties (T1) and (T2) in the commutative case. Moreover, if φ and α are in addition positive, we shall construct a deformed CAS with the same spaces X, D but deformed Markov-kernels. On the level of hypergroups this deformation is just the known deformation of a hypergroup by a positive semicharacter in [8, 36]. In the case of commutative CAS associated with noncompact symmetric spaces X, the eigenfunctions φ are closely related to the joint eigenfunctions of the invariant differential operators on X which are completely classified; see [19, 25]. In Section 9 we shall mainly study the deformation of commutative CAS which appear via orbits when some compact group acts continuously on some locally compact abelian group.

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Section 10 is devoted to the deformation of a concrete class of examples, namely of the infinite association schemes associated with infinite distance-transitive graphs. For this recapitulate that the set of these graphs extend the class of all homogeneous trees only slightly and is parametrized by two parameters. We show how boundary points of these graphs lead to deformations. In Section 11 we present several further standard constructions which lead from given CAS to new ones. Typical examples are direct products and joins, which are well known in the theory of hypergroups by [8, 22].

Section 12 contains an introduction into random walks on X for a CAS (X, D, K); we in particular show that the canonical projections of these random walks to D are random walks on the hypergroup $(D, *)$. This observation may be used to transfer limit theorems for random walks on $(D, *)$ like (strong) laws of large numbers and central limit theorems (see [8, Ch. 7] and references therein) to random walks on X in future. This seems to be interesting in particular for examples which appear as deformations of group CAS, as here random walks on X may be seen as 'radial random walks with additional drift' on the homogeneous space X. Finally, Section 13 contains a short list of central open problems for CAS.

2. Hypergroups

In this section we recapitulate some facts on hypergroups with a focus on the commutative case mainly from [8, 15, 22]. Only some results at the end of this section are new.

Hypergroups form an extension of locally compact groups. For this, remember that the group multiplication on a locally compact group $(G, \cdot)$ leads to the convolution $\delta_x * \delta_y = \delta_{xy}$ $(x, y \in G)$ of point measures. Bilinear, weakly continuous extension of this convolution together with the canonical involution with $\delta_x \mapsto \delta_{x^{-1}}$ then lead to a Banach-$*$-algebra structure on the Banach space $M_b(G)$ of all signed bounded regular Borel measures with the total variation norm $\|.\|_{TV}$ as the norm.

In the case of hypergroups we usually do not have an algebraic operation on the basis space, and we only require a convolution $*$ for bounded complex measures which admits most properties of a group convolution.

Definition 2.1. A hypergroup $(D, *)$ is a locally compact Hausdorff space D with a weakly continuous, associative, bilinear convolution $*$ on the Banach space $M_b(D)$ of all bounded, complex regular Borel measures with the following properties.

(1) For all $x, y \in D$, $\delta_x * \delta_y$ is a compactly supported probability measure on D such that the support supp $(\delta_x * \delta_y)$ depends continuously on x, y with respect to the so-called Michael topology on the space of all compacta in X (see [22] for details).
(2) There exists a neutral element $e \in D$ with $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$ for $x \in D$.
(3) There exists a continuous involution $x \mapsto \bar{x}$ on D such that for all $x, y \in D$, $e \in$ supp $(\delta_x * \delta_y)$ holds if and only if $y = \bar{x}$.
(4) If for $\mu \in M_b(D)$, μ^- denotes the image of μ under the involution, then $(\delta_x * \delta_y)^- = \delta_{\bar{y}} * \delta_{\bar{x}}$ for all $x, y \in D$.

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A hypergroup is called commutative if the convolution $*$ is commutative. It is called symmetric if the involution is the identity.

If D is finite, then Definition 2.1 agrees with that of the introduction.

REMARKS 2.2.

(1) The identity e and the involution $.^{-}$ above are unique.
(2) Each symmetric hypergroup is commutative.
(3) For each hypergroup $(D, *)$, $(M_b(D), *)$ is a Banach-$*$-algebra with the involution $\mu \mapsto \mu^*$ with $\mu^*(A) := \overline{\mu(A^-)}$ for Borel sets $A \subset D$.
(4) For a second countable locally compact space D, the Michael topology agrees with the well-known Hausdorff topology; see [28].

The most prominent examples of hypergroups are double coset hypergroups $G//H := \{HgH : g \in G\}$ for compact subgroups H of locally compact groups G. This extends the discussion in the introduction.

EXAMPLE 2.3. Let H be a compact subgroup of a locally compact group G with identity e and unique normalized Haar measure $\omega_H \in M^1(H) \subset M^1(G)$. Then the space

$$M_b(G\|H) := \{\mu \in M_b(G) : \mu = \omega_H * \mu * \omega_H\}$$

of all H-biinvariant measures in $M_b(G)$ is a Banach-$*$-subalgebra of $M_b(G)$. With the quotient topology, $G//H$ is locally compact, and the canonical projection $p_{G//H} : G \to G//H$ is continuous, proper, and open. Now consider the push forward $\tilde{p}_{G//H} : M_b(G) \to M_b(G//H)$ with $\tilde{p}_{G//H}(\mu)(A) = \mu(p^{-1}_{G//H}(A))$ for $\mu \in M_b(G)$ and Borel sets $A \subset G//H$. Then $\tilde{p}_{G//H}$ is an isometric isomorphism between the Banach spaces $M_b(G\|H)$ and $M_b(G//H)$ with respect to the total variation norms, and the transfer of the convolution on $M_b(G\|H)$ to $M_b(G//H)$ leads to a hypergroup $(G//H, *)$ with identity HeH and involution $HgH \mapsto Hg^{-1}H$, see [22].

The pair (G, H) is called a Gelfand pair if the double coset hypergroup is commutative. For the theory of Gelfand pairs we refer to [13] and [16].

The notion of Haar measures on hypergroups generalizes that on groups.

DEFINITION 2.4. Let $(D, *)$ be a hypergroup, $x, y \in D$, and $f \in C_c(D)$ a continuous function with compact support. We write

$$_xf(y) := f(x * y) := \int_K f \, d(\delta_x * \delta_y) \quad \text{and} \quad f_x(y) := f(y * x)$$

where, by the hypergroup axioms, f_x, $_xf \in C_c(D)$ holds by [22].

A nontrivial positive Radon measure $\omega \in M^+(D)$ is a left or right Haar measure if

$$\int_D {_xf} \, d\omega = \int_D f \, d\omega \quad \text{or} \quad \int_D f_x \, d\omega = \int_D f \, d\omega \quad (f \in C_c(D),\ x \in D)$$

respectively. Thus, ω is called a Haar measure if it is a left and right Haar measure. If $(D, *)$ admits a Haar measure, then it is called unimodular.

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The uniqueness of left and right Haar measures and their existence for particular classes are known by Dunkl, Jewett, and Spector; see [8] for details. The general existence was settled only recently by Chapovsky [11].

Theorem 2.5. *Each hypergroup admits a left and a right Haar measure. Both are unique up to normalization.*

Examples 2.6.

(1) Let $(D, *)$ be a discrete hypergroup. Then, by [22], left and right Haar measures are given by

$$\omega_l(\{x\}) = \frac{1}{(\delta_{\bar{x}} * \delta_x)(\{e\})}, \quad \omega_r(\{x\}) = \frac{1}{(\delta_x * \delta_{\bar{x}})(\{e\})} \qquad (x \in D).$$

Notice that discrete hypergroups are not necessarily unimodular; see, [23] for examples of double coset hypergroups.

(2) If $(G//H, *)$ is a double coset hypergroup and ω_G a left Haar measure of G, then its canonical projection to $G//H$ is a left Haar measure of $(G//H, *)$.

We next recapitulate some facts on Fourier analysis on commutative hypergroups from [8, 22]. For the rest of Section 2 let $(D, *)$ be a commutative hypergroup with Haar measure ω. For $p \geq 1$ consider the L^p-spaces $L^p(D) := L^p(D, \omega)$. Moreover, $C_b(D)$ and $C_0(D)$ are the Banach spaces of all bounded continuous functions on D and those which vanish at infinity respectively. For a function $f : D \to \mathbb{C}$ and $x \in X$ we put $f^-(x) := f(\bar{x})$ and $f^*(x) := \overline{f(\bar{x})}$.

Definitions and facts 2.7.

(1) The spaces of all (bounded) nontrivial multiplicative continuous functions on $(D, *)$ are

$$\chi(D, *) := \{\alpha \in C(D) : \alpha \not\equiv 0, \quad \alpha(x * y) = \alpha(x) \cdot \alpha(y) \quad \text{for all } x, y \in D\}$$

and $\chi_b(D, *) := \chi(D, *) \cap C_b(D)$. Moreover,

$$\hat{D} := (D, *)^\wedge := \{\alpha \in \chi_b(D, *) : \alpha(\bar{x}) = \overline{\alpha(x)} \quad \text{for all } x \in D\}$$

is the dual space of $(D, *)$. Its elements are called characters.
All spaces will be equipped with the topology of compact-uniform convergence. $\chi_b(D, *)$ and $\hat{D}$ are then locally compact.
If D is discrete, then $\hat{D}$ is compact, and if D is compact, then $\hat{D}$ is discrete.
All characters $\alpha \in \hat{D}$ satisfy $\|\alpha\|_\infty = 1$ and $\alpha(e) = 1$.

(2) For $f \in L^1(D)$ and $\mu \in M_b(D)$, their Fourier(–Stieltjes) transforms are defined by

$$\hat{f}(\alpha) := \int_D f(x)\overline{\alpha(x)}\, d\omega(x), \quad \hat{\mu}(\alpha) := \int_D \overline{\alpha(x)}\, d\mu(x) \quad (\alpha \in \hat{D}).$$

We have $\hat{f} \in C_0(\hat{D})$, $\hat{\mu} \in C_b(\hat{D})$ and $\|\hat{f}\|_\infty \leq \|f\|_1$, $\|\hat{\mu}\|_\infty \leq \|\mu\|_{TV}$.

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(3) There exists a unique positive measure $\pi \in M^+(\hat{D})$ such that the Fourier transform $.^\wedge : L^1(D) \cap L^2(D) \to C_0(\hat{D}) \cap L^2(\hat{D}, \pi)$ is an isometry. Thus, π is called the Plancherel measure on $\hat{D}$. The Fourier transform $.^\wedge$ can be extended uniquely to an isometric isomorphism between $L^2(D)$ and $L^2(\hat{D}, \pi)$.
Notice that, different from locally compact abelian groups, the support $S := \operatorname{supp} \pi$ may be a proper closed subset of $\hat{D}$. Quite often, we even have $\mathbf{1} \notin S$.

(4) For $f \in L^1(\hat{D}, \pi)$, $\mu \in M_b(\hat{D})$, their inverse Fourier transforms are given by

$$\check{f}(x) := \int_S f(\alpha)\alpha(x)\, d\pi(\alpha), \quad \check{\mu}(x) := \int_{\hat{D}} \alpha(x)\, d\mu(\alpha) \quad (x \in D)$$

with $\check{f} \in C_0(D)$, $\check{\mu} \in C_b(D)$ and $\|\check{f}\|_\infty \le \|f\|_1$, $\|\check{\mu}\|_\infty \le \|\mu\|_{TV}$.

(5) $f \in C_b(D)$ is called positive definite on the hypergroup D if for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in D$ and $c_1, \ldots, c_n \in \mathbb{C}$, $\sum_{k,l=1}^n c_k \bar{c}_l \cdot f(x_k * \bar{x}_l) \ge 0$. Obviously, all characters $\alpha \in \hat{D}$ are positive definite.

We collect further results.

Facts 2.8.

(1) (Bochner's theorem, [22]) A function $f \in C_b(D)$ is positive definite if and only if $f = \check{\mu}$ for some $\mu \in M_b^+(\hat{D})$. In this case, μ is a probability measure if and only if $\check{\mu}(e) = 1$.

(2) For $f, g \in L^2(D)$, the convolution product $f * g(x) := \int f(x * \bar{y}) g(y)\, d\omega(y)$ $(x \in D)$ satisfies $f * g \in C_0(D)$. Moreover, for $f \in L^2(D)$, $f^*(x) = \overline{f(\bar{x})}$ satisfies $f^* \in L^2(D)$, and $f * f^* \in C_0(D)$ is positive definite; see [8, 22].

(3) (Refining Bochner's theorem, [38]) For a positive definite function $f \in C_b(D)$ with $f = \check{\mu}$ for some $\mu \in M_b^+(\hat{D})$, the following statements are equivalent:
   (i) $\operatorname{supp} \mu \subset S$;
   (ii) f is the compact-uniform limit of positive definite functions in $C_c(D)$;
   (iii) f is the compact-uniform limit of functions of the form $h * h^*$ with $h \in C_c(D)$.

(4) There exists precisely one positive character $\alpha_0 \in S$ by [8, 36].

(5) If $\mu \in M^1(\hat{D})$ satisfies $\check{\mu} \ge 0$ on D, then its support $\operatorname{supp} \mu$ contains at least one positive character; see [39].

In contrast to locally compact abelian groups, pointwise products of positive definite functions on D are not necessarily positive definite; see [22, Section 9.1C] for an example with $|D| = 3$. However, in some cases positive definiteness is preserved under pointwise products.

If for all $\alpha, \beta \in \hat{D}$ (or a subset of $\hat{D}$ like S) the products $\alpha\beta$ are positive definite, then by Bochner's theorem 2.8(1), there are probability measures $\delta_\alpha \hat{*} \delta_\beta \in M^1(\hat{D})$ with $(\delta_\alpha \hat{*} \delta_\beta)^\vee = \alpha\beta$, that is, we obtain dual positive product formulas as claimed in Section 1. Under additional conditions, $(\hat{D}, \hat{*})$ then carries a dual hypergroup structure

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with **1** as identity and complex conjugation as involution. This for instance holds for all compact commutative double coset hypergroups $G//H$ by [15]. For noncompact Gelfand pairs (G, H) it is known that there are dual positive convolutions on S; see [22, 39] for details. These convolutions usually do not generate a dual hypergroup structure. Moreover, it is possible here that $\alpha\beta$ is not positive definite on D for some $\alpha, \beta \in \hat{D}$; see [42] for discrete examples.

The following result extends [39, Theorem 2.1(4)] and is needed below.

Proposition 2.9. *Let $(D, *)$ be a commutative hypergroup. Let $\alpha \in C_b(D)$ be a function on D such that $\alpha \cdot \beta$ is positive definite for each character $\beta \in S$ in the support of the Plancherel measure. Then for each $\beta \in S$ there is a unique measure $\mu \in M_b^+(S)$ with $\alpha \cdot \beta = \check{\mu}$.*

Proof. Fix $\beta \in S$. By 2.8(3) there exists a sequence of positive definite functions f_n in $C_c(D)$ which tend locally uniformly to β. Moreover, again by 2.8(3), each f_n has the form $f_n = \check{\mu}_n$ for some $\mu_n \in M_b^+(S)$. We conclude easily from the assumption of the proposition that the functions $\alpha \cdot f_n = \alpha \cdot \check{\mu}_n = \int_S \alpha\gamma \, d\mu(\gamma) \in C_c(D)$ are positive definite for all n. As these functions tend locally uniformly to $\alpha \cdot \beta$, we obtain from 2.8(3) that $\alpha \cdot \beta = \check{\mu}$ for some $\mu \in M_b^+(S)$ as claimed. □

3. Discrete association schemes

In this section we briefly recapitulate two discrete generalizations of classic finite association schemes from [43] as announced in the introduction. For classic finite association schemes we refer to the monographs [4, 5, 48]. This section is useful to understand the nondiscrete generalization in the next section which is technically more involved. The first extension from the finite to the possibly infinite case is canonical.

Definition 3.1. Let X, D be nonempty, at most countable sets and $(R_i)_{i\in D}$ a disjoint partition of $X \times X$ with $R_i \neq \emptyset$ for $i \in D$ and the following properties.

(1) There exists $e \in D$ with $R_e = \{(x, x) : x \in X\}$.
(2) There exists an involution $i \mapsto \bar{i}$ on D such that for $i \in D$, $R_{\bar{i}} = \{(y, x) : (x, y) \in R_i\}$.
(3) For all $i, j, k \in D$ and $(x, y) \in R_k$, the number

$$p^k_{i,j} := |\{z \in X : (x, z) \in R_i \text{ and } (z, y) \in R_j\}|$$

is finite and independent of $(x, y) \in R_k$.

Then $\Lambda := (X, D, (R_i)_{i\in D})$ is called an association scheme with intersection numbers $(p^k_{i,j})_{i,j,k\in D}$ and identity e.

An association scheme is called commutative if $p^k_{i,j} = p^k_{j,i}$ for all $i, j, k \in D$. It is called symmetric (or hermitian) if the involution on D is the identity. Moreover, it is called finite, if so are X and D.

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FACTS 3.2. Now let $\Lambda := (X, D, (R_i)_{i\in D})$ be an association scheme according to Definition 3.1. Following [5, 43], we form the adjacency matrices $A_i \in \mathbb{R}^{X\times X}$ $(i \in D)$ with

$$(A_i)_{x,y} := \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{otherwise} \end{cases} \qquad (i \in D,\ x, y \in X).$$

The adjacency matrices A_i have the following obvious properties.

(1) A_e is the identity matrix I_X.
(2) $\sum_{i\in D} A_i$ is the matrix J_X whose entries are all equal to 1.
(3) $A_i^T = A_{\bar{i}}$ for $i \in D$.
(4) For all $i \in D$ and all rows and columns of A_i, all entries are equal to zero except for finitely many cases.
(5) For $i, j \in D$, the usual matrix product A_iA_j exists, and $A_iA_j = \sum_{k\in D} p^k_{i,j}A_k$.
(6) Λ is commutative if and only if $A_iA_j = A_jA_i$ for all $i, j \in D$.
(7) Λ is symmetric if and only if all A_i are symmetric.
(8) For $i, j, k \in D$, $p^k_{i,j} = p^{\bar{k}}_{\bar{j},\bar{i}}$.

The valency of R_i or $i \in D$ is defined as

$$\omega_i := p^e_{i,\bar{i}} = |\{z \in X :\ (x, z) \in R_i\}| \in \mathbb{N}$$

where ω_i is independent of $x \in X$. Therefore, the renormalized matrices $S_i := (1/\omega_i)A_i \in \mathbb{R}^{X\times X}$ are stochastic, that is, all row sums are equal to 1. The stochastic matrices S_i satisfy

$$S_iS_j = \sum_{k\in D} \frac{\omega_k}{\omega_i\omega_j} p^k_{i,j}S_k \quad \text{for } i, j \in D. \tag{3.1}$$

We next discuss a property of association schemes which is always valid in the finite case, but not necessarily in infinite cases.

DEFINITION 3.3. An association scheme with valencies ω_i is called unimodular if $\omega_i = \omega_{\bar{i}}$ for all $i \in D$.

We collect some facts from [43, Section 3].

FACTS 3.4.

(1) If an association scheme is commutative or finite, then it is unimodular.
(2) An association scheme is unimodular if and only if the associated discrete hypergroup is unimodular.
(3) If $(X, D, (R_i)_{i\in D})$ is unimodular, then $S_i^T = S_{\bar{i}}$ for all $i \in D$.
(4) There exist nonunimodular association schemes.

The observations above and in particular Equation (3.1) were used in [43, Section 5] for the following generalization of Definition 3.1 of association schemes.

DEFINITION 3.5. Let X, D be nonempty, at most countable sets and $(R_i)_{i\in D}$ a disjoint partition of $X \times X$ with $R_i \neq \emptyset$ for $i \in D$. Let $\tilde{S}_i \in \mathbb{R}^{X\times X}$ for $i \in D$ be stochastic matrices. Assume the following conditions.

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(1) For all $i, j, k \in D$ and $(x, y) \in R_k$, the number

$$p^k_{i,j} := |\{z \in X : \ (x, z) \in R_i \text{ and } (z, y) \in R_j\}|$$

is finite and independent of $(x, y) \in R_k$.

(2) For all $i \in D$ and $x, y \in X$, $\tilde{S}_i(x, y) > 0$ if and only if $(x, y) \in R_i$.

(3) For all $i, j, k \in D$ there exist (necessarily nonnegative) numbers $\tilde{p}^k_{i,j}$ with $\tilde{S}_i\tilde{S}_j = \sum_{k\in D} \tilde{p}^k_{i,j}\tilde{S}_k$.

(4) There exists an identity $e \in D$ with $\tilde{S}_e = I_X$ as the identity matrix.

(5) There exists a positive measure $\omega_X \in M^+(X)$ with $\operatorname{supp} \omega_X = X$ and an involution $i \mapsto \bar{i}$ on D such that for all $i \in D$, $x, y \in X$,

$$\omega_X(\{y\})\tilde{S}_{\bar{i}}(y, x) = \omega_X(\{x\})\tilde{S}_i(x, y).$$

Then $\Lambda := (X, D, (R_i)_{i\in D}, (\tilde{S}_i)_{i\in D})$ is called a generalized association scheme.

Λ is called commutative if $\tilde{S}_i\tilde{S}_j = \tilde{S}_j\tilde{S}_i$ for all $i, j \in D$. It is called symmetric if the involution is the identity. Λ is called finite, if so are X and D.

Remarks 3.6.

(1) If $\Lambda = (X, D, (R_i)_{i\in D})$ is a unimodular association scheme with the associated stochastic matrices $(S_i)_{i\in D}$ as above, then $(X, D, (R_i)_{i\in D}, (S_i)_{i\in D})$ is a generalized association scheme. In fact, axioms (1)–(4) are clear, and for axiom (5) we take the involution of Λ and ω_X as the counting measure on X. Fact 3.4(3) and unimodularity then imply axiom (5). Clearly, notions like commutativity and symmetry are preserved. For details we refer to [43].

(2) If $(X, D, (R_i)_{i\in D}, (\tilde{S}_i)_{i\in D})$ is a generalized association scheme, then $(X, D, (R_i)_{i\in D})$ is an association scheme. Moreover, if this scheme is unimodular, then we may form the two generalized association schemes

$$(X, D, (R_i)_{i\in D}, (\tilde{S}_i)_{i\in D}) \quad \text{and} \quad (X, D, (R_i)_{i\in D}, (S_i)_{i\in D})$$

on the same spaces X, D where the second one is formed as in (1).

For examples of infinite commutative association schemes and of generalized association schemes, which are not association schemes, we refer to [43, 44] and to Section 10 below.

Generalized association schemes always lead to discrete hypergroups; see [43, Proposition 5.4].

Proposition 3.7. *Let $\Lambda := (X, D, (R_i)_{i\in D}, (\tilde{S}_i)_{i\in D})$ be a generalized association scheme with deformed intersection numbers $\tilde{p}^k_{i,j}$. Then the product $\tilde{*}$ with*

$$\delta_i \tilde{*} \delta_j := \sum_{k\in D} \tilde{p}^k_{i,j}\delta_k$$

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can be extended uniquely to an associative, bilinear, $\|\,.\,\|_{TV}$-continuous, probability-preserving mapping on $M_b(D)$. $(D, \tilde{*})$ is a discrete hypergroup with identity e and the involution on D from Definition 3.5(5). $(D, \tilde{*})$ has the left and right Haar measure

$$\Omega_l := \sum_{i\in D} \omega_i \delta_i \quad \text{and} \quad \Omega_r := \sum_{i\in D} \omega_{\bar{i}} \delta_i \quad \text{with } \omega_i := \frac{1}{\tilde{p}^e_{i,\bar{i}}} > 0 \quad (i \in D)$$

respectively. $(D, \tilde{*})$ is commutative or symmetric if and only if so is Λ.

For association schemes there is a corresponding result; see [43, Proposition 3.8]. In fact, the associated hypergroup convolution algebras are just the Bose–Mesner algebras for finite association schemes in [5].

PROPOSITION 3.8. *Let $\Lambda := (X, D, (R_i)_{i\in D})$ be an association scheme with intersection numbers $p^k_{i,j}$ and valencies ω_i. Then the product $*$ with*

$$\delta_i * \delta_j := \sum_{k\in D} \frac{\omega_k}{\omega_i \omega_j} \cdot p^k_{i,j} \delta_k$$

can be extended uniquely to an associative, bilinear, $\|\,.\,\|_{TV}$-continuous mapping on $M_b(D)$. $(D, *)$ is a discrete hypergroup with the left and right Haar measure

$$\Omega_l := \sum_{i\in D} \omega_i \delta_i \quad \text{and} \quad \Omega_r := \Omega_l^* := \sum_{i\in D} \omega_{\bar{i}} \delta_i$$

respectively. $(D, *)$ is commutative, symmetric, or unimodular if and only if so is Λ.

A generalized association scheme is called unimodular if so is the associated hypergroup. Clearly, unimodular association schemes always lead to unimodular generalized association schemes in Remark 3.6(1).

4. Continuous association schemes

In this section we propose and discuss a system of axioms which extends the notion of generalized association schemes above to a continuous setting where we replace the stochastic matrices $(S_i)_{i\in D}$ by Markov-kernels. We briefly recapitulate some well-known notations on Markov-kernels.

DEFINITION 4.1. Let X, Y be locally compact spaces equipped with the associated Borel σ-algebras $\mathcal{B}(X), \mathcal{B}(Y)$. A Markov-kernel K from X to Y is as a mapping $K : X \times \mathcal{B}(Y) \to [0, 1]$ such that the following holds.

(1) For all $x \in X$, the mapping $K(x, .) : \mathcal{B}(Y) \to [0, 1]$, $A \mapsto K(x, A)$ is a probability measure on $(Y, \mathcal{B}(Y))$.
(2) For $A \in \mathcal{B}(Y)$, the mapping $K(., A) : X \to [0, 1]$, $x \mapsto K(x, A)$ is measurable.

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Consider the Banach spaces $F_b(X), F_b(Y)$ of all $\mathbb{C}$-valued bounded measurable functions on X, Y with the supremum norm. Then for any Markov-kernel K from X to Y we define the associated transition operator

$$T_K : F_b(Y) \to F_b(X) \quad \text{with } T_K f(x) := \int_Y f(y)\, K(x, dy) \quad (x \in X).$$

Clearly, K is determined uniquely by the operator T_K.

We say that a Markov-kernel K is continuous if $T_K(C_b(Y)) \subset C_b(X)$. If $T_K(C_0(Y)) \subset C_0(X)$, then K is called a Feller kernel.

Let us also recapitulate the composition

$$K_1 \circ K_2(x, A) := \int_X K_2(y, A)\, K_1(x, dy) \quad (x \in X,\ A \in \mathcal{B}(X))$$

of Markov-kernels K_1, K_2 on X, that is, from X to X. We then have $T_{K_1 \circ K_2} = T_{K_1} \circ T_{K_2}$, and the composition of kernels and transition operators are associative.

A (nontrivial) positive Radon measure $\omega \in M^+(X)$ is called K-invariant with respect to a Markov-kernel K on X, if $\int_B K(x, A)\, d\omega(x) = \omega(A) \in [0, \infty]$ for all Borel sets $A \in \mathcal{B}(X)$.

We now turn to continuous association schemes. We here consider two second countable, locally compact spaces X, D together with a continuous Markov-kernel K from $X \times D$ to X with transition operator

$$T_K : C_b(X) \to C_b(X \times D).$$

For each $h \in D$, we then define the Markov-kernels

$$K_h(x, A) := K(x, h; A) := K((x, h), A) \quad (x \in X, A \in \mathcal{B}(X))$$

on X with transition operators

$$\begin{aligned} T_h :& C_b(X) \to C_b(X), \\ & T_h f(x) := T_{K_h} f(x) = \int_X f(y)\, K(x, h; dy) = T_K f(x, h). \end{aligned}$$

With these notions we now define continuous association schemes. Unfortunately, the definition is more involved than in the discrete case due to the additional continuity assumptions and some other restrictions which are satisfied in the discrete case automatically.

Definition 4.2. Let X and D be second countable, locally compact spaces, and K a continuous Markov-kernel from $X \times D$ to X. Then (X, D, K) is called a continuous association scheme (or, for short, **CAS**), if the following holds.

(1) **(Compact support)** For $x \in X$, $h \in D$, the support supp $K(x, h; .)$ is compact, and the mapping $(x, h) \mapsto \text{supp } K(x, h; .)$ from $X \times D$ into the space $C(X)$ of all compact subsets of X is continuous with respect to the Hausdorff topology on $C(X)$.

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(2) **(Partition property)** For each $x \in X$, the compacta $\operatorname{supp} K(x, h; .)$ $(h \in D)$ form a partition of X, and the associated unique map $\pi : X \times X \to D$ with $y \in \operatorname{supp} K(x, \pi(x, y); .)$ for $x, y \in X$ is continuous.

(3) **(Hypergroup property)** D carries a hypergroup structure $(D, *)$ such that for all $h_1, h_2 \in D$, $x \in X$, and $A \in \mathcal{B}(X)$

$$K_{h_1} \circ K_{h_2}(x, A) = \int_D K_h(x, A)\, d(\delta_{h_1} * \delta_{h_2})(h). \tag{4.1}$$

Moreover, the identity $e \in D$ satisfies $K(x, e; .) = \delta_x$ for $x \in X$.

(4) **(Invariant measure on** X**)** There exists a positive Radon measure $\omega_X \in M^+(X)$ with $\operatorname{supp} \omega_X = X$, such that for the continuous hypergroup involution $\bar{.} : D \to D$ and all $h \in D$, $f, g \in C_c(X)$,

$$\int_X T_{\bar{h}} f \cdot g \, d\omega_X = \int_X f \cdot T_h g \, d\omega_X. \tag{4.2}$$

A CAS (X, D, K) is called commutative, symmetric, or unimodular if so is the hypergroup $(D, *)$. It is called compact or discrete if so are D and X.

Clearly, a CAS (X, D, K) is commutative if and only the Markov-kernels K_h $(h \in D)$ commute.

We present some standard examples of CAS. Further examples are given later on.

Proposition 4.3. *Let* $(X, D, (R_h)_{h\in D}, (\tilde{S}_h)_{h\in D})$ *be a generalized association scheme as in Definition 3.5. Then*

$$K((x, h), A) := \sum_{y\in A} \tilde{S}_h(x, y) \quad (x \in X,\ h \in D,\ A \subset X)$$

defines a Markov-kernel from $X \times D$ *to* X*, and* (X, D, K) *is a CAS.*

Proof. K is a obviously a Markov-kernel from $X \times D$ to X. Moreover, as X and D are discrete, all topological axioms are trivial. Furthermore, fact (4) after Definition 3.1 in combination with Definition 3.5(2) shows that $\operatorname{supp} K((x, h), .)$ is finite for all $x \in X$ and $h \in D$. This shows 4.2(1). Moreover, 4.2(2) is obvious, and 4.2(3) follows from Proposition 3.7. Finally, 4.2(3) follows from 3.5(5). □

Remark 3.6(1) and Proposition 4.3 imply the following corollary.

Corollary 4.4. *Let* $(X, D, (R_h)_{h\in D})$ *be a unimodular association scheme with the stochastic matrices* $(S_h)_{h\in D}$ *as defined after Definition 3.1. Then*

$$K((x, h), A) := \sum_{y\in A} S_h(x, y) \quad (x \in X,\ h \in D,\ A \subset X)$$

defines a Markov-kernel from $X \times D$ *to* X*, and* (X, D, K) *is a unimodular CAS.*

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Remark 4.5. Proposition 4.3 admits the following partial converse statement.

Let (X, D, K) be a discrete CAS. Define the stochastic matrices

$$(\tilde{S}_h)_{x,y} := K_h(x, \{y\}) \quad \text{for } h \in D, \quad x, y \in X$$

as well as the sets $R_h := \{(x, y) \in X \times X : (\tilde{S}_h)_{x,y} > 0\}$. Then the tuple $(X, D, (R_h)_{h\in D}, (\tilde{S}_h)_{h\in D})$ satisfies almost all axioms of a generalized association scheme in Definition 3.5. In fact $(R_h)_{h\in D}$ forms a partition of $X \times X$, and the axioms (2)–(5) of Definition 3.5 hold. We do not know at the moment whether also (1) in Definition 3.5 holds automatically. We come back to this problem later on in the finite case.

Here is a further standard class of examples of CAS.

Proposition 4.6. *Let H be a compact subgroup of a locally compact unimodular group G with normalized Haar measure $\omega_H \in M^1(H) \subset M^1(G)$. Then the quotient $X := G/H$ and the double coset space $D := G//H$ are locally compact with respect to the quotient topology, and the canonical projections*

$$p_G : G \to G/H, \quad p_G(x) := xH, \quad p_{G/H} : G/H \to G//H, \quad p_{G/H}(xH) := HxH$$

are continuous, open, and closed. Moreover,

$$K((xH, HhH), A) := p_G(\delta_x * \omega_H * \delta_h * \omega_H)(A) \quad (x, h \in G, A \in \mathcal{B}(X))$$

establishes a well-defined Markov-kernel from $X \times D$ to X, and (X, D, K) is a unimodular CAS.

Proof. The topological statements about $p_G, p_{G/H}$ are well known from the theory of locally compact groups. We next check that K is a well-defined continuous Markov-kernel, and that (X, D, K) is a CAS. Clearly, by its construction, K is a probability measure with respect to the variable A, and the definition of $K((xH, HhH), A)$ is independent of the representatives x, h of xH and HhH respectively. Before checking that the maps

$$D_A : X \times D \to [0, 1], \quad (x, h) \mapsto K((x, h), A)$$

are measurable for all of Borel set A, we investigate the associated transition operator T_K. For $f \in C_b(X)$ we have that

$$\begin{aligned} T_K f(xH, HhH) &= \int_G f(yH)\, d(\delta_x * \omega_H * \delta_h * \omega_H)(y) \\ &= \int_G \int_G f(xz_1hz_2H)\, d\omega_H(z_1)\, d\omega_H(z_2) \end{aligned} \tag{4.3}$$

is continuous in $x, h \in G$. As the projections $p_G, p_{G/H}$ are open, it follows that the map $(xH, HhH) \mapsto T_K f(xH, HhH)$ is continuous. If we have proved that the maps D_A are measurable for all of Borel set A, we conclude that K is a continuous Markov-kernel as claimed. To prove that the maps D_A are measurable, we first choose a compact set A. As the characteristic function 1_A of A is a monotone limit of functions $f_n \in C_c(X)$,

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we obtain from the theorem of monotone convergence that the continuous functions $(xH, HhH) \mapsto T_K f_n(xH, HhH)$ tend to

$$(xH, HhH) \mapsto T_K 1_A(xH, HhH) = K((xH, HhH), A) = D_A(xH, HhH).$$

This proves that D_A is measurable for A compact. For the general case we use Dynkin systems. In fact,

$$\mathcal{D} := \{A \in \mathcal{B}(X) : \; D_A \text{ is measurable}\}$$

is a Dynkin system which contains the set $\mathcal{K}$ of all compacta in X, where $\mathcal{K}$ is closed under intersections. Therefore, the σ-algebra $\sigma(\mathcal{K})$ and the Dynkin system $\mathcal{D}(\mathcal{K})$ generated by $\mathcal{K}$ satisfy $\mathcal{B}(X) = \sigma(\mathcal{K}) = \mathcal{D}(\mathcal{K}) \subset \mathcal{D}$. Hence, D_A is measurable for all of Borel set A as claimed.

It is now standard to check the axioms (1)–(4) of Definition 4.2. In fact, the compact support in (1) is clear, and the continuity with respect to the Hausdorff topology follows in the same way as in [22] for double coset hypergroups. Moreover, the projection π in (2) is given by $\pi(xH, yH) := Hx^{-1}yH$ and thus continuous, while the partition property in (2) is clear. Furthermore, it is well known that $D = G//H$ is a double coset hypergroup with the convolution

$$\delta_{Hh_1H} * \delta_{Hh_2H} = \int_H \delta_{Hh_1hh_2H}\, d\omega_H(h);$$

see 2.3 and [22] for details. Moreover, for $x, h_1, h_2 \in G$ and Borel sets $A \subset X$,

$$\begin{aligned}
K_{Hh_1H} \circ K_{Hh_2H}(xH, A) &= \int_X K_{Hh_2H}(w, A)\, K_{Hh_1H}(xH, dw) \\
&= \int_G (\delta_y * \omega_H * \delta_{h_2} * \omega_H)(p_G^{-1}(A))\, d(\delta_x * \omega_H * \delta_{h_1} * \omega_H)(y) \\
&= (\delta_x * \omega_H * \delta_{h_1} * \omega_H * \delta_{h_2} * \omega_H)(p_G^{-1}(A))
\end{aligned}$$

and

$$\begin{aligned}
\int_D K_w(xH, A)\, d(\delta_{Hh_1H} * \delta_{Hh_2H})(w) &= \int_H K_{Hh_1hh_2H}(xH, A)\, d\omega_H(h) \\
&= \int_H (p_G(\delta_x * \omega_H * \delta_{h_1hh_2} * \omega_H))(A)\, d\omega_H(h) \\
&= (\delta_x * \omega_H * \delta_{h_1} * \omega_H * \delta_{h_2} * \omega_H)(p_G^{-1}(A)),
\end{aligned}$$

which proves Equation (4.1).

We next turn to 4.2(4). Let ω_G be some Haar measure ω_G of G and take its projection $\omega_X := p_G(\omega_G)$ as the invariant measure on X. To check (4.2), we take $f, g \in C_c(X)$, $h \in G$ and observe from (4.3) and unimodularity that

$$\begin{aligned}
\int_X f \cdot T_h g\, d\omega_X &= \int_G \int_G f(xH) g(xyH)\, d(\omega_H * \delta_h * \omega_H)(y)\, d\omega_G(x) \\
&= \int_G \int_G f(xyH) g(xH)\, d(\omega_H * \delta_{h^{-1}} * \omega_H)(y)\, d\omega_G(x) \\
&= \int_X T_{\bar{h}} f \cdot g\, d\omega_X.
\end{aligned}$$

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This completes the proof of 4.2(4). Finally, as the canonical projection of the Haar measure ω_G to $G//H = D$ is a Haar measure of the double coset hypergroup $(D; *)$ (see [22]), (X, D, K) is unimodular. □

We next proceed with the theory of CAS. We first collect some obvious consequences from Definition 4.2 for a CAS (X, D, K).

FACTS 4.7.

(1) Property 4.2(2) ensures that for $h_1, h_2 \in D$ with $K_{h_1} = K_{h_2}$ we have $h_1 = h_2$. Therefore, the convolution $*$ of $(D, *)$ is determined uniquely by the kernels K_h, $h \in D$.

(2) T_e is the identity operator.

(3) It can be easily shown that the adjoint relation (4.2) holds for further classes of functions. In particular, it can be easily seen that (4.2) holds for all $f \in C_b(X)$ and $g \in L^1(X, \omega_X)$. Taking in particular $f = \mathbf{1}$, then

$$\int_X g\, d\omega_X = \int_X T_h g\, d\omega_X \tag{4.4}$$

for all $h \in D$ and $g \in L^1(X, \omega_X)$. Taking $g = \mathbf{1}_A$ for measurable sets $A \subset X$, we conclude that ω_X is K_h-invariant for all $h \in D$.

(4) For all $h \in D$ and $p \in [1, \infty[$, the operator T_h associated with the kernel K_h on X is a continuous linear operator on $L^p(X, \omega_X)$ with $\|T_h\| \le 1$. In fact, for $f \in L^p(X, \omega_X)$, the Hölder inequality and the invariance of ω_X imply

$$\begin{aligned}\|T_h f\|_p^p &= \int_X \Big| \int_X f(y)\, K_h(x, dy) \Big|^p d\omega_X(x) \\ &\le \int_X \Big(\int_X |f(y)|^p\, K_h(x, dy) \Big) \Big(\int_X 1\, K_h(x, dy) \Big)^{p/q} d\omega_X(x) \\ &= \|f\|_p^p.\end{aligned}$$

(5) For all $h \in D$, T_h clearly is a continuous operator on $(C_b(X), \|.\|_\infty)$.

(6) As $C_c(X)$ is $\|.\|_2$-dense in $L^2(X, \omega_X)$, and as T_h is $\|.\|_2$-continuous, the adjoint relation (4.2) implies that $T_{\bar{h}}$ is the adjoint operator T_h^* on $L^2(X, \omega_X)$.

(7) For all $x, y \in X$, $\pi(y, x) = \overline{\pi(x, y)}$.
In fact, for each $h \in D$, $\pi(x, y) \neq h$ means that there is a neighborhood U_y of y in x such that for all $g \in C_c(X)$ with supp $g \subset U_y$ we have $T_h g(x) = 0$. As $T_h g$ is continuous by our assumptions, we conclude that $\pi(x, y) \neq h$ is equivalent to the fact that for all $\varepsilon > 0$ there are neighborhoods U_x, U_y of x, y respectively such that for all $f, g \in C_c(X)$ with supp $f \subset U_x$, supp $g \subset U_y$, and with $\|f\|_{1,\omega_X} = \|g\|_{1,\omega_X} = 1$ we have $|\int_X f(w)\, T_h g(w)\, d\omega_X(w)| \le \varepsilon$. This and the adjoint relation 4.2(4) lead to the claim.

We next study some topological properties of π. For this recapitulate that in our setting, the Hausdorff topology on $C(X)$ agrees with the so-called Michael topology

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in [22] or [8, Section 1.1]; see, for example, [28]. In particular, by these references (see in particular (2.5F) of [22]), $C(X)$ is locally compact, and for each compactum $\Omega \subset C(X)$, the set $\bigcup_{A\subset\Omega} A \subset X$ is compact. With these preparation we obtain the following lemma.

Lemma 4.8.

(1) *For compact sets $K \subset X$ and $L \subset D$, the set $\bigcup_{h\in L, x\in K} \operatorname{supp} K_h(x,.) \subset X$ is compact.*

(2) *For each $x \in X$, the projection $\pi_x : X \to D$, $\pi_x(y) := \pi(x,y)$ is open, closed, and proper, that is, $\pi_x^{-1}(A) \subset X$ is compact for each compactum $A \subset D$.*

Proof. Part (1) follows from 4.2(1) and the remark about $C(X)$ above.

These facts also imply that π_x is proper, as for each compactum $A \subset D$, the set

$$\pi_x^{-1}(A) = \bigcup_{y\in A} \pi_x^{-1}(y) = \bigcup_{y\in A} \operatorname{supp} K_h(x,.)$$

is compact. Problem 5 of Section XI.6 of [14] now implies that π_x is also closed.

We finally show that π_x is open. For this assume that there is some neighborhood $U \subset X$ of some $y \in X$ such that $\pi_x(U) \subset D$ is no neighborhood of $\pi_x(y)$. This means that there is a sequence $(h_n)_n \subset D \setminus \pi_x(U)$ with $h_n \to \pi(x,y)$. Hence, by 4.2(1), the compacta $\pi_x^{-1}(h_n) = \operatorname{supp} K_{h_n}(x,.)$ tend to $\pi_x^{-1}(\pi(x,y))$. On the other hand, $\pi_x^{-1}(h_n) \cap U = \emptyset$, and $\pi_x^{-1}(\pi(x,y))$ contains $x \in U$. This leads to a contradiction. Hence, π_x is open. □

Lemma 4.9. *Each $g \in C_0(X)$ is uniformly continuous in the sense that for each $\varepsilon > 0$ there exists a neighborhood $U \subset D$ of the identity e such that for all $x, y \in X$ with $\pi(x,y) \in U$, $|g(x) - g(y)| \le \varepsilon$.*

Proof. The proof is similar to a corresponding result for hypergroups; see, for example, [8, 1.2.28].

Fix some $\varepsilon > 0$. Choose some compactum $G \subset X$ such that $|g(x)| \le \varepsilon/2$ for $x \in X \setminus G$. For each $x \in X$ we take some open neighborhood $W_x \subset X$ with $|g(y) - g(x)| \le \varepsilon/2$ for $y \in W_x$. Now choose open neighborhoods $\tilde{U}_x \subset D$ of e with $\{y \in X : \pi(x,y) \in \tilde{U}_x\} \subset W_x$. By a basic result on hypergroups we find open symmetric neighborhoods $U_x \subset D$ of e with $U_x * U_x \subset \tilde{U}_x$. We now consider the open set $V_x := \{y \in X : \pi(x,y) \in U_x\}$ which cover the compactum G. Choose $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$ with $G \subset \bigcup_{l=1,\ldots,n} V_{x_l}$, and define the open neighborhood $U := \bigcap_{l=1,\ldots,n} U_{x_l} \subset D$.

Now consider $x \in G$ and $y \in X$ with $\pi(x,y) \in U$. We find l with $x \in V_{x_l} \subset W_{x_l}$. As then $\pi(x_l, x) \in U_{x_l}$ and $\pi(x,y) \in U_{x_l}$, we obtain $\pi(x_l, y) \in U_{x_l} * U_{x_l} \subset \tilde{U}_{x_l}$, which implies $y \in W_{x_l}$. Hence, by the definition of W_{x_l}, $|g(y) - g(x)| \le \varepsilon$. As this also holds for all $x, y \in X \setminus G$, the proof is complete. □

Lemma 4.10. *Let $h \in D$. Then the following conditions hold:*

(1) *if $f \in C_c(X)$, then $T_h f \in C_c(X)$;*

(2) *if $f \in C_0(X)$, then $T_h f \in C_0(X)$; in other words, the kernels K_h on X are Feller-kernels.*

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PROOF. The continuity of $T_h f$ is clear in both cases.

Now let $f \in C_c(X)$. Let $x \in X$ with $T_h f(x) \neq 0$. Then, by the definition of the projection π, $\operatorname{supp} f \cap \{y \in X : \pi(x, y) = h\} \neq \emptyset$, and thus, by 4.7(7), $\operatorname{supp} f \cap \{y \in X : \pi(y, x) = \bar{h}\} \neq \emptyset$. This yields $x \in \bigcup_{y \in \operatorname{supp} f} \operatorname{supp} K_{\bar{h}}(y, .)$. As this set is compact by Lemma 4.8, we obtain $T_h f \in C_c(X)$.

Part (2) follows from part (1) and the continuity of T_h with respect to $\|.\|_\infty$. □

We next study integrals over the operators T_h.

LEMMA 4.11. *Let $\mu \in M_b(D)$.*

(1) *For each $f \in C_b(X)$,*

$$T_\mu f(x) := \int_D T_h f(x)\, d\mu(h) \quad (x \in X)$$

defines a function $T_\mu f \in C_b(X)$. The operator T_μ is a continuous linear operator on $C_b(X)$ with $\|T_\mu\| \leq \|\mu\|_{TV}$.

(2) *If* supp *μ is compact, then T_μ maps $C_c(X)$ into $C_c(X)$.*

(3) *The operator T_μ maps $C_0(X)$ into $C_0(X)$.*

(4) *For each $p \in [1, \infty[$, the operator T_μ from (1) may be also regarded as a continuous linear operator on $L^p(X, \omega_X)$ with $\|T_\mu\| \leq \|\mu\|_{TV}$.*

PROOF.

(1) $T_\mu f$ is continuous by Definition 4.2. The further statements are clear.

(2) Follows from Lemma 4.8 in the same way as Lemma 4.10(1).

(3) If $\operatorname{supp} \mu$ is compact, then (3) follows from (2) and the continuity of T_μ. On the other hand, for each $\varepsilon > 0$ and $\mu \in M_b(D)$ there exists a measure $\mu_\varepsilon \in M_b(D)$ with compact support and $\|\mu - \mu_\varepsilon\|_{TV} \leq \varepsilon$. Hence $\|T_\mu - T_{\mu_\varepsilon}\| \leq \varepsilon$. Thus, $T_\mu f$ is a uniform limit of functions in $C_0(X)$ which yields the claim.

(4) This follows from 4.7(4) and standard facts on operator-valued integrals. □

We now consider the C^*-algebra $\mathcal{B}(L^2(X, \omega_X))$ of all bounded linear operators on $L^2(X, \omega_X)$ as well as the closed subspace

$$A(X) := \overline{\operatorname{span}\{T_h : h \in D\}}.$$

The space $A(X)$ is closed under the involution $.^*$ on $\mathcal{B}(L^2(X, \omega_X))$ by 4.7(5). Moreover, by Lemma 4.11(4), we have $T_\mu \in A(X)$ for all $\mu \in M_b(D)$. In summary, the following proposition holds.

PROPOSITION 4.12.

(1) *$A(X)$ is a C^*-subalgebra of $\mathcal{B}(L^2(X, \omega_X))$.*

(2) *The map*

$$T : (M_b(D), *, .^*, \|.\|_{tv}) \to A(X) \subset \mathcal{B}(L^2(X, \omega_X)), \quad \mu \mapsto T_\mu,$$

is a norm-decreasing Banach-$*$-algebra homomorphism, that is, T is a $*$-representation of the hypergroup $(D, *)$ on the Hilbert space $L^2(X, \omega_X)$.

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Proof.

(1) Let $h_1, h_2 \in D$. Then, by 4.2(3), $T_{h_1} T_{h_2} = T_{\delta_{h_1} * \delta_{h_1}} \in A(X)$. This yields that $A(X)$ is closed under multiplication. All further facts are clear.

(2) Is also clear by the same arguments and Lemma 4.11. □

We next study a couple of linear operators $A : C_c(X) \to C(X)$. For each such A we form the scalar products

$$\langle Ag_1, g_2 \rangle_X := \int_X Ag_1(x) \cdot \overline{g_2(x)}\, d\omega_X(x) \quad \text{for } g_1, g_2 \in C_c(X). \tag{4.5}$$

Some examples are as follows: let $F \in C(X \times X)$ and form $T^F : C_c(X) \to C(X)$ by

$$T^F g(x) := \int_X F(x, y)\, g(y)\, d\omega_X(y) \quad (x \in X,\ g \in C_c(X)).$$

Moreover, for each $\mu \in M_b(D)$, $A := T_\mu$ is an operator as in (4.5).

We now fix some left Haar measure ω_D of the hypergroup $(D, *)$. Then $L^1(D, \omega_D)$ is a Banach-$*$-algebra with the convolution and involution

$$f * g(x) := \int_D f(x * y) g(\bar{y})\, d\omega_D(y), \qquad f^*(x) = \overline{f(\bar{x})} \quad (x \in D).$$

Moreover, the map $L^1(D, \omega_D) \to M_b(D)$, $f \mapsto f\, \omega_D$ is an embedding of the Banach-$*$-algebra $L^1(D, \omega_D)$ into the Banach-$*$-algebra $M_b(D)$; see [22]. For each $f \in L^1(D, \omega_D)$ we thus may define the linear operators $T_f := T_{f\,\omega_D}$, for which the results of Lemma 4.11 and Proposition 4.12 hold.

Moreover, even for $f \in C(D)$, the linear operators $T_f : C_c(X) \to C(X)$ with

$$T_f g(x) := \int_D \int_X g(y)\, K_h(x, dy)\, f(h)\, d\omega_D(h) \quad (x \in X,\ g \in C_c(X))$$

are well-defined, as by 4.2(2) for $g \in C_c(X)$, the set $\pi(x, \operatorname{supp} g) \subset D$ is compact. It is also clear that for $f \in C(D)$ and $f_1, f_2 \in C_c(D)$, we have $f_1 * f * f_2 \in C(D)$ with $T_{f_1 * f * f_2} = T_{f_1} T_f T_{f_2}$. We shall use these facts for relations between positive definite functions on $(D, *)$ and X in Section 6. For this we need additional properties for CAS, which we discuss in the next section.

Before doing this, we study the linear operators T_α for multiplicative functions $\alpha \in C(D)$, that is, $\alpha(h * l) = \alpha(h) \cdot \alpha(l)$ for all $h, l \in D$.

Lemma 4.13. *Let (X, D, K) be a CAS, $g \in C_c(X)$, and let $\alpha \in C(D)$ be multiplicative. Then the function $\varphi := T_\alpha g \in C(X)$ satisfies the equation $T_l \varphi(x) = \alpha(\bar{l}) \cdot \varphi(x)$ for all $l \in D$ and $x \in X$.*

Proof. Let ω_D be a left Haar measure of $(D, *)$ as above. Fix $l \in D$, $x \in X$ and consider the function $g_x(w) := \int_X g(y)\, K_w(x, dy)$ $(w \in D)$. Then $g_x \in C_c(D)$ by the considerations above, and, by [22, Lemma 5.5G],

$$\begin{aligned}\int_D \int_D g_x(h)\, d(\delta_l * \delta_w)(h)\, \alpha(w)\, d\omega_D(w) &= \int_D g_x(l * w) \alpha(w)\, d\omega_D(w) \\ &= \int_D g_x(w) \alpha(\bar{l} * w)\, d\omega_D(w)\end{aligned}$$

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$$= \alpha(\bar{l}) \int_D g_x(w)\alpha(w)\, d\omega_D(w)$$
$$= \alpha(\bar{l}) T_\alpha g(x).$$

Hence, as claimed,

$$\begin{aligned} T_l(T_\alpha g)(x) &= \int_D \int_X \int_X g(y)\, K_h(z, dy)\, K_l(x, dz)\, \alpha(h)\, d\omega_D(h) \\ &= \int_D \int_D \int_X g(y)\, K_h(x, dy)\, d(\delta_l * \delta_w)(h)\, \alpha(w)\, d\omega_D(w) \\ &= \int_D \int_D g_x(h)\, d(\delta_l * \delta_w)(h)\, \alpha(w)\, d\omega_D(w) = \alpha(\bar{l}) T_\alpha g(x). \end{aligned}$$ □

Lemma 4.13 can be applied to the uniqueness of the adjoint measure $\omega_X \in M^+(X)$ at least in the compact case.

Lemma 4.14. *Let (X, D, K) be a compact CAS. Then the following conditions hold.*

(1) *If $g \in C(X)$ satisfies $T_h g = g$ for all $h \in D$, then g is constant.*
(2) *For all $g \in C(X)$, $T_{\mathbf{1}} g$ is constant, and there is a unique measure $\omega \in M_b^+(X)$ with $\omega(A) = \int_D K_h(x, A)\, d\omega_D(h)$ for all Borel sets $A \subset X$, where the right-hand side is independent of $x \in X$.*
(3) *The measure ω from (2) is equal to the adjoint measure $\omega_X \in M^+(X)$ from 4.2(4) up to a positive multiplicative constant. In particular, ω_X is unique up to a positive multiplicative constant.*

Proof. For (1) assume without loss of generality that g is $\mathbb{R}$-valued with $T_h g = g$ for all $h \in D$. Take $x_0 \in X$ such that $g(x_0)$ is maximal. As $g(x_0) = \int_X g(y)\, K_h(x_0, dy)$ we see that $g(y) = g(x_0)$ for all $y \in \operatorname{supp} K_h(x_0, .)$. As this holds for all $h \in D$, it follows that g is constant.

For (2) we conclude from Lemma 4.13 for $\alpha = \mathbf{1}$ that for $l \in D$, we have $T_l(T_{\mathbf{1}} g) = T_{\mathbf{1}} g(x)$. Hence, by part (1), $T_{\mathbf{1}} g$ is constant. As

$$T_{\mathbf{1}} g(x) = \int_D \int_X g(y)\, K_w(x, dy)\, d\omega_D(w),$$

we see from the representation theorem of Riesz that there exists a unique measure $\omega \in M_b^+(X)$ with $T_{\mathbf{1}} g(x) = \int_X g\, d\omega$ for all $g \in C(X)$ and $x \in X$.

For (3) we assume without loss of generality that $\omega_X \in M^1(X)$ and $\omega_D \in M^1(D)$. We use the invariance condition (4.4) for ω_X and obtain for $g \in C(X)$ and each $x \in X$ that

$$\int_X g\, d\omega = T_{\mathbf{1}} g(x) = \int_X T_{\mathbf{1}} g\, d\omega_X = \int_D \int_X T_h g(x)\, d\omega_X(x)\, d\omega_D(h) = \int_X g\, d\omega_X.$$

This proves $\omega_X = \omega$ and the claim. □

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5. Strong continuous association schemes

Let (X, D, K) be a CAS with associated hypergroup $(D, *)$. We first recapitulate the translations $f_h(l) := f(l * h) := \int_D f\, d(\delta_l * \delta_h)$ of functions $f \in C(D)$ for $h, l \in D$ as well as the projection maps $\pi_x : X \to D$ for $x \in X$ from the preceding section.

Definition 5.1. Let (X, D, K) be a CAS.

(1) We say that (X, D, K) has the translation property (T1) if for all $h \in D$, $x \in X$, and $f \in C_c(D)$,

$$f_h \circ \pi_x = T_h(f \circ \pi_x).$$

(2) We say that (X, D, K) has the translation property (T2), if for all $f \in C_c(D)$, $T_f = T^{f\circ\pi}$, where we assume that the invariant measure ω_X and the left Haar measure ω_D of $(D, *)$ are chosen with suitable normalizations.

(3) We say that (X, D, K) is strong, if (T1) and (T2) hold.

We shall prove below that in the discrete case and in the commutative case, property (T2) implies (T1); see Theorems 5.4 and 8.5. This indicates that generally, (T2) seems to be the stronger condition. Otherwise we do not know further relations between these conditions. In several sections below we present examples where (T1) and (T2) do not hold. On the other hand, there are several standard classes of strong CAS; here is the first one.

Proposition 5.2. *Let H be a compact subgroup of a locally compact unimodular group G. Then the associated unimodular CAS*

$$(X := G/H, D := G//H, K)$$

as in Proposition 4.6 is strong.

Proof. Let $x, y, h \in G$, and $f \in C_c(G//H)$. The proof of Proposition 4.6 yields

$$(f_{HhH} \circ \pi_{xH})(yH) = \int_{G//H} f\, d(\delta_{Hx^{-1}yH} * \delta_{HhH}) = \int_H f(Hx^{-1}ywhH)\, d\omega_H(w)$$

and

$$\begin{aligned} T_{HhH}(f \circ \pi_{xH})(yH) &= \int_X f(Hx^{-1}zH)\, K_{HhH}(y, d(zH))) \\ &= \int_H f(Hx^{-1}ywhH)\, d\omega_H(w). \end{aligned}$$

This proves (T1). Moreover, for $f \in C_c(G//H)$, $g \in C_c(G/H)$, and $x \in G$ we have with the notations of Proposition 4.6 that

$$\begin{aligned}
T_f g(xH) &= \int_D \int_X g(yH) K_{HhH}(xH, yH)\, f(HhH)\, d\omega_{G//H}(HhH) \\
&= \int_G \int_X g(yH)\, d(p_G(\delta_x * \omega_H * \delta_h * \omega_H))(yH)\, f(HhH)\, d\omega_G(h) \\
&= \int_H \int_G g(xrhH)\, f(HhH)\, d\omega_G(h)\, d\omega_H(r) \\
&= \int_H \int_G g(xhH)\, f(Hr^{-1}hH)\, d\omega_G(h)\, d\omega_H(r) \\
&= \int_G g(xhH)\, f(HhH)\, d\omega_G(h) \\
&= \int_G f(Hx^{-1}yH)\, g(yH)\, d\omega_G(y) \\
&= \int_D f(\pi(xH, yH))\, g(yH)\, d\omega_{G/H}(yH) = T^{f\circ\pi} g(xH)
\end{aligned}$$

which proves (T2). □

Here is a second standard class of strong CAS.

Proposition 5.3. *Let $(X, D, (R_i)_{i\in D})$ be a unimodular association scheme. Then the associated unimodular discrete CAS (X, D, K) of Corollary 4.4 is strong.*

Proof. By linearity, it suffices to check (T1) for characteristic functions $f = \mathbf{1}_{\{\mathbf{r}\}}$ with $r \in D$. For $h \in D$ and $x, y \in X$ we obtain from the axioms and basic properties of an association scheme and the definition of the kernels K_h that

$$\begin{aligned}
T_h(\mathbf{1}_{\{r\}} \circ \pi_x)(y) &= \int_X \mathbf{1}_{\{r\}}(\pi(x, z))\, K_h(y, dz) \\
&= K_h(y, \{z \in X : \pi(x, z) = r\}) \\
&= \frac{1}{\omega_h} |\{z \in X : \pi(x, z) = r,\ \pi(y, z) = h\}| \\
&= \frac{1}{\omega_h} |\{z \in X : \pi(z, x) = \bar{r},\ \pi(y, z) = h\}| \\
&= \frac{1}{\omega_h} p^{\pi(y,x)}_{h,\bar{r}} = \frac{1}{\omega_h} p^{\pi(x,y)}_{r,\bar{h}}.
\end{aligned}$$

On the other hand, we see from [43, Proposition 3.8 and Lemma 3.5(4)] that

$$\begin{aligned}
(f_h \circ \pi_x)(y) &= \int_D \mathbf{1}_{\{r\}}\, d(\delta_{\pi(x,y)} * \delta_h) = (\delta_{\pi(x,y)} * \delta_h)(\{r\}) \\
&= \frac{\omega_r}{\omega_h \omega_{\pi(x,y)}} p^r_{\pi(x,y),h} = \frac{1}{\omega_h} p^{\pi(x,y)}_{r,\bar{h}},
\end{aligned}$$

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which completes the proof of (T1). For (T2) we again use linearity and check (T2) for $f = \mathbf{1}_{\{\mathbf{r}\}}$ and $g = \mathbf{1}_{\{\mathbf{z}\}}$ with $r \in D$ and $z \in X$. Let $x \in X$. With the Kronecker-δ we obtain

$$T_f g(x) = \omega_r \cdot \int_X g(y)\, K_r(x, dy) = \delta_{r,\pi(x,z)} = T^{f\circ\pi} g(x)$$

which yields the claim (T2). □

Proposition 5.3 has the following converse statement.

THEOREM 5.4. *Let (X, D, K) be a discrete unimodular CAS with property (T2). Then there is a unimodular association scheme $(X, D, (R_r)_{r\in D})$ such that (X, D, K) is the associated CAS according to Corollary 4.4. In particular, for discrete unimodular CAS, (T2) implies (T1).*

PROOF. Assume that the measure $\omega_X \in M^+(X)$ and the Haar measure ω_D are normalized such that (T2) holds. Let $r \in D$ and $x, z \in X$ and put $f = \mathbf{1}_{\{\mathbf{r}\}}$ and $g = \mathbf{1}_{\{\mathbf{z}\}}$. Then, as in the proof of the preceding result, $T^{f\circ\pi} g(x) = \omega_X(\{z\})\, \delta_{r,\pi(x,z)}$ and

$$T_f g(x) = K_r(x, \{z\})\, \omega_D(\{r\}) = K_r(x, \{z\})\, \omega_D(\{r\})\, \delta_{r,\pi(x,z)}.$$

Hence, by (T2),

$$K_r(x, \{z\}) = \frac{\omega_X(\{z\})}{\omega_D(\{r\})}\, \delta_{r,\pi(x,z)} \quad (x, z \in X,\ r \in D). \tag{5.1}$$

This in particular shows that for $r \in D$ and $x \in X$,

$$\omega_D(\{r\}) = \sum_{z\in X:\, \pi(x,z)=r} \omega_X(\{z\}) \tag{5.2}$$

and, as $K_r(x, .)$ is a probability measure,

$$(K_r \circ K_{\bar{r}})(x, \{x\}) = \sum_{z\in X:\, \pi(x,z)=r} \frac{\omega_X(\{z\})}{\omega_D(\{r\})} \cdot \frac{\omega_X(\{x\})}{\omega_D(\{\bar{r}\})}\, \delta_{\bar{r},\pi(z,x)} = \frac{\omega_X(\{x\})}{\omega_D(\{\bar{r}\})}. \tag{5.3}$$

As by the definition of a discrete CAS $K_r \circ K_{\bar{r}}$ is a finite convex combination of the K_s $(s \in D)$ where the identity kernel K_e appears with a positive coefficient, we conclude from (5.3) that $\omega_X(\{x\}) > 0$ is independent of $x \in X$. Therefore, after renormalization of ω_X and ω_D, we may assume that ω_X is the counting measure. We then see from (5.2) that $\omega_D(\{r\}) = |\mathrm{supp}\, K_r(x, .)|$ for $r \in D$ and all $x \in X$. Moreover, by (5.1),

$$K_r(x, \{z\}) = \frac{\delta_{r,\pi(x,z)}}{\omega_D(\{r\})} \quad (r \in D,\ x, z \in X). \tag{5.4}$$

We now define the partition $(R_r)_{r\in D}$ of $X \times X$ via

$$R_r := \{(x, y) :\ y \in \mathrm{supp}\, K_r(x, .)\}.$$

This is a partition by the partition property of a CAS for which clearly property (1) of 3.1 holds. Moreover, 3.1(2) follows from (5.4) and $\omega(\{\bar{r}\}) = \omega(\{r\})$. We finally

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check 3.1(3). For this let $i, j, k \in D$ and $x, y \in X$ with $\pi(x,y) = k$. Then

$$K_i \circ K_j(x, \{y\}) = \sum_{z \in X : \pi(x,z)=i,\, \pi(z,y)=j} \frac{1}{\omega_D(\{i\})\omega_D(\{j\})}$$
$$= \frac{|\{z \in X : \pi(x,z) = i,\ \pi(z,y) = j\}|}{\omega_D(\{i\})\omega_D(\{j\})}$$

and

$$K_i \circ K_j(x, \{y\}) = (\delta_i * \delta_j)(\{k\}) \cdot K_k(x, \{y\}) = \frac{(\delta_i * \delta_j)(\{k\})}{\omega_D(\{k\})}.$$

A comparison of both formulas shows that

$$|\{z \in X : \pi(x,z) = i,\ \pi(z,y) = j\}|$$

depends only on $\pi(x,y) = k$ and not on the choice of x, y as claimed.

In summary, we see that $(X, D, (R_r)_{r\in D})$ is an association scheme with (X, D, K) as associated CAS by (5.4). The last statement of the theorem follows from Proposition 5.3. □

In summary, in the unimodular case, strong discrete CAS are precisely classical association schemes. Moreover, discrete commutative CAS may be seen as generalizations of generalized association schemes.

Theorem 5.4 suggests that in general (T2) implies (T1). Unfortunately, we do not see any approach for the proof of this conjecture in the nondiscrete case.

We next rewrite (T2) as the following lemma.

Lemma 5.5. *Let (X, D, K) be a unimodular CAS with the Haar measure ω_D of $(D, *)$ and the adjoint measure $\omega_X \in M^+(X)$. Then (T2) holds if and only if $\omega_X(A) = \int_D K_h(x, A)\, d\omega_D(h)$ for all Borel sets $A \subset X$ and $x \in X$.*

In particular, for each unimodular CAS with (T2), ω_X is unique up to a positive constant.

Proof. Assume first that (T2) holds. It can be easily seen (see Lemma 5.9 below) that then for all $g \in C_c(X)$ and $x \in X$, $T^{\mathbf{1}\circ\pi} g = T_{\mathbf{1}} g$ and thus

$$\int_D \int_X g(y)\, K_h(x, dy)\, d\omega_D(h) = T_{\mathbf{1}} g(x) = T^{\mathbf{1}\circ\pi} g(x) = \int_X g(y)\, d\omega_X(y).$$

This shows that $\omega_X = \int_D K_h(x, .)\, d\omega_D(h)$ for $x \in X$.

Conversely, this representation of ω_X shows for $f \in C_c(D)$, $g \in C_c(X)$, and $x \in X$ that

$$T^{f\circ\pi} g(x) = \int_X g(y) \cdot f(\pi(x,y))\, d\omega_X(y)$$
$$= \int_D \int_X g(y) \cdot f(\pi(x,y))\, K_h(x, dy)\, d\omega_D(h)$$
$$= \int_D \int_X g(y) \cdot f(h)\, K_h(x, dy)\, d\omega_D(h) = T_f g(x).$$

Hence, (T2) holds. The uniqueness assertion is clear. □

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Lemmas 5.5 and 4.14 now lead to the following proposition.

PROPOSITION 5.6. *Each compact CAS has property (T2).*

If we combine Proposition 5.6 with Theorem 5.4 and Proposition 4.3, we obtain the following classification of finite CAS.

THEOREM 5.7. *Each finite CAS (X, D, K) is related to an association scheme according to Corollary 4.4. In particular, each finite generalized association scheme is in fact an association scheme.*

We notice that this classification does not hold in the infinite case. Examples are given in [44] and in Section 10 below.

We now return to (T1) and (T2) and study CAS with the below properties.

LEMMA 5.8. *Let (X, D, K) be a CAS with (T1). Then the following hold.*

(1) *For all $x \in X$, the push forward $\pi_x(\omega_X) \in M^+(D)$ is a right Haar measure of $(D, *)$.*
(2) *For all $\mu \in M_b(D)$, $f \in C_c(D)$ and $x \in X$, $T_\mu(f \circ \pi_x) = (f * \mu^-) \circ \pi_x$.*
(3) *For all $\varphi \in C(D)$, $f \in C_c(D)$ and $x \in X$, $T_\varphi(f \circ \pi_x) = (f * \varphi^-) \circ \pi_x$.*

PROOF.

(1) For all $h \in D$ and $f \in C_c(D)$ we obtain from 4.7(3) that, as claimed,

$$\begin{aligned}(\pi_x(\omega_X))(f_h) &= \omega_X(f_h \circ \pi_x) = \omega_X(T_h(f \circ \pi_x)) \\ &= \omega_X(f \circ \pi_x) = (\pi_x(\omega_X))(f).\end{aligned}$$

(2), (3) Follow simply by integration of the equation in Definition 5.1. □

LEMMA 5.9. *Let (X, D, K) be a CAS with (T2). Then the following hold.*

(1) *For all $x \in X$, the push forward $\pi_x(\omega_X) \in M^+(D)$ is a right Haar measure of $(D, *)$.*
(2) *For all $f \in C(D)$ and $g \in C_c(X)$, $T_f g = T^{f\circ\pi} g \in C(X)$.*
(3) *For all $f \in C_c(D)$ and $g \in C(X)$, $T_f g = T^{f\circ\pi} g \in C(X)$.*

PROOF. (2) is clear, and (3) follows from Lemma 4.8 similar to the proof of Lemma 4.10. For the proof of (1) we use (3) with $g \equiv 1$ and $f \in C_c(D)$. Hence, for $x \in X$,

$$\begin{aligned}\int_X f(\pi(x, y))\, d\omega_X(y) &= T^{f\circ\pi} g(x) = T_f g(x) \\ &= \int_D \int_X 1\, K_h(x, dy)\, f(h)\, d\omega_D(h) = \int_D f\, d\omega_D(h)\end{aligned}$$

which proves the claim. □

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Remark 5.10. There exist commutative CAS without (T1) and (T2). For this consider the discrete generalized association schemes associated with homogeneous trees of [44] or Section 10 below with the parameter $c \neq 1$ there. As shown in [44, Remark 2.4] or Remark 10.3 below, there the measure ω_X is unique up to a positive multiplicative constant for which the push forward statements of Lemmas 5.8(1) and 5.9(1) are not correct. This means that (T1) and (T2) do not hold there; see Remark 10.3 for the details.

Lemma 5.11. *Let (X, D, K) be a strong unimodular CAS. Let $f \in C_c(D)$ and $f \in C(D)$, or $f \in C(D)$ and $g \in C_c(D)$, or $f, g \in L^2(D, \omega_D)$. Then, for all $x, z \in X$,*

$$\int_X f(\pi(x,y))\,\overline{g(\pi(z,y))}\,d\omega_X(y) = \int_D f(h)\,\overline{g(\pi(z,x) * h)}\,d\omega_D(h).$$

Proof. Let $f \in C_c(D)$ and $f \in C(D)$. Then, by (T2) and Lemma 5.8(3),

$$\int_X f(\pi(x,y))\,\overline{g(\pi(z,y))}\,d\omega_X(y) = T^{f\circ\pi}(\overline{g\circ\pi_z})(x) = T_f(\overline{g\circ\pi_z})(x)$$
$$= ((f^- * \bar{g})\circ\pi_z)(x) = (f^- * \bar{g})(\pi(z,x)) = \int_D f(h)\,\overline{g(\pi(z,x) * h)}\,d\omega_D(h)$$

as claimed. The same computation works for $f \in C(D)$ and $g \in C_c(D)$ as well as for $f, g \in L^2(D, \omega_D)$ by density. Notice here that due to Lemma 5.8(1), for all $z \in X$ the map $f \mapsto f \circ \pi_z$ is an L^2-isometry from $L^2(D, \omega_D)$ into $L^2(X, \omega_X)$. □

We next present some orthogonality result which is well known in the group case.

Corollary 5.12. *Let (X, D, K) be a compact, commutative strong CAS. Then for $\alpha, \beta \in (D, *)^\wedge$ and $x, z \in X$,*

$$\int_X \alpha(\pi(x,y))\,\overline{\beta(\pi(z,y))}\,d\omega_X(y) = \delta_{\alpha,\beta}\cdot\overline{\alpha(\pi(z,x))}\cdot\|\alpha\|^2_{2,\omega_D}.$$

Proof. Lemma 5.11 yields

$$\int_X \alpha(\pi(x,y))\,\overline{\beta(\pi(z,y))}\,d\omega_X(y) = \int_D \alpha(h)\,\overline{\beta(\pi(z,x) * h)}\,d\omega_D(h)$$
$$= \int_D \alpha(h)\,\overline{\beta(h)}\,d\omega_D(h)\cdot\overline{\beta(\pi(z,x))}.$$

As the characters of the compact commutative hypergroup $(D, *)$ form an orthogonal basis of $L^2(D, \omega_D)$, the proof is complete. □

We finally remark that for given spaces X, D and given projection $\pi : X \times X \to D$, there is at most one CAS with property (T2), that is, (T2) is a quite strong condition.

Proposition 5.13. *Let (X, D, K) and $(X, D, \tilde{K})$ be CAS with property (T2) with the same X, D, π. Then $(X, D, K) = (X, D, \tilde{K})$ and $(D, *) = (D, \tilde{*})$.*

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PROOF. (T2) implies that for all $f \in C_c(D)$ and $g \in C_c(X)$

$$T_f g = T^{f\circ\pi} g = \tilde{T}_f g \tag{5.5}$$

with $\tilde{T}_f g$ as the operator associated with the kernels $\tilde{K}_h$. As for $x \in X$ and $g \in C_c(X)$ the map $D \to \mathbb{C}$, $h \mapsto \int_X g(y)\, K_h(x, dy)$ is continuous, (5.5) implies by a limit that $T_h g = \tilde{T}_h g$ for all $g \in C_c(D)$ and $h \in D$. This also readily shows that $K_h = \tilde{K}_h$ for all h. Fact 4.7(1) finally proves $(D, *) = (D, \tilde{*})$. □

Propositions 5.13 and 5.6 yield the following corollary.

COROLLARY 5.14. *Let (X, D, K) and $(X, D, \tilde{K})$ be compact CAS with the same X, D, π. Then $(X, D, K) = (X, D, \tilde{K})$ and $(D, *) = (D, \tilde{*})$.*

Variants of 5.13 and 5.14 will be given in Section 7.

6. Positive definite functions

In this section we study several concepts of positive definiteness on CAS. We restrict our attention to the commutative case for simplicity even if some results remain valid in a slightly more general setting. Therefore, (X, D, K) will always be a commutative CAS.

DEFINITION 6.1.

(1) Let $A : C_c(X) \to C(X)$ be a linear operator. A is called positive definite if $\langle Ag, g\rangle_X \in [0, \infty[$ for all $g \in C_c(X)$.

(2) A continuous function $F : X \times X \to \mathbb{C}$ is called positive definite, if for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $c_1, \ldots, c_n \in \mathbb{C}$, $\sum_{k,l=1}^n c_k \bar{c}_l\, F(x_k, x_l) \geq 0$.

Both concepts are closely related.

LEMMA 6.2. *A continuous function $F : X \times X \to \mathbb{C}$ is positive definite if and only if the linear operator $T^F : C_c(X) \to C(X)$ is positive definite.*

PROOF. This follows from standard density arguments similar to corresponding results for hypergroups; see, for example, [8, Lemma 4.1.4]. □

As the pointwise products of positive semidefinite matrices are again positive semidefinite (see, for example, [6, Lemma 3.2]), we have the following well-known result.

LEMMA 6.3. *If $F, G : X \times X \to \mathbb{C}$ are positive definite, then the pointwise product $F \cdot G : X \times X \to \mathbb{C}$ is also positive definite.*

We now study for which $f \in C_b(D)$ the operators T_f are positive definite. The following more or less obvious result will be needed later on.

LEMMA 6.4. *For a function $f \in C_b(D)$, the operator T_f is positive definite if and only if for each step function $g = \sum_{i=1}^n c_i \mathbf{1}_{A_i} : X \to \mathbb{C}$ with $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{C}$, and disjoint, relatively compact Borel sets $A_1, \ldots, A_n \subset X$, the inequality $\langle T_f g, g\rangle_X \in [0, \infty[$ holds.*

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Proof. For the only-if-part, we notice that each step function g as required in the lemma is the pointwise limit of functions in $C_c(X)$ whose supports are contained in some fixed compactum in X. The result then follows from dominated convergence. The if-part follows by the same arguments. □

We now collect some relations between positive definite functions on D and positive definiteness on X.

Lemma 6.5. *Let $f \in C_c(D)$. Then, $f * f^*$ is positive definite on D, and T_{f*f^*} is positive definite.*

Proof. The first statement is well known; see 2.8(2). The second one is clear as $T_{f*f^*} = T_f T_f^*$ by 4.12. □

Corollary 6.6. *For each character $\alpha \in (D, *)^\wedge$ in the support S of the Plancherel measure of $(D, *)$, the operator T_α is positive definite. Moreover, if $f \in C_b(D)$ is positive definite on $(D, *)$ such that f has the form $f = \check{\mu}$ for some $\mu \in M_b^+(S)$, then T_f is positive definite.*

Proof. By Fact 2.8(3), each $\alpha \in S$ is a locally uniform limit of functions of the form $f * f^*$ with $f \in C_c(D)$. It follows from the axioms of a continuous association scheme and the definition of T_α that for each $g \in C_c(X)$, $T_\alpha g$ is a locally uniform limit of $T_{f*f^*} g$. Lemma 6.5 thus implies that T_α is positive definite. The second statement follows in the same way. □

If (X, D, K) has property (T2), then the preceding results can be rewritten.

Corollary 6.7. *Let (X, D, K) be a commutative CAS with (T2). Then the following holds.*

(1) *For each $f \in C_c(D)$, $(f * f^*) \circ \pi : X \times X \to \mathbb{C}$ is positive definite.*
(2) *For each character $\alpha \in S \subset \hat{D}$ in the support of the Plancherel measure of $(D, *)$, $\alpha \circ \pi : X \times X \to \mathbb{C}$ is positive definite.*

Proof. (1) follows from Lemma 6.5, property (T2), and Lemma 6.2.

For (2) we again use that $\alpha \in S$ is a locally uniform limit of functions of the form $f * f^*$ with $f \in C_c(D)$. Hence, by part (1), $\alpha \circ \pi$ is a locally uniform limit of positive definite functions on $X \times X$ and thus also positive definite. □

We now turn to the converse statement of Lemma 6.5 and Corollary 6.6.

Lemma 6.8. *Let $f \in C(D)$ such that T_f is positive definite. Then the following hold:*

(1) *$f(e) \in [0, \infty[$;*
(2) *f is positive definite on $(D, *)$.*

Proof. For part (1) assume that $f(e) \in \mathbb{C} \setminus [0, \infty[$ holds, that is, $\arg f(e) \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ with a branch of the arg-function on $\mathbb{C} \setminus \{0\}$ which is continuous in $f(e)$. Now choose $\varepsilon > 0$ such that for all $z \in \mathbb{C}$ with $|z - f(e)| < \varepsilon$ we have $z \neq 0$, $\arg z \notin 2\pi\mathbb{Z}$, and $|\arg z - \arg f(e)| < 1/2$ (or another small positive constant). As f is continuous, we

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find a neighborhood $W_e \subset D$ of e with $|f(x) - f(e)| < \varepsilon$ for $x \in W_e$. We thus obtain that for all $\varphi \in C(D)$ with values in $[0, \infty[$ and with $\varphi(e) > 0$,

$$\int_{W_e} \varphi(x)\, f(x)\, d\omega_D(x) \in \mathbb{C} \setminus [0, \infty[. \tag{6.1}$$

On the other hand, we now fix some $z \in X$. As $\pi : X \times X \to D$ is continuous with $\pi(z, z) = e$, we find a neighborhood $U_z \subset X$ of z with $\pi(U_z, U_z) \subset W_e$. Choose some $g \in C_c(X)$ with values in $[0, \infty[$ and with $g(z) > 0$ and $\operatorname{supp} g \subset U_z$. Then $K_h(x, U_z) = 0$ for all $x \in U_z$ and $h \in D \setminus W_e$. As T_f is positive definite, then

$$\begin{aligned} 0 \le \langle Ag, g\rangle_X &= \int_D \int_X \int_X g(x)g(y)\, K_h(x, dy)\, d\omega_X(x) \cdot f(h)\, d\omega_D(h) \\ &= \int_{W_e} \int_{U_z} \int_{U_z} g(x)g(y)\, K_h(x, dy)\, d\omega_X(x) \cdot f(h)\, d\omega_D(h) \\ &=: \int_{W_e} \varphi(x)\, f(x)\, d\omega_D(x), \end{aligned}$$

where φ is continuous with values in $[0, \infty[$ and with $\varphi(e) > 0$. This contradicts (6.1) and completes the proof of part (1).

For (2) consider any $\varphi \in C_c(D)$ and $g \in C_c(X)$. Then, by 4.11(2), $T_\varphi g \in C_c(X)$, and by our preceding considerations,

$$\langle T_{\varphi^* * f * \varphi} g, g\rangle_X = \langle T_\varphi^* T_f T_\varphi g, g\rangle_X = \langle T_f T_\varphi g, T_\varphi g\rangle_X \ge 0.$$

This shows that $T_{\varphi^* * f * \varphi}$ is positive definite, and we obtain from a standard computation for hypergroups and part (1) that

$$\int_D \int_D f(h_1 * \bar{h}_2) \cdot \varphi(h_1) \cdot \overline{\varphi(h_2)}\, d\omega_D(h_1)\, d\omega_D(h_2) = \varphi^* * f * \varphi(e) \in [0, \infty[.$$

As this holds for all $\varphi \in C_c(D)$, it follows from standard arguments for hypergroups (see [8, Lemma 4.1.4]) that f is positive definite on $(D, *)$. □

Corollary 6.7 and Lemmas 6.3 and 6.8 now lead to the following result. As was given for association schemes in [43, Theorem 4.6].

Theorem 6.9. *Let $(D, *)$ be a commutative hypergroup which is associated with some CAS (X, D, K) with (T2). Then, for all $\alpha, \beta \in S \subset \hat{D}$ in the support of the Plancherel measure, $\alpha \cdot \beta$ is positive definite on D, and there is a unique probability measure $\delta_\alpha \hat{*} \delta_\beta \in M^1(\hat{D})$ with $(\delta_\alpha \hat{*} \delta_\beta)^\vee = \alpha \cdot \beta$. The support of this measure is contained in S.*

Furthermore, for all $\alpha \in S$, the unique positive character α_0 in S according to 2.8(4) is contained in the support of $\delta_\alpha \hat{*} \delta_{\bar{\alpha}}$.

Proof. Corollary 6.7, property (T2), and Lemmas 6.3 and 6.8 show that $\alpha \cdot \beta$ is positive definite on $(D, *)$. Bochner's theorem 2.8(1) now leads to the probability measure $\delta_\alpha \hat{*} \delta_\beta \in M^1(\hat{D})$. Furthermore, Proposition 2.9 ensures that the support of this measure is contained in S. The assertion about the support of $\delta_\alpha \hat{*} \delta_{\bar{\alpha}}$ follows from [39, Theorem 2.1]. □

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The methods of the proof of Theorem 6.9 can be used to prove the following equivalence of different concepts of positive definiteness.

PROPOSITION 6.10. *Let (X, D, K) be a commutative CAS with property (T2) such that* $\mathbf{1}$ *is contained in the support S of the Plancherel measure of the associated hypergroup $(D, *)$. Then for $\alpha \in (D, *)^\wedge$ the following facts are equivalent:*

(1) $\alpha \in S$;
(2) *the operator T_α is positive definite;*
(3) *$\alpha \circ \pi \in C_b(X \times X)$ is positive definite;*
(4) *for each $\beta \in S$, the product $\alpha \cdot \beta$ is positive definite on $(D, *)$.*

PROOF. (1) $\Longrightarrow$ (2) follows from Corollary 6.6, and (2) $\Longrightarrow$ (3) is a consequence of (T2) and Lemma 6.2. (3) $\Longrightarrow$ (4) follows from Lemma 6.3 with the methods of the proof of Theorem 6.9. Finally, (4) $\Longrightarrow$ (1) is a consequence of $\mathbf{1} \in S$ and [38, Corollary 7]. □

For compact CAS, Theorem 6.9 can be improved.

THEOREM 6.11. *Let $(D, *)$ be a compact commutative hypergroup which is associated with some compact commutative CAS (X, D, K). Then (T2) holds by Proposition 5.6, and, with the dual convolution $\hat{*}$ of Theorem 6.9, $(\hat{D}, \hat{*})$ satisfies all hypergroup axioms possibly except for the condition that* $\operatorname{supp}(\delta_\alpha \hat{*} \delta_\beta)$ *is compact (that is, finite) for all $\alpha, \beta \in \hat{D}$.*

PROOF. For compact commutative hypergroups we have $S = \hat{D}$, $\hat{D}$ is discrete, and the unique positive character in S is the identity $\mathbf{1}$; see, for example, [8]. Therefore, if we take $\mathbf{1}$ as identity and complex conjugation as involution, almost all hypergroup axioms of $(\hat{D}, \hat{*})$ follow from Theorem 6.9. In fact, as $\hat{D}$ is discrete, the topological axioms hold automatically. Moreover, the bilinear, weakly continuous extension of the dual convolution $\hat{*}$ from the set of point measures to $M_b(\hat{D})$ is associative as the inverse Fourier transform is injective; see [8].

We thus only have to check that for $\alpha \neq \beta \in \hat{D}$, the character $\mathbf{1}$ is not contained in the support of $\delta_\alpha \hat{*} \delta_{\bar{\beta}}$. For this we recapitulate that for all $\gamma, \rho \in \hat{D}$, $\hat{\gamma}(\rho) = \int_D \gamma \bar{\rho}\, d\omega_D = \|\gamma\|_2^2 \delta_{\gamma,\rho}$ with the Kronecker-δ. Therefore, with [22, 12.16],

$$\begin{aligned}(\delta_\alpha \hat{*} \delta_{\bar{\beta}})(\{\mathbf{1}\}) &= \int_{\hat{D}} \mathbf{1}_{\{\mathbf{1}\}}\, d(\delta_\alpha \hat{*} \delta_{\bar{\beta}}) = \int_{\hat{D}} \hat{\mathbf{1}}\, d(\delta_\alpha \hat{*} \delta_{\bar{\beta}}) \\ &= \int_D \mathbf{1}\, (\delta_\alpha \hat{*} \delta_{\bar{\beta}})^\vee\, d\omega_D = \int_D \alpha \bar{\beta}\, d\omega_D = \|\gamma\|_2^2 \delta_{\alpha,\beta} = 0.\end{aligned}$$

This completes the proof. □

In the finite case, Theorem 6.11 is as follows; see also [43, Theorem 4.7].

COROLLARY 6.12. *Let $(D, *)$ be a finite commutative hypergroup which is associated with some finite commutative CAS (X, D, K). Then $\hat{D}$ carries a dual hypergroup structure.*

We next turn to the problem of whether the conclusions of 6.9, 6.11, and 6.12 on positive dual convolutions also hold for commutative CAS without (T2). We here

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follow [43, Section 5] and assume that we have two commutative CAS (X, D, K) and $(X, D, \tilde{K})$ with the same spaces X, D and the same projection $\pi : X \times X \to D$. We denote the associated commutative hypergroups by $(D, *)$ and $(D, \tilde{*})$ and the supports of the associated Plancherel measures by S and $\tilde{S}$. Assume that (X, D, K) has property (T2), and that all characters $\tilde{\alpha} \in \tilde{S}$ of $(D, \tilde{*})$ in the support of the Plancherel measure have the form

$$\tilde{\alpha}(h) = \int_S \alpha(h)\, d\mu(\alpha) \quad \text{for all } h \in D$$

for some $\mu \in M^1(S)$. It was proved in [43, Theorem 5.10] for commutative generalized association schemes that then $(D, \tilde{*})$ also admits a positive dual convolution on $\tilde{S}$. We shall extend this result to CAS in Theorem 7.1 which has some unexpected consequences: It turns out that under some additional conditions like property (T2) for **one** of the CAS and a support condition, the hypergroup structures $(D, *)$ and $(D, \tilde{*})$ are equal; compare this with the assertions of 5.7, 5.13, and 5.14.

7. A comparison of different CAS on the same spaces

As before, let (X, D, K) and $(X, D, \tilde{K})$ be commutative CAS with the same spaces X, D and the same projection $\pi : X \times X \to D$. Let again $(D, *)$ and $(D, \tilde{*})$ be the associated commutative hypergroups and S and $\tilde{S}$ the associated Plancherel measures respectively. The following extension of Theorem 6.9 is the main result of this section.

THEOREM 7.1. *Assume that (X, D, K) has property (T2) in the setting above. Then for all characters $\tilde{\alpha} \in \tilde{S}$ and $\beta \in S$, the product $\tilde{\alpha} \cdot \beta$ is positive definite on $(D, \tilde{*})$, and there is a unique $\mu \in M^1(\tilde{S})$ with*

$$\tilde{\alpha}(h)\beta(h) = \int_{\tilde{S}} \alpha(h)\, d\mu(\alpha) \quad \textit{for all } h \in D.$$

In the discrete case, the proof is quite simple and similar to that of [43, Theorem 5.10], while it will be more involved in the continuous case due to some approximation procedure. To highlight the idea of the proof, we first give the proof in the discrete case.

PROOF OF THEOREM 7.1 IN THE DISCRETE CASE. Let $\tilde{\alpha} \in \tilde{S}$ and $\beta \in S$. Let $\tilde{T}_{\tilde{\alpha}}$ be the linear operator associated with $\tilde{\alpha}$ and the CAS $(X, D, \tilde{K})$. Then, by Corollary 6.6, $\tilde{T}_{\tilde{\alpha}}$ is positive definite. Now let $g \in C_c(X)$. Choose $x_1, \ldots, x_n \in X$ different with $\operatorname{supp} g = \{x_1, \ldots, x_n\}$, and let $\tilde{\omega}_D$ be a Haar measure of $(D, \tilde{*})$, $\tilde{\omega}_X \in M^+(X)$ the associated measure, and $\langle ., . \rangle_{\tilde{X}}$ the associated scalar product on $L^2(X, \tilde{\omega}_X)$. The positive definiteness of $\tilde{T}_{\tilde{\alpha}}$ and the properties of $\operatorname{supp} K_h(x, .)$ for $h \in D$ and $x \in X$ imply that

$$\begin{aligned} 0 \le \langle \tilde{T}_{\tilde{\alpha}} g, g \rangle_{\tilde{X}} &= \sum_{h \in D} \sum_{k,l=1}^{n} \overline{g(x_k)}\, g(x_l)\, \tilde{K}_h(x_k, \{x_l\})\, \tilde{\omega}_X(\{x_k\}) \cdot \tilde{\alpha}(h) \tilde{\omega}_D(\{h\}) \\ &= \sum_{k,l=1}^{n} \overline{g(x_k)}\, g(x_l)\, \tilde{K}_{\pi(x_k,x_l)}(x_k, \{x_l\})\, \tilde{\omega}_X(\{x_k\}) \tilde{\alpha}(\pi(x_k, x_l)) \tilde{\omega}_D(\{\pi(x_k, x_l)\}). \end{aligned} \tag{7.1}$$

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On the other hand, as (X, D, K) has property (T2), $\beta \circ \pi : X \times X \to \mathbb{C}$ is positive definite, that is, the matrix $(\beta(\pi(x_k, x_l)))_{k,l}$ is positive semidefinite. Equation (7.1) and the fact that pointwise products of positive semidefinite matrices are positive semidefinite yield that

$$\sum_{k,l=1}^{n} \overline{g(x_k)} g(x_l) \tilde{K}_{\pi(x_k,x_l)}(x_k, \{x_l\})\, \tilde{\omega}_X(\{x_k\}) \\ \cdot \tilde{\alpha}(\pi(x_k, x_l))\beta(\pi(x_k, x_l))\tilde{\omega}_D(\{\pi(x_k, x_l)\}) \geq 0.$$

As in the computation of Equation (7.1), we obtain that $\langle \tilde{T}_{\tilde{\alpha}\cdot\beta} g, g\rangle_{\tilde{X}} \geq 0$, that is, $\tilde{T}_{\tilde{\alpha}\cdot\beta}$ is positive definite. Hence, by Lemma 6.8, $\tilde{\alpha}\beta$ is positive definite on $(D, \tilde{*})$. Finally, the support condition follows from Proposition 2.9. □

We now turn to the general case by using Lemma 6.4.

Proof of Theorem 7.1 in the general case. We keep the notations of the discrete case. Let $\tilde{\alpha} \in \tilde{S}$ and $\beta \in S$. Consider some step function $g = \sum_{i=1}^{n} c_i \mathbf{1}_{A_i} : X \to \mathbb{C}$ with $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{C}$, and disjoint, relatively compact, nonempty Borel sets $A_1, \ldots, A_n \subset X$ as in Lemma 6.4. Let $\varepsilon > 0$ be a small constant.

As $\beta \circ \pi : X \times X \to \mathbb{C}$ is continuous and thus uniformly continuous on the compactum $\operatorname{supp} g \times \operatorname{supp} g \subset X \times X$, we may decompose the sets $A_1, \ldots, A_n$ into finitely many, disjoint, nonempty Borel sets such that, after denoting these finitely many sets again by $A_1, \ldots, A_n$ we have the following additional property.

$$\text{For all} \quad i, j = 1, \ldots, n,\ u, v \in A_i,\ x, y \in A_j : \quad |\beta(\pi(u, x)) - \beta(\pi(v, y))| \leq \varepsilon. \tag{7.2}$$

We thus now assume without loss of generality that the step function g has a representation where this property holds, and where the sets A_i are fixed. For the functions $f = \tilde{\alpha}, \tilde{\alpha} \cdot \beta \in C_b(D)$ we put

$$\Phi_f(i, j) := \int_D \int_{A_i} \tilde{K}_h(x, A_j)\, d\tilde{\omega}_X(x) \cdot f(h)\, d\tilde{\omega}_D(h).$$

A short computation and the definition of g then yield that

$$\langle \tilde{T}_f g, g\rangle_{\tilde{X}} = \sum_{i,j=1}^{n} \overline{c_i}\, c_j\, \Phi_f(i, j). \tag{7.3}$$

As $\tilde{T}_{\tilde{\alpha}}$ is positive definite, we obtain that $\langle \tilde{T}_{\tilde{\alpha}} g, g\rangle_{\tilde{X}} \geq 0$ for all choices of $c_1, \ldots, c_n \in \mathbb{C}$. This means that the matrix $(\Phi_{\tilde{\alpha}}(i, j))_{i,j=1,\ldots,n}$ is positive semidefinite.

On the other hand, as (X, D, K) has property (T2), we know that $\beta \circ \pi : X \times X \to \mathbb{C}$ is positive definite, that is, the matrix $(\beta(\pi(x_i, x_j)))_{i,j}$ is positive semidefinite for

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all choices of points $x_i \in A_i$, $i = 1, \ldots, n$. As pointwise products of positive semidefinite matrices are positive semidefinite, we obtain that the matrix $(\Phi_f(i,j) \cdot \beta(\pi(x_i, x_j)))_{i,j=1,\ldots,n}$ is positive semidefinite. This means that for all choices of $c_1, \ldots, c_n \in \mathbb{C}$,

$$\begin{aligned} 0 &\le \sum_{i,j=1}^{n} \overline{c_i}\, c_j\, \Phi_{\tilde{\alpha}}(i,j)\beta(\pi(x_i, x_j)) \\ &= \sum_{i,j=1}^{n} \overline{c_i}\, c_j \int_D \int_{A_i} \tilde{K}_h(x, A_j)\, d\tilde{\omega}_X(x) \cdot \tilde{\alpha}(h)\, \beta(\pi(x_i, x_j))\, d\tilde{\omega}_D(h). \end{aligned} \tag{7.4}$$

Notice that for $i, j = 1, \ldots, n$ and $h \in D$, $\int_{A_i} \tilde{K}_h(x, A_j)\, d\tilde{\omega}_X(x) > 0$. This yields $\sup_{x \in A_i} \tilde{K}_h(x, A_j) > 0$, and this implies that $h \in \pi(A_i, A_j)$. This, the estimate (7.2), and $\|\tilde{\alpha}\|_\infty = 1$ imply that

$$\begin{aligned} &\Big|\sum_{i,j=1}^{n} \overline{c_i}\, c_j \int_D \int_{A_i} \tilde{K}_h(x, A_j)\, d\tilde{\omega}_X(x) \cdot \tilde{\alpha}(h)\, \beta(\pi(x_i, x_j))\, d\tilde{\omega}_D(h) \\ &\qquad - \sum_{i,j=1}^{n} \overline{c_i}\, c_j \int_D \int_{A_i} \tilde{K}_h(x, A_j)\, d\tilde{\omega}_X(x) \cdot \tilde{\alpha}(h)\, \beta(h)\, d\tilde{\omega}_D(h)\Big| \\ &\le \varepsilon \|g\|_\infty^2 \cdot \tilde{\omega}_X(\mathrm{supp}\, g)^2. \end{aligned}$$

We conclude from (7.4) and (7.3) that

$$\langle \tilde{T}_{\tilde{\alpha}\beta} g, g \rangle_{\tilde{X}} = \sum_{i,j=1}^{n} \overline{c_i}\, c_j\, \Phi_{\tilde{\alpha}\beta}(i,j) \in \mathbb{C}$$

has a distance from $[0, \infty[\subset \mathbb{C}$ which is at most $C\varepsilon$ for some constant $C \ge 0$ depending on g only. As in our approximation $\varepsilon > 0$ may be arbitrarily small, we obtain $\langle \tilde{T}_{\tilde{\alpha}\beta} g, g \rangle_{\tilde{X}} \in [0, \infty[$. As this holds for all step functions g, Lemma 6.4 shows that $\tilde{T}_{\tilde{\alpha}\beta}$ is positive definite as claimed. Again, the support condition follows from Proposition 2.9. □

We now present some applications of Theorem 7.1.

Corollary 7.2. *Let (X, D, K) and $(X, D, \tilde{K})$ be commutative CAS with the same X, D, and π. Assume that (X, D, K) has property (T2), and that the identity $\mathbf{1}$ is contained in $\tilde{S}$. Then each character $\beta \in S$ is positive definite on $(D, \tilde{*})$, and there is a unique $\mu \in M^1(\tilde{S})$ with*

$$\beta(h) = \int_{\tilde{S}} \tilde{\beta}(h)\, d\mu(\tilde{\beta}) \quad \text{for all } h \in D.$$

Proof. Use Theorem 7.1 with $\tilde{\alpha} = \mathbf{1}$. □

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In Remark 9.9 we present an example which shows that the technical condition $\mathbf{1} \in \tilde{S}$ in Corollary 7.2 is necessary.

Here is a further consequence of Theorem 7.1 which generalizes [43, Theorem 5.10].

Corollary 7.3. *Let (X, D, K) and $(X, D, \tilde{K})$ be commutative CAS with the same X, D, and π. Assume that (X, D, K) has property (T2), and that each character $\tilde{\beta} \in \tilde{S}$ of $(D, \tilde{*})$ has the form*

$$\tilde{\beta}(h) = \int_S \beta(h)\, d\mu(\beta) \quad (h \in D) \tag{7.5}$$

for some $\mu \in M^1(S)$.

Then, for all $\tilde{\alpha}, \tilde{\beta} \in \tilde{S}$, the product $\tilde{\alpha} \cdot \tilde{\beta}$ is positive definite on $(D, \tilde{*})$, and there is a unique probability measure $\delta_{\tilde{\alpha}} \hat{*} \delta_{\tilde{\beta}} \in M^1(\tilde{S})$ with $(\delta_{\tilde{\alpha}} \hat{*} \delta_{\tilde{\beta}})^{\vee} = \tilde{\alpha} \cdot \tilde{\beta}$.

Proof. Theorem 7.1 shows that for $\tilde{\alpha} \in \tilde{S}$ and $\beta \in S$ the product $\tilde{\alpha}\beta$ is positive definite on $(D, \tilde{*})$ with $\tilde{\alpha}\beta = \int_{\tilde{S}} \gamma\, d\mu_{\tilde{\alpha},\beta}(\gamma)$. Equation (7.5) now implies that for all $\tilde{\alpha}, \tilde{\beta} \in \tilde{S}$, the product $\tilde{\alpha}\tilde{\beta}$ is positive definite on $(D, \tilde{*})$, and that the claimed integral representation holds. □

Clearly, Corollary 7.1 is also a generalization of Theorem 6.9. However, in practice it does not go far beyond of Theorem 6.9 by the following theorem which is closely related to 5.13.

Theorem 7.4. *Let (X, D, K) and $(X, D, \tilde{K})$ be commutative CAS with the same X, D, and π. Assume that (X, D, K) has property (T2) and that $\mathbf{1} \in \operatorname{supp} \tilde{S}$ holds. Assume in addition that each character $\tilde{\beta} \in \tilde{S}$ has the form*

$$\tilde{\beta}(h) = \int_S \beta(h)\, d\mu(\beta) \quad (h \in D) \tag{7.6}$$

for some $\mu \in M^1(S)$. Then, $(D, *) = (D, \tilde{*})$.

Proof. We first recapitulate some facts on commutative hypergroups from [22] which are well known for locally compact abelian groups and Gelfand pairs. For this let $(D, *)$ be any commutative hypergroup. Then the dual $\hat{D}$ can be identified with the symmetric spectrum

$$\Delta_s(L^1(D, \omega_D)) := \{\varphi \in L^1(D, \omega_D)^* : \varphi \quad \text{multiplicative}, \\ \varphi(f^*) = \overline{\varphi(f)} \text{ for } f \in L^1(D)\}$$

of the commutative Banach-$*$-algebra $(L^1(D, \omega_D), *, .^*)$ via

$$\alpha \mapsto \varphi_\alpha \quad \text{with } \varphi_\alpha(f) := \int_D \varphi(x)\alpha(x)\, d\omega_D(x).$$

In particular, if $\Delta_s(L^1(D, \omega_D))$ carries the Gelfand topology and $\hat{D}$ the topology of compact-uniform convergence, then this mapping is a homeomorphism. We also

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recapitulate the well-known fact that $\Delta_s(L^1(D,\omega_D))$ is the set of all extremal points in the set $P(D,*)$ of all positive linear functionals on $L^1(D,\omega_D))$ with dual norm equal to 1; see, for example, Rudin [33].

We now apply these facts to our theorem. We first conclude from (7.6) that each $\tilde\beta \in \tilde S \subset C_b(D)$ leads to a positive linear functional $\varphi_{\tilde\beta} \in P(D,*)$. Moreover, by Corollary 7.2, each $\beta \in S \subset C_b(D)$ has the form

$$\beta(h) = \int_{\tilde S} \tilde\beta(h)\, d\mu(\tilde\beta) \quad (h \in D)$$

for some $\mu_\beta \in M^1(S)$. As φ_β is an extremal point, we conclude that μ_β is a point measure. In fact, if μ_β fails to be a point measure, then we may write μ_β as $\mu_\beta = \lambda\mu_1 + (1-\lambda)\mu_2$ with different measures $\mu_1, \mu_2 \in M^1(S)$ and $\lambda \in]0,1[$, that is, φ_β would be a nontrivial convex combination of elements of $P(D,*)$ contradicting extremality.

In summary μ_β is a point measure for each $\beta \in S$ which proves $S \subset \tilde S$. The same arguments also yield $\tilde S \subset S$, that is, we have $S = \tilde S$. Therefore, for all $x, y \in D$ and $\alpha \in S = \tilde S$,

$$(\delta_x * \delta_y)^\wedge(\alpha) = \int_D \bar\alpha\, d(\delta_x * \delta_y) = \overline{\alpha(x)\alpha(y)} = \cdots = (\delta_x \tilde{*} \delta_y)^\wedge(\alpha).$$

As the restricted Fourier transform $M^1(D) \to C_b(S) = C_b(\tilde S)$, $\mu \mapsto \hat\mu$ is also injective (see [22]), we obtain that $\delta_x * \delta_y = \delta_x \tilde{*} \delta_y$ for all $x, y \in D$ as claimed. □

REMARK 7.5. We briefly discuss some implication of the preceding results. For this we recapitulate the original motivation in the introduction of [43] for the study of generalizations of classical commutative association schemes. Consider a sequence $(G_n, H_n)_{n\in\mathbb{N}}$ of Gelfand pairs such that the double coset spaces $G_n//H_n$ are homeomorphic with some fixed locally compact space D. Modulo these homeomorphims, we obtain associated double coset hypergroups $(D, *_n)$. The spherical functions of (G_n, H_n) then may be regarded as nontrivial continuous multiplicative functions on $(D, *_n)$. For many examples of series $(G_n, H_n)_n$, these functions are parameterized by some spectral parameter set $\chi(D)$ independent on n, and the associated functions $\varphi_n : \chi(D) \times D \to \mathbb{C}$ can be embedded into a family of special functions which depend analytically on n in some parameter domain $A \subset \mathbb{C}$. In many cases, these special functions are well known, and the product formulas for spherical functions can be written down explicitly on D with $n \in \mathbb{N}$ as the parameter. Based on Carleson's theorem, a principle of analytic continuation (see, for example, [35], page 186), one can often easily extend these positive product formulas to a continuous range of parameters, say $n \in [1, \infty[$ such that for all these n associated commutative hypergroup structures $(D, *_n)$ exist.

Besides positive product formulas for $\varphi_n(\lambda, .)$ on D, there also exist dual product formulas for the functions $\varphi_n(., x)$ $(x \in D)$ on suitable subsets of $\chi(D)$ for the group cases, that is, for $n \in \mathbb{N}$. In particular, positive dual convolutions on the supports

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$S_n \subset \chi(D)$ of the Plancherel measures of the double coset hypergroups $G_n//H_n$ exist; see, for example, Theorem 6.9. For many examples, these dual convolutions are known and can be extended again by Carleson's theorem to positive dual convolutions for all $n \in [1, \infty[$. However, for symmetric spaces of rank ≥ 2, this dual convolution is usually a difficult business, and not very much is known in this respect. When writing (the introduction of) [43], the author hoped that a theory of continuous association schemes might lead to examples of commutative CAS associated with $(D, *_n)$ for all $n \in [1, \infty[$ such that Theorem 6.9 or Corollary 7.3 leads at least to the existence of dual positive convolutions on S_n in these cases.

This idea was motivated by natural families $(K_h)_{h\in D}$ of Markov kernels on concrete spaces X which are associated with commutative hypergroup structures on D in [26] and [7]. In fact, Kingman [26] studies the Euclidean case $X = \mathbb{R}^2$ with $D = [0, \infty[$ where the associated hypergroups are the Bessel–Kingman hypergroups indexed by a continuous parameter. Moreover, Bingham [7] studies the spherical case $X = S^2 := \{x \in \mathbb{R}^3 : \|x\|_2 = 2\}$ with $D = [-1, 1]$ where the associated hypergroups on $[-1, 1]$ are related to ultraspherical polynomials. Unfortunately, the kernels $(K_h)_{h\in D}$ in [26] and [7] do not lead to commutative CAS such that the theory of our paper cannot be applied there. This becomes clear from Theorem 7.4 without discussing any details of these kernels from Theorem 7.4, as almost all conditions of Theorem 7.4 are satisfied for these examples. In fact, if the structures in [26] and [7] would lead to commutative CAS, then Theorem 7.4 could be applied to this structure as the CAS $(X, D, \tilde{K})$ where we would have to take the CAS (X, D, K) as a group case with a suitable smaller group parameter than for $(X, D, \tilde{K})$. Here, we notice that then in particular (7.6) holds by well-known explicit positive integral representations of the associated Bessel functions and ultraspherical polynomials; see, for example, the survey of Askey [2].

8. Multiplicative functions and deformations

Following the well-known notion of multiplicative functions, semicharacters, and characters on commutative hypergroups (see [8, 22], and Section 2 above), we here introduce a corresponding concept for commutative continuous association schemes. We in particular use it to construct deformed continuous association schemes $(X, D, \tilde{K})$ from a given scheme (X, D, K) with the same spaces X, D and modified kernels K. This construction leads to examples of CAS which are beyond double coset examples and classical discrete association schemes where usually (T1) and (T2) do not hold.

Definition 8.1. A pair $(\alpha, \varphi) \in C(D) \times C(X)$ of continuous functions is called multiplicative on a commutative continuous association scheme $(X, D, \tilde{K})$ if $\alpha \not\equiv 0$ and

$$T_h\varphi(x) = \int_X \varphi(y)K_h(x, dy) = \varphi(x) \cdot \alpha(h) \quad \text{for all } h \in D,\ x \in X. \tag{8.1}$$

A multiplicative pair (α, φ) is called a semicharacter of $(X, D, \tilde{K})$, if in addition

$$\alpha(\bar{h}) = \overline{\alpha(h)} \quad \text{for all } h \in D.$$

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A semicharacter (α, φ) is called a character, if α and φ are bounded, and positive, if α and φ are $]0, \infty[$-valued.

REMARKS 8.2.

(1) Equation (8.1) means that φ is a joint eigenfunction of all mean value operators T_h, $h \in D$.

(2) If $\alpha \equiv 1$, then (8.1) is a mean value condition, that is, φ may be seen as a harmonic function. Notice that for a compact CAS, all harmonic functions are constant by Lemma 4.14. We shall see in Remark 9.10 that usually there might exist unbounded positive harmonic functions. It might be interesting to explore under which conditions on a CAS, all bounded harmonic functions are constant.

(3) For a multiplicative pair (α, φ), α is determined uniquely by φ. The converse statement is not correct as for any $\alpha \in C(D)$, the joint eigenspace

$$E_\alpha := \{\varphi \in C(X) : \ (\alpha, \varphi) \quad \text{multiplicative}\}$$

is a vector space.

(4) Let $(X = G/H, D = G//H, K)$ be a commutative continuous association scheme which comes from some Gelfand pair (G, H) with a connected Lie group G. Then, any $\varphi \in C(X)$ is contained in some joint eigenspace E_α for $\alpha \in C(D)$ if and only if φ is a joint eigenfunction of all G-invariant differential operators on $X = G/H$; see, for example, Helgason [19, Proposition IV.2.4].

We next study relations between α and φ for multiplicative pairs.

PROPOSITION 8.3. *If (α, φ) is multiplicative on (X, D, K) with $\varphi \not\equiv 0$, then α is a multiplicative function of $(D, *)$, that is, for all $h_1, h_2, h \in D$, $\alpha(h_1 * h_2) = \alpha(h_1)\alpha(h_2)$. Moreover, if (α, φ) is in addition a semicharacter or character, then so is α on $(D, *)$.*

Conversely, for each nontrivial multiplicative function $\alpha \in C(D)$ on $(D, *)$ there exist functions $\varphi \in C(X)$ with $\varphi \not\equiv 0$ such that (α, φ) is multiplicative on (X, D, K).

PROOF. Let (α, φ) be multiplicative as described. For $x \in X$, $h_1, h_2 \in D$,

$$\begin{aligned}\varphi(x) \cdot \alpha(h_1 * h_2) = \varphi(x) \cdot \int_D \alpha(h)\, d(\delta_{h_1} * \delta_{h_1})(h) &= \int_D T_h\varphi(x)\, d(\delta_{h_1} * \delta_{h_1})(h) \\ = T_{h_1} \circ T_{h_1}\varphi(x) = \varphi(x) \cdot \alpha(h_1)\alpha(h_2).\end{aligned}$$

Taking $x \in X$ with $\varphi(x) \neq 0$ leads to the first claim. The second statement is clear.

For the last statement we conclude from Lemma 4.13 that for each $g \in C_c(X)$ and each nontrivial multiplicative function $\alpha \in C(D)$ on $(D, *)$, the function $\varphi := T_{\alpha^-} g$ with $\alpha^-(h) := \alpha(\bar{h})$ satisfies $T_h\varphi = \alpha(h) \cdot \varphi$. We still have to check that we can choose g such that $\varphi \not\equiv 0$ holds. As $\alpha(e) = 1$, we find a neighborhood $U \subset D$ of e on which $\Re\alpha \geq 0$ holds. Now fix some $x \in X$ and a neighborhood $W \subset X$ of x with $\pi(x, W) \subset U$. Now choose $g \in C_c(G)$ with $g \geq 0$, $g \not\equiv 0$, and supp $g \subset W$. Then

$$\Re\varphi(x) = \int_U \int_W g(y)\, K_h(x, dy)\, \Re\alpha(h)\, d\omega_D(h) > 0$$

as claimed. □

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(T1) and (T2) lead to a further standard construction of multiplicative pairs.

Lemma 8.4. *Let (X, D, K) be a commutative CAS with (T1) or (T2). Then for all multiplicative functions $\alpha \in C(D)$ on $(D, *)$ and $z \in X$, the pair $(\alpha, \tilde{\alpha} := \alpha \circ \pi_z)$ is multiplicative on (X, D, K).*

Proof. Assume first that (T1) holds. Then for $x \in X$, $h \in D$,

$$T_h\tilde{\alpha}(x) = T_h(\alpha \circ \pi_z)(x) = \alpha(\pi_z(x) * h) = \alpha(\pi_z(x))\alpha(h) = \tilde{\alpha}(x)\alpha(h) \tag{8.2}$$

as claimed.

Assume now that (T2) holds. We conclude from the last assertion of Proposition 8.3 that for all $g \in C_c(X)$, the function $\varphi_g := T^{\alpha^- \circ \pi} g = T_{\alpha^-} g$ satisfies $T_h \varphi_g = \alpha(h) \cdot \varphi_g$ for $h \in D$. On the other hand, as $\alpha \circ \pi$ is uniformly continuous on compact subsets of $X \times X$, we find relatively compact neighborhoods $U_n \subset X$ of z with $U_{n+1} \subset U_n$ for all n and $\bigcap_n U_n = \{z\}$ such that for all $g_n \in C_c(X)$ with $\operatorname{supp} g_n \subset U_n$, $g_n \geq 0$, and $\int_X g_n \, d\omega_X = 1$,

$$\begin{aligned}\varphi_{g_n}(x) &= \int_X \alpha^-(\pi(x, y))\, g_n(y)\, d\omega_X(y) \\ &\longrightarrow \alpha^-(\pi(x, z)) = \alpha(\pi(z, x)) = \alpha(\pi_z(x))\end{aligned}$$

uniformly on compacta with respect to x. Hence, the limit $\tilde{\alpha} := \alpha \circ \pi_z$ also satisfies $T_h \tilde{\alpha} = \alpha(h) \cdot \tilde{\alpha}$ for $h \in D$ as claimed. □

The arguments of the preceding proof and in particular in (8.2) can be combined in a different way and lead finally to the following theorem.

Theorem 8.5. *For each commutative CAS, (T2) implies (T1).*

Proof. Let (X, D, K) be a commutative CAS with (T2) and with associated commutative hypergroup $(D, *)$. Let $\alpha \in \hat{D}$ be a character, and let $z \in X$ and $\tilde{\alpha} := \alpha \circ \pi_z$. Then, by property (T2) and Lemma 8.4, $T_h \tilde{\alpha} = \alpha(h) \cdot \tilde{\alpha}$ for $h \in D$. Hence, for $x \in X$ and $h \in D$,

$$T_h(\alpha \circ \pi_z)(x) = T_h\tilde{\alpha}(x) = \tilde{\alpha}(x)\alpha(h) = \alpha(\pi_z(x))\alpha(h) = \alpha(\pi_z(x) * h).$$

As this equation is linear in α, then

$$T_h(f \circ \pi_z) = f_h \circ \pi_z$$

for all $h \in D$, $z \in X$, and $f \in C_b(D)$ of the form $f = \check{\mu}$ with $\mu \in M_b(\hat{D})$ in the notation of Section 2.7(4). On the other hand, it is well known from hypergroup theory (see, for example, [8, Theorem 2.2.32(vii)]) that, again with the notion of Section 2.7(4), the set $\{\check{g} : g \in C_c(\hat{D})\}$ is a $\|.\|_\infty$-dense subspace of $C_0(D)$. We thus conclude that $T_h(f \circ \pi_z) = f_h \circ \pi_z$ for all $f \in C_c(D)$, $h \in D$, and $z \in X$ as claimed. □

Theorem 8.5 and Proposition 5.6 imply the following result.

Corollary 8.6. *Each compact commutative CAS is strong.*

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REMARK 8.7.

(1) Notice that in the proof of the first part of Lemma 8.4, (T1) is needed for the specific $z \in X$ with $\tilde{\alpha} := \alpha \circ \pi_z$ only.
(2) If $(X = G/H, D = G//H, K)$ is a commutative CAS associated with the Gelfand pair (G, H), then (T1) and (T2) hold by 5.2. Hence, for a multiplicative function α of $(D, *)$ and $z \in G$, we may take $\tilde{\alpha} \in C(X)$ with $\tilde{\alpha}(xH) := \alpha(\pi_{zH}(xH)) = \alpha(Hz^{-1}xH)$ for $x \in G$.
(3) Let $(X = G/H, D = G//H, K)$ be a commutative CAS which comes from some symmetric space G/H. Then for each multiplicative $\alpha \in C(D)$, $E_\alpha \subset C(X)$ is the joint eigenspace of the algebra $D(G/H)$ of all G-invariant differential operators on $X = G/H$ (with suitable eigenvalues). In this case E_α is completely known by Kashiwara *et al.* [25]; see also the description of the results in [19, Section II.4.1]. As this goes beyond the scope of this paper, we skip details. For some interesting concrete examples of functions in E_α on hyperbolic planes we refer to the introduction of [19].
(4) We expect that the well-established representation theory of compact hypergroups and the arguments of the proof of Theorem 8.5 yield that for all compact CAS, (T2) implies (T1). Proposition 5.6 then would imply that each compact CAS is strong.

We now restrict our attention to positive semicharacters (α_0, φ_0) of some commutative CAS (X, D, K). It is well known from [36] or [8, Section 2.3] that then the positive semicharacter α_0 of $(D, *)$ leads to a deformed commutative hypergroup $(D, \tilde{*})$ with the deformed convolution of point measures

$$\delta_{h_1} \tilde{*} \delta_{h_2} := \frac{\alpha_0}{\alpha_0(h_1) \cdot \alpha_0(h_2)} \cdot (\delta_{h_1} * \delta_{h_2}) \in M^1(D) \quad (h_1, h_2 \in D) \tag{8.3}$$

where the identity and involution of $(D, *)$ are not changed. Moreover, if ω_D is a Haar measure of $(D, *)$, then

$$\tilde{\omega}_D := \alpha_0^2 \cdot \omega_D \in M^+(D)$$

is a Haar measure of $(D, \tilde{*})$ by [36]. We now show that (α_0, φ_0) also leads to a deformed commutative CAS $(X, D, \tilde{K})$ which is associated with $(D, \tilde{*})$.

PROPOSITION 8.8. *Let (α_0, φ_0) be a positive semicharacter as above. Define the deformed kernel $\tilde{K}$ from $X \times D$ to X with*

$$\tilde{K}_h(x, A) := \frac{1}{\alpha_0(h)\varphi_0(x)} \int_A \varphi_0(y)\, K_h(x, dy) \quad (h \in D,\ x \in X,\ A \in \mathcal{B}(X)).$$

Then $(X, D, \tilde{K})$ is a commutative CAS which is associated with $(D, \tilde{*})$ above. Moreover, the following holds.

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(1) *If $\omega_X \in M^+(X)$ is an invariant measure of (X, D, K), then $\tilde{\omega}_X := \varphi_0^2 \cdot \omega_X \in M^+(X)$ is an invariant measure of $(X, D, \tilde{K})$.*

(2) *Assume that (X, D, K) has property (T1), and let $\tilde{\alpha}_0 \in C(D)$ be a positive semicharacter and $z \in X$. Consider the positive semicharacter $(\alpha_0, \tilde{\alpha}_0 := \alpha_0 \circ \pi_z)$ of (X, D, K) according to Lemma 8.4. Then for all $f \in C(D)$, $h \in D$, and $y \in X$,*

$$f(h \tilde{*} \pi_z(y)) = (\tilde{T}_h(f \circ \pi_z))(y)$$

where $\tilde{T}_h$ is the operator associated with the kernel $\tilde{K}_h$. This means that (T1) holds for $(X, D, \tilde{K})$ for the specific $z \in X$.

PROOF. Notice that $\tilde{K}$ satisfies $\tilde{K}_h(x, X) = 1$ for $h \in D$, $x \in X$. This normalization and the continuity of K show that $\tilde{K}$ is a continuous Markov-kernel from $X \times D$ to X. Moreover, $\tilde{K}$ clearly satisfies the conditions of 4.2(1) and (2) with the projection π of the scheme (X, D, K). We next check axiom 4.2(3). For $h_1, h_2 \in D$, $x \in X$, and $A \in \mathcal{B}(X)$,

$$\begin{aligned}
\tilde{K}_{h_1} \circ \tilde{K}_{h_2}(x, A) &= \int_X \tilde{K}_{h_2}(y, A)\, \tilde{K}_{h_1}(x, dy) \\
&= \frac{1}{\alpha_0(h_1)\varphi_0(x)} \int_X \tilde{K}_{h_2}(y, A)\, \varphi_0(y)\, K_{h_1}(x, dy) \\
&= \frac{1}{\alpha_0(h_1)\alpha_0(h_2)\varphi_0(x)} \int_X \int_A \varphi_0(z)\, K_{h_2}(y, dz)\, K_{h_1}(x, dy) \\
&= \frac{1}{\alpha_0(h_1)\alpha_0(h_2)\varphi_0(x)} \int_D \int_A \varphi_0(z)\, K_h(x, dz)\, d(\delta_{h_1} * \delta_{h_2})(h) \\
&= \frac{1}{\alpha_0(h_1)\alpha_0(h_2)} \int_D \tilde{K}_h(x, A)\, \alpha_0(h)\, d(\delta_{h_1} * \delta_{h_2})(h) \\
&= \int_D \tilde{K}_h(x, A)\, d(\delta_{h_1} \tilde{*} \delta_{h_2})(h).
\end{aligned}$$

For the adjoint relation in 4.2(4), we observe for $f_1, f_2 \in C_c(X)$, and $h \in D$ that

$$\begin{aligned}
\int_X f_1 \cdot \tilde{T}_h f_2\, d\tilde{\omega}_X &= \int_X \int_X f_1(x) f_2(y) \tilde{K}_h(x, dy)\, d\tilde{\omega}_X(x) \\
&= \int_X \int_X \frac{\varphi_0(y)\, f_1(x)\, f_2(y)}{\alpha_0(h)\, \varphi_0(x)} K_h(x, dy)\, \varphi_0(x)^2\, d\omega_X(x) \\
&= \frac{1}{\alpha_0(h)} \int_X \varphi_0 f_1 \cdot T_h(\varphi_0 f_2)\, d\omega_X.
\end{aligned}$$

This, $\alpha_0(\bar{h}) = \alpha_0(h)$, and the adjoint relation 4.2(4) for (X, D, K) now lead to 4.2(4) in the deformed case. This completes the proof of the main statement and of part (1).

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Finally, for the proof of (2) we observe that (T1) for (X, D, K) and (8.3) imply that

$$\begin{aligned} f(h \tilde{*} \pi_z(y)) &= \frac{1}{\alpha_0(h)\alpha_0(\pi_z(y))} \cdot (\delta_h * \delta_{\pi_z(y)})(\alpha_0 f) \\ &= \frac{1}{\alpha_0(h)\tilde{\alpha}_0(y)} (\alpha_0 f)_h(\pi_z(y)) \\ &= \frac{1}{\alpha_0(h)\tilde{\alpha}_0(y)} T_h(\alpha_0 f)(\pi_z(y)) \\ &= \int_X f(\pi_x(w))\, \tilde{K}_h(y, dw) = (\tilde{T}_h(f \circ \pi_z))(y). \end{aligned}$$ □

Remark 8.9. Let (X, D, K) be a commutative strong CAS, $\alpha_0 \in C(D)$ a positive semicharacter of $(D, *)$, and $z \in X$. Then $(\alpha_0, \alpha_0 \circ \pi_z)$ is a positive semicharacter of (X, D, K) by 8.4. Consider the associated deformed CAS $(X, D, \tilde{K})$ which has property (T1) for $z \in X$ by 8.8(2). A short computation similar to the proof of 8.8(2) shows that

$$\tilde{T}_f g(x) = \tilde{T}^{f\circ\pi} g(x) \quad \text{for all } f \in C_c(D), \quad g \in C_c(X),$$

that is, (T2) also holds for $(X, D, \tilde{K})$ and $z \in X$.

In the setting of Preposition 8.8, the semicharacters of (X, D, K) and $(X, D, \tilde{K})$ are closely related. This is well known for hypergroup deformations from [36] or [8, Section 2.3].

Lemma 8.10. *Let (X, D, K) and $(X, D, \tilde{K})$ be related as in Proposition 8.8. Then*

$$\{(\alpha/\alpha_0, \varphi/\varphi_0) : (\alpha, \varphi) \textit{ a semicharacter of } (X, D, K)\}$$

is the set of all semicharacters of $(X, D, \tilde{K})$.

Proof. Let (α, φ) be a semicharacter of (X, D, K). Then, for $h \in D$, $x \in X$,

$$\tilde{T}_h(\varphi/\varphi_0)(x) = \frac{1}{\alpha_0(h)\varphi_0(x)} \int_X \varphi(y) K_h(x, dy) = \frac{\alpha(h)\varphi(x)}{\alpha_0(h)\varphi_0(x)}.$$

This shows (8.1) for $(\alpha/\alpha_0, \varphi/\varphi_0)$ and thus the first part of the lemma.

For the converse statement we notice that $(1, 1)$ is a positive character of (X, D, K). Hence, $(1/\alpha_0, 1/\varphi_0)$ is a positive semicharacter of $(X, D, \tilde{K})$ by the first part of the lemma. We now apply the first part of the lemma to $(1/\alpha_0, 1/\tilde{\alpha}_0)$ where the rules of (X, D, K) and $(X, D, \tilde{K})$ are interchanged. This readily proves that each semicharacter of $(X, D, \tilde{K})$ has the form as stated in the lemma. □

9. Orbit schemes and their deformations

We now study examples of semicharacters (α, φ) and associated deformations beyond the case $\varphi = \alpha \circ \pi_z$ for $z \in X$ under condition (T1) or (T2). We know from 4.6 and 5.2 that Gelfand pairs (G, H) lead to commutative, strong CAS $(G/H, G//H, K)$. Typical examples are given by the following orbit construction; see [22] for the background.

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ORBIT SCHEMES 9.1. Let G be a locally compact abelian group and $H \subset \mathrm{Aut}(G)$ a compact group of automorphisms which acts continuously. Form the semidirect product $G \rtimes H$ which contains H as a compact subgroup canonically. Then $(G \rtimes H, H)$ is a Gelfand pair, and we may identify $(G \rtimes H)/H$ with G via $(g, h)H \sim g$ $(g \in G, h \in H)$, and $(G \rtimes H)//H$ with the space G^H of all H-orbits in G via $H(g, h)H \sim g^H := \{h(g) : h \in H\}$ where all spaces carry the quotient topology. Consider the associated commutative strong CAS

$$\Lambda := (X := (G \rtimes H)/H = G, \quad D := (G \rtimes H)//H = G^H,\ K)$$

where the double coset hypergroup $(D, *)$ has the identity $\{e\}$ (e the identity of G), the involution $\overline{g^H} = (g^{-1})^H$, and the convolution

$$\delta_{g_1^H} * \delta_{g_2^H} := \int_H \delta_{(g_1 \cdot h(g_2))^H}\, d\omega_H(h) \quad (g_1, g_2 \in G)$$

for the normalized Haar measure ω_H of H. The Markov-kernel K is given by

$$K_{g_1^H}(g_2, A) := \int_H \delta_{g_2 \cdot h(g_1)}(A)\, d\omega_H(h) = \omega_H(\{h \in H : \ g_2 \cdot h(g_1) \in A\})$$

for $g_1, g_2 \in G$, $A \in \mathcal{B}(G)$. By the proof of Proposition 4.6, the map $\pi : X \times X \to D$ is given by $\pi(g_1 H, g_2 H) := (g_1^{-1} g_2)^H$. Moreover, if ω_G is some Haar measure of G, then $\omega_G \times \omega_H$ is a Haar measure of $G \rtimes H$, and we may choose the measures ω_X, ω_D of Λ as $\omega_X := \omega_G$ and $\omega_D := \varphi(\omega_G)$ for the orbit map $\varphi : G \to G^H$ with $\varphi(g) = g^H$.

We call Λ the orbit scheme associated with (G, H).

In this setting we have multiplicative pairs as follows.

LEMMA 9.2. *Let $\varphi \in C(G)$ be multiplicative, that is, $\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$ for $g_1, g_2 \in G$, with $\varphi \not\equiv 0$. Form $\alpha \in C(D)$ with*

$$\alpha(g^H) := \int_H \varphi(h(g))\, d\omega_H(h) \quad (g \in G). \tag{9.1}$$

Then (α, φ) is multiplicative on Λ. Moreover, the following conditions hold.

(1) *If $\varphi \in C(G)$ is a character of G, then so is the pair (α, φ) on Λ.*
(2) *Let $\varphi_0 \in C(G)$ be positive and multiplicative such that the associated α_0 satisfies $\alpha_0(\overline{g^H}) = \alpha_0(g^H)$ for $g \in G$. Then (α_0, φ_0) is a positive semicharacter on Λ. Hence, with the associated kernel $\tilde{K}$ from Proposition 8.8, $(X, D, \tilde{K})$ is a commutative CAS.*

PROOF. Thus, φ satisfies $\varphi(e) = 1$ and $\varphi(g) \neq 0$ with $\varphi(g^{-1}) = \varphi(g)^{-1}$ for $g \in G$. In particular, we obtain $\alpha \not\equiv 0$. Moreover, as for $g_1, g_2 \in G$,

$$\begin{aligned} T_{g_1^H}\varphi(g_2) &= \int_G \varphi(y)\, K_{g_1^H}(g_2, dy) = \int_H \varphi(g_2 \cdot h(g_1))\, d\omega_H(h) \\ &= \int_H \varphi(g_2) \cdot \varphi(h(g_1))\, d\omega_H(h) = \varphi(g_2) \cdot \alpha(g_1^H), \end{aligned}$$

the first statement is clear. Parts (1) and (2) are then clear. □

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Remark 9.3. Let $\varphi_0 \in C(G)$ and $\alpha_0 \in C(D)$ be as in the setting of Lemma 9.2(2).

(1) We are mainly interested in nontrivial φ_0, that is, $\varphi_0 \not\equiv 1$, as otherwise 9.2(2) does not lead to a nonidentical deformation. This clearly works for noncompact groups G only.

(2) The push forwards $\pi_x(\tilde{\omega}_X) \in M^+(D)$ of invariant measures $\tilde{\omega}_X$ as in 8.8(2) for $x \in X$ usually will not be Haar measures on $(D, \tilde{*})$; for examples see below. Therefore, by Lemmas 5.8 and 5.9, the deformed CAS of Proposition 9.2 usually do not have (T1) and (T2).

(3) Consider the original orbit scheme Λ as in Section 9.1. Let $\hat{G}$ be the dual group on which H also acts via $h(\alpha)(g) := \alpha(h(g))$. Consider the orbit maps $\Phi : G \to G^H = D$, $g \mapsto g^H$ and $\hat{\Phi} : \hat{G} \to (\hat{G})^H$, $\alpha \mapsto \alpha^H$. It is well known that $(\hat{G})^H$ can be identified with the dual $(D, *)^\wedge$ via $\hat{\Phi}(\alpha)(g^H) := \int_H \alpha(h(g))\, d\omega_H(h)$, and that for a Haar measure ω_G and its associated Plancherel measure $\omega_{\hat{G}}$ (which is a Haar measure of $\hat{G}$), the push forwards $\Phi(\omega_G) \in M^+(D)$ and $\hat{\Phi}(\omega_{\hat{G}}) \in M^+((D, *)^\wedge)$ are a Haar measure of $(D, *)$ and its Plancherel measure respectively; see [22]. In particular, the support S of the Plancherel measure is equal to $(D, *)^\wedge$ for these examples.
On the other hand, if $\varphi_0 \in C(G)$ is a positive multiplicative function with $\alpha_0 \not\equiv 1$ as in 9.2(2), then for the associated deformed hypergroup $(D, \tilde{*})$ the support $\tilde{S}$ of the Plancherel measure is a proper subset of $(D, \tilde{*})^\wedge$ for $\tilde{\alpha}_0 \not\equiv 1$. In fact, we even have $\mathbf{1} \notin \tilde{S}$ by [36].

It is an interesting problem whether $(D, \tilde{*})^\wedge$ or $\tilde{S}$ carry dual positive convolutions. Generally, the answer is negative for $(D, \tilde{*})^\wedge$; see below. On the other hand, for $\tilde{S}$ there exist some positive results. In fact, for $\tilde{S}$, this problem is closely related to a property of α_0.

Lemma 9.4. *In the setting of Lemma 9.2(2), the following statements are equivalent:*

(1) $1/\alpha_0 \in C_b(D)$ *is positive definite on the orbit hypergroup* $(D, *)$*;*

(2) *for all* $\tilde{\alpha}, \tilde{\beta} \in \tilde{S}$ *there exists* $\tilde{\mu}_{\tilde{\alpha},\tilde{\beta}} \in M^1(\tilde{S})$ *with*

$$\tilde{\alpha}(x)\tilde{\beta}(x) = \int_{\tilde{S}} \gamma(x)\, d\tilde{\mu}_{\tilde{\alpha},\tilde{\beta}}(\gamma) \quad \textit{for } x \in D.$$

(3) *each character* $\tilde{\alpha} \in \tilde{S}$ *is positive definite on* $(D, *)$.

Proof. For (1) $\Longrightarrow$ (3) assume that $1/\alpha_0$ is positive definite on $(D, *)$. For $\tilde{\alpha} \in \tilde{S}$ we find a unique $\alpha \in (D, *)^\wedge$ with $\tilde{\alpha} = \alpha/\alpha_0$ by [36]. As $(D, *)^\wedge$ carries a dual positive convolution on $(D, *)^\wedge$, we see that $\tilde{\alpha} = \alpha \cdot (1/\alpha_0)$ is positive definite on $(D, *)$ as claimed.

(3) $\Longrightarrow$ (2) follows from (T2) for (X, D, K), $S = (D, *)^\wedge$, and Corollary 7.3.

Finally, for (2) $\Longrightarrow$ (1) we take $\tilde{\alpha} := \tilde{\beta} := 1/\alpha_0 \in \tilde{S}$ in (2). The homeomorphism $S \to \tilde{S}$, $\alpha \mapsto \alpha/\alpha_0$ then yields that $1/\alpha_0$ is the inverse Fourier transform of some $\mu \in M^1(S)$ as claimed. □

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REMARK 9.5. Consider some example in the setting of Lemma 9.2(2) with $\alpha \not\equiv \mathbf{1}$ such that one and thus all statements of Lemma 9.4 hold. Then $(X, D, K) \neq (X, D, \tilde{K})$ and $(D, *) \neq (d, \tilde{*})$. This shows that the technical condition $\mathbf{1} \in \tilde{S}$ in Theorem 7.4 is essential.

We now present some examples for the theory of Sections 9.1–9.4.

EXAMPLES 9.6. Fix an integer $d \geq 1$ and put $G := (\mathbb{R}^d, +)$ and $H := O(d)$ as the orthogonal group acting on G. We use the canonical identification $D = [0, \infty[$. Then $(D, *)$ is the so-called Bessel–Kingman hypergroup of index $\alpha = d/2 - 1$; see, for example, [8, 22], and [26].

The multiplicative functions on G have the form

$$\varphi(x) = \varphi_z(x) := e^{i\langle z,x\rangle} := e^{i\sum_{k=1}^d z_k x_k}$$

with $z \in \mathbb{C}^d$. Then φ_z is a character precisely for $z \in \mathbb{R}^d$, and φ_z is positive precisely for $z \in i \cdot \mathbb{R}^d$.

In the first case, the character $\alpha_z \in (D, *)^\wedge$ associated with φ_z ($z \in \mathbb{R}^d$) according to (9.1) is given by $\alpha_z(w) = j_\alpha(w \cdot \|z\|_2)$ with the modified Bessel functions

$$j_\alpha(y) :=_0 F_1(\alpha + 1; -y^2/4) \quad (y \in \mathbb{C}).$$

Please be careful with the different meaning of the parameter α and the functions α_z.

In the second case the positive multiplicative function $\alpha_z \in C(D)$ associated with φ_z ($z \in i \cdot \mathbb{R}^d$) is given by $\alpha_z(w) = j_\alpha(iw \cdot \|z\|_2)$. In particular, as the hypergroup $(D, *)$ is symmetric, all conditions of 9.2 are satisfied in this case, that is, (α_z, φ_z) is a positive semicharacter of our orbit CAS $(\mathbb{R}^d, [0, \infty[, K)$, and (α_z, φ_z) leads to a deformation for each $z \in i \cdot \mathbb{R}^d$. We now study examples for the equivalent conditions of 9.4, and we discuss whether the complete dual $(D, \tilde{*})^\wedge$ carries a dual positive convolution.

Before doing this, we notice that for each $c > 0$, the map $x \mapsto cx$ is a hypergroup automorphism on $(D = [0, \infty[, *)$. This ensures that we may restrict our attention to $z \in i \cdot \mathbb{R}^d$ with $\|z\|_2 = 1$ without loss of generality.

EXAMPLES 9.7.

(1) Let $d = 1$, that is, $\alpha = -1/2$ and $j_{-1/2}(x) = \cos x$. Let $z = \pm i$. The deformed hypergroup $(D = [0, \infty[, \tilde{*})$ is then the so-called cosh-hypergroup; see [47] and [8, Sections 3.4.7 and 3.5.72]. The characters are given by

$$\alpha_\lambda(x) := \frac{\cos(\lambda x)}{\cosh x} \quad (x \in [0, \infty[, \lambda \in [0, \infty[\cup i \cdot [0, 1])$$

where in this parameterization, α_λ is in the support of the Plancherel measure precisely for $\lambda \in [0, \infty[$. Using

$$\frac{\cos(\lambda x)}{\cosh x} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos(tx)}{\cosh((t + \lambda)\pi/2)}\, dt \quad \text{for } \lambda \in \mathbb{C}, \quad |\Im\lambda| < 1$$

(see (1) in [47] and references therein), we see that the first condition of 9.2 holds. Hence, by 9.2, the support of the Plancherel measure of $(D, \tilde{*})$ carries

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a positive dual convolution. This convolution was computed explicitly in [47]. We remark, that by [47], there does not exist a positive dual convolution on the complete dual space.

(2) Let $d = 3$, that is, $\alpha = 1/2$ and $j_{1/2}(x) = \sin x/x$. Let $\|z\|_2 = 1$. In this case, the deformed hypergroup $(D, \tilde{*})^\wedge$ is the Naimark hypergroup with convolution

$$\delta_x \tilde{*} \delta_y = \frac{1}{\sinh x \sinh y} \int_{|x-y|}^{x+y} \sinh t \, \delta_t \, dt \quad (x, y \in [0, \infty[);$$

see [8, 22] and [47]. This example is also isomorphic with the double coset hypergroup SU(1, 1)//SU(2), and it follows from the work of Flensted-Jensen and Koornwinder [17, 18, 27] that all bounded hermitian spherical functions are positive definite on SU(1, 1) and that thus the complete dual $(D, \tilde{*})^\wedge$ carries a dual positive convolution. The dual convolution was computed explicitly in [47].

Here is a short list of further examples for 9.1–9.4.

Examples 9.8.

(1) Put $G := (\mathbb{Z}, +)$ and $H := \{\pm 1\}$ which acts multiplicatively on G. Then $D = \mathbb{N}_0$ in a canonical way, and $(D, *)$ is the so-called discrete polynomial hypergroup associated with T-polynomials of the first kind; see, for example, [29] and [8]. The associated transition matrices are given by

$$S_0 = I_{\mathbb{Z}}, \quad S_k(x, y) = \tfrac{1}{2}\delta_{k,|x-y|} \quad (k \in \mathbb{N}, \ x, y \in \mathbb{Z})$$

with the Kronecker-δ.
Similar to Examples 9.6 and 9.7(1), we consider $\varphi_z(k) := e^{zk}$ for $k \in \mathbb{Z}, z \in \mathbb{R}$. Then $\alpha_z(n) = \cosh(zn)$ for $n \in \mathbb{N}_0$, and we obtain deformed CAS similar to 9.7(1). For further details on this discrete example see also [43, Example 5.11].

(2) Fix integers $p \geq q \geq 1$ as well as one of the division algebras $\mathbb{F} := \mathbb{R}, \mathbb{C}$, or quaternions $\mathbb{H}$. Take $G := (M_{p,q}(\mathbb{F}), +)$ as the additive group of $p \times q$ matrices over $\mathbb{F}$ on which the unitary group $H := U_p(\mathbb{F})$ acts from the left. G is a real Euclidean vector space of dimension dpq with real scalar product $(x|y) = \Re\mathrm{tr}(x^*y)$ where $x^* = \overline{x}^t$, $\Re t = \frac{1}{2}(t + \overline{t})$ is the real part of $t \in \mathbb{F}$, and tr the trace in $M_{q,q}(\mathbb{F})$. The action of H is orthogonal with respect to this scalar product, and $x, y \in G$ are in the same H-orbit if and only if $x^*x = y^*y$. Thus, the space of H-orbits is naturally parametrized by the cone $D := \Pi_q(\mathbb{F})$ of positive semidefinite $q \times q$-matrices over $\mathbb{F}$.
For $q = 1$ and $\mathbb{F} = \mathbb{R}$, we just have $\Pi_1 = [0, \infty[$, and we end up with the one-dimensional examples in Section 9.6. For $q \geq 2$, the associated orbit hypergroup structures were discussed in [32] where the associated multiplicative functions are Bessel functions of matrix argument.
Similar to Section 9.6, we now fix $z \in G$, and consider the positive multiplicative function $\varphi_z(x) := e^{(x|z)}$ on G. The associated positive semicharacter α_z on $(D, *)$ can be written down explicitly in terms of Bessel functions of matrix argument. The associated deformed CAS $(M_{p,q}(\mathbb{F}), \Pi_q(\mathbb{F}), \tilde{K})$ may now be written down explicitly.

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(3) We mention a further example. Fix an integer $q \geq 1$ as well as $\mathbb{F}$ as above. Let $G := (H_q(\mathbb{F}), +)$ be the vector space of all $\mathbb{F}$-hermitian $q \times q$-matrices on which the unitary group $H := U_p(\mathbb{F})$ acts by conjugation. Here, two matrices $x, y \in G$ are in the same H-orbit if and only if x and y have the same (ordered) spectrum, that is, we may identify the space of H-orbits with the Weyl chamber

$$C_q := \{(x_1, \ldots, x_q) \in \mathbb{R}^q : \; x_1 \geq \cdots \geq x_q\}$$

of type A. Again we may write down the multiplicative functions φ_z on G explicitly, where the associated positive multiplicative functions α_z on $(D = C_q, *)$ are Bessel functions of type A.

REMARK 9.9. The example in 9.7(1) shows that the condition $\mathbf{1} \in \tilde{S}$ in Corollary 7.2 is necessary. To explain this, define (X, D, K) as the orbit CAS from 9.6(1) for $d = 1$. Then (T2) holds for (X, D, K). Now let $(X, D, \tilde{K})$ be the deformation of this CAS considered in 9.6(2). If Corollary 7.2 would be correct, we would find some $\mu \in M^1([0, \infty[)$ with

$$\cos x = \int_0^\infty \frac{\cos(\lambda x)}{\cosh x}\, d\mu(\lambda) \quad \text{for all } x \in [0, \infty[.$$

If we write the factor $\cosh x$ on the left-hand side, it becomes clear that such a measure μ does not exist.

REMARK 9.10. Let $d \geq 1$ be an integer. Consider the orbit CAS $(X = \mathbb{R}^d, D = [0, \infty[, K)$ from 9.6. Fix vectors $z_1, z_2 \in i \cdot \mathbb{R}^d$ with $\|z_1\|_2 = \|z_2\|_2 = 1$ and $z_1 \neq z_2$. Consider the associated multiplicative pairs $(\alpha_{z_j}, \varphi_{z_j})$ $(j = 1, 2)$ with $\varphi_{z_j} := e^{-i\langle z_j, x\rangle}$ and $\alpha_{z_1} = \alpha_{z_2}$.

Now consider the deformation $(X, D, \tilde{K})$ of (X, D, K) associated with $(\alpha_{z_1}, \varphi_{z_1})$. Then, by Lemma 8.10, $(\varphi_{z_2}/\varphi_{z_1}, \alpha_{z_2}/\alpha_{z_1} = \mathbf{1})$ is a positive semicharacter of (X, D, K), that is, $\varphi_{z_2}/\varphi_{z_1}$ is a positive, unbounded harmonic function of $(X, D, \tilde{K})$.

This construction of positive, unbounded harmonic functions can be extended to other classes of examples like the discrete ones in the next section.

10. Examples associated with infinite distance-transitive graphs

The set of all infinite distance transitive graphs of finite valency can be parametrized by two parameters as follows by Macpherson [31].

Let $a, b \geq 2$ be integers. Let C_b the complete undirected graph with b vertices, that is, all vertices of C_b are connected. Consider the infinite graph $\Gamma := \Gamma(a, b)$ where precisely a copies of the graph C_b are tacked together at each vertex in a tree-like way, that is, there are no other cycles in Γ than those in a copy of C_b. For $b = 2$, Γ is the homogeneous tree of valency a. We denote the natural distance function on Γ by d.

After drawing a picture, it is clear that the group $G := \mathrm{Aut}(\Gamma)$ of all graph automorphisms acts on Γ in a distance-transitive way, that is, for all $v_1, v_2, v_3, v_4 \in \Gamma$ with $d(v_1, v_3) = d(v_3, v_4)$ there exists $g \in \Gamma$ with $g(v_1) = v_3$ and $g(v_2) = v_4$. $\mathrm{Aut}(\Gamma)$ is a totally disconnected, locally compact group with respect to the topology of pointwise

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convergence, and the stabilizer subgroup $H \subset G$ of any fixed vertex $e \in \Gamma$ is compact and open. We identify G/H with Γ, and $G//H$ with $\mathbb{N}_0$ by distance transitivity. We now study the association scheme $\Lambda = (\Gamma \simeq G/H, \mathbb{N}_0 = G//H, (R_i)_{i\in\mathbb{N}_0})$ with (T1) and (T2) as well as the associated double coset hypergroup $(\mathbb{N}_0 \simeq G//H, *)$. As in the case of finite distance-transitive graphs in [5], Λ and $(\mathbb{N}_0, *)$ are symmetric and associated with a sequence of orthogonal polynomials in the Askey scheme [3].

More precisely, it can be seen by some counting (see [37]) that the hypergroup convolution is given by

$$\delta_m * \delta_n = \sum_{k=|m-n|}^{m+n} g_{m,n,k}\delta_k \in M^1(\mathbb{N}_0) \quad (m, n \in \mathbb{N}_0)$$

with

$$g_{m,n,m+n} = \frac{a-1}{a} > 0, \quad g_{m,n,|m-n|} = \frac{1}{a(a-1)^{m\wedge n-1}(b-1)^{m\wedge n}} > 0,$$

$$g_{m,n,|m-n|+2k+1} = \frac{b-2}{a(a-1)^{m\wedge n-k-1}(b-1)^{m\wedge n-k}} \geq 0 \quad (k = 0, \ldots, m\wedge n-1),$$

$$g_{m,n,|m-n|+2k+2} = \frac{a-2}{a(a-1)^{m\wedge n-k-1}(b-1)^{m\wedge n-k-1}} \geq 0 \quad (k = 0, \ldots, m\wedge n-2).$$

The Haar weights are given by $\omega_0 := 1$, $\omega_n = a(a-1)^{n-1}(b-1)^n \quad (n \geq 1)$. Using

$$g_{n,1,n+1} = \frac{a-1}{a}, \quad g_{n,1,n} = \frac{b-2}{a(b-1)}, \quad g_{n,1,n-1} = \frac{1}{a(b-1)},$$

we define a sequence of orthogonal polynomials $(P_n^{(a,b)})_{n\geq 0}$ by

$$P_0^{(a,b)} := 1, \quad P_1^{(a,b)}(x) := \frac{2}{a} \cdot \sqrt{\frac{a-1}{b-1}} \cdot x + \frac{b-2}{a(b-1)},$$

and the three-term-recurrence relation

$$P_1^{(a,b)}P_n^{(a,b)} = \frac{1}{a(b-1)}P_{n-1}^{(a,b)} + \frac{b-2}{a(b-1)}P_n^{(a,b)} + \frac{a-1}{a}P_{n+1}^{(a,b)} \quad (n \geq 1). \tag{10.1}$$

Then,

$$P_m^{(a,b)}P_n^{(a,b)} = \sum_{k=|m-n|}^{m+n} g_{m,n,k}P_k^{(a,b)} \quad (m, n \geq 0).$$

We discuss some properties of the $P_n^{(a,b)}$ from [37, 42]. Equation (10.1) yields

$$P_n^{(a,b)}\Big(\frac{z+z^{-1}}{2}\Big) = \frac{c(z)z^n + c(z^{-1})z^{-n}}{((a-1)(b-1))^{n/2}} \quad \text{for } z \in \mathbb{C} \setminus \{0, \pm 1\} \tag{10.2}$$

with

$$c(z) := \frac{(a-1)z - z^{-1} + (b-2)(a-1)^{1/2}(b-1)^{-1/2}}{a(z-z^{-1})}.$$

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We define

$$s_0 := s_0^{(a,b)} := \frac{2-a-b}{2\sqrt{(a-1)(b-1)}}, \quad s_1 := s_1^{(a,b)} := \frac{ab-a-b+2}{2\sqrt{(a-1)(b-1)}}.$$

Then

$$P_n^{(a,b)}(s_1) = 1, \quad P_n^{(a,b)}(s_0) = (1-b)^{-n} \quad (n \geq 0).$$

It is shown in [42] that the $P_n^{(a,b)}$ fit into the Askey-Wilson scheme ([3, pages 26–28]). By the orthogonality relations in [3], the normalized orthogonality measure $\rho = \rho^{(a,b)} \in M^1(\mathbb{R})$ is

$$d\rho^{(a,b)}(x) = w^{(a,b)}(x)dx\Big|_{[-1,1]} \quad \text{for } a \geq b \geq 2$$

and

$$d\rho^{(a,b)}(x) = w^{(a,b)}(x)dx\Big|_{[-1,1]} + \frac{b-a}{b}d\delta_{s_0} \quad \text{for } b > a \geq 2$$

with

$$w^{(a,b)}(x) := \frac{a}{2\pi} \cdot \frac{(1-x^2)^{1/2}}{(s_1-x)(x-s_0)}.$$

For $a, b \in \mathbb{R}$ with $a, b \geq 2$, the numbers s_0, s_1 satisfy

$$-s_1 \leq s_0 \leq -1 < 1 \leq s_1.$$

By Equation (10.2), we have the dual space

$$\hat{D} \simeq \{x \in \mathbb{R} : (P_n^{(a,b)}(x))_{n\geq 0} \text{ is bounded}\} = [-s_1, s_1].$$

This interval contains the support

$$S := \operatorname{supp} \rho^{(a,b)} = \begin{cases} [-1,1] & \text{for } a \geq b \geq 2 \\ \{s_0\} \cup [-1,1] & \text{for } b > a \geq 2 \end{cases} \tag{10.3}$$

of the orthogonality measure, which is also the Plancherel measure; see [29]. We have $S = \hat{D}$ precisely for $a = b = 2$. The following theorem from [42] shows that for these examples several interesting phenomena appear, and that Theorem 6.9 cannot be extended considerably from S to a bigger subset of $\hat{D}$.

THEOREM 10.1. *In the setting above the following statements are equivalent for $x \in \mathbb{R}$.*

(1) $x \in [s_0, s_1]$.

(2) *The kernel $\Gamma \times \Gamma \to \mathbb{R}$, $(v_1, v_2) \longmapsto P_{d(v_1,v_2)}^{(a,b)}(x)$ is positive definite.*

(3) *The mapping $g \longmapsto P_{d(gH,e)}^{(a,b)}(x)$ is positive definite on G.*

Moreover, for all $x, y \in [s_0, s_1]$ there exists a unique $\mu_{x,y} \in M^1([-s_1, s_1])$ with

$$P_n^{(a,b)}(x) \cdot P_n^{(a,b)}(y) = \int_{-s_1}^{s_1} P_n^{(a,b)}(z)\, d\mu_{x,y}(z) \quad \text{for all } n \in \mathbb{N}_0. \tag{10.4}$$

Finally, there are $b > a$ and $x, y \in [-s_1^{(a,b)}, s_0^{(a,b)}[$ for which no $\mu_{x,y} \in M^1(\mathbb{R})$ exists with (10.4).

For homogeneous trees, Theorem 10.1 was derived by Letac [30].

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We next construct examples of positive semicharacters of Λ and study the associated deformed CAS. The approach will be similar to [44] for homogeneous trees. However, we shall use the results of Section 8 which will simplify some computations.

Fix some constant $c \in \mathbb{R}$ as well as some point B in the boundary $\partial\Gamma$, that is, B is a sequence $(v_n)_{n\in\mathbb{N}_0} \subset \Gamma$ of vertices with $d(v_{n+m}, v_n) = m$ for $n, m \in \mathbb{N}_0$ where v_0 is the vertex above which is stabilized by H. We define some kind of 'distance' $d(v, B) \in \mathbb{Z}$ of a vertex $v \in \Gamma$ from B as follows: for $v \in \Gamma$ there is a unique index $n_0 \in \mathbb{N}_0$ such that $d(v, v_n) \in \mathbb{N}_0$ is minimal for $n = n_0$. We then put $d(v, B) := d(v, v_{n_0}) - n_0$. We in particular have the normalization $d(v_0, B) = 0$. We now define the function $\varphi := \varphi_{B,c} : \Gamma \to]0, \infty[$ with

$$\varphi(v) := e^{c\cdot d(v,B)}.$$

Proposition 10.2. *Thus, φ is a joint eigenfunction of all transition operators T_h, $h \in \mathbb{N}_0$. More precisely, for all $v \in \Gamma$,*

$$T_h\varphi(v) = \frac{1}{|\{w \in \Gamma : \ d(v, w) = h\}|} \sum_{w\in\Gamma:\ d(v,w)=h} \varphi(w) = P_h^{(a,b)}(x_c) \cdot \varphi(v)$$

with

$$x_c := \frac{1}{2}\Big(e^c \sqrt{(a-1)(b-1)} + \frac{1}{e^c \sqrt{(a-1)(b-1)}}\Big) \in [1, \infty[.$$

Proof. The assertion is trivial for $h = 0$.

Assume now that $h \geq 1$. We first observe by counting that

$$|S(v, h)| = a(a-1)^{h-1}(b-1)^h \quad \text{for } S(v, h) := \{w \in \Gamma : \ d(v, w) = h\}.$$

Moreover, again by counting we have the following facts:
there is 1 vertex $w \in S(v, h)$ with $\varphi(w) = e^{c\cdot d(w,B)} = e^{c\cdot(d(v,B)-h)}$;
there are $b - 2$ vertices $w \in S(v, h)$ with $\varphi(w) = e^{c\cdot(d(v,B)-h+1)}$;
there are $(a-2)(b-1)$ vertices $w \in S(v, h)$ with $\varphi(w) = e^{c\cdot(d(v,B)-h+2)}$ and so on.
In general, we see that for $k = 0, 1, \ldots, h-1$, there are $(b-2)(a-1)^k(b-1)^k$ vertices $w \in S(v, h)$ with $\varphi(w) = e^{c\cdot(d(v,B)-h+2k+1)}$,
and for $k = 0, 1, \ldots, h-2$, there are $(a-2)(b-1)^{k+1}(a-1)^k$ vertices $w \in S(v, h)$ with $\varphi(w) = e^{c\cdot(d(v,B)-h+2k+2)}$.
Finally, there are $(a-1)^h(b-1)^h$ vertices $w \in S(v, h)$ with $\varphi(w) = e^{c\cdot(d(v,B)+h)}$.

If we insert these facts into the definition of T_h and use the formula for a geometric sum twice, we arrive at

$$T_h\varphi(v) = \alpha(h, a, b, c) \cdot \varphi(v)$$

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with

$$\begin{aligned}\alpha(h,a,b,c) :=& \frac{1}{a(a-1)^{h-1}(b-1)^h}\Big(1\cdot e^{-ch} + (a-1)^h(b-1)^h e^{ch}\\ &+ (b-2)e^{-ch}\sum_{k=0}^{h-1}(a-1)^k(b-1)^k e^{(2k+1)c}\\ &+ (a-2)(b-1)e^{-ch}\sum_{k=0}^{h-2}(a-1)^k(b-1)^k e^{(2k+2)c}\Big)\\ =& \frac{e^{-ch}}{a(a-1)^{h-1}(b-1)^h}\Big(1 + (a-1)^h(b-1)^h e^{2ch}\\ &+ \frac{(b-2)e^c[(a-1)^h(b-1)^h e^{2ch}-1]}{(a-1)(b-1)e^{2c}-1}\\ &+ (a-2)(b-1)e^{2c}\frac{(a-1)^{h-1}(b-1)^{h-1}e^{2c(h-1)}-1}{(a-1)(b-1)e^{2c}-1}\Big).\end{aligned}\tag{10.5}$$

In particular, φ is a joint eigenfunction of all T_h. We now conclude from Proposition 8.3 that the mapping $\mathbb{N}_0 \to]0,\infty[$, $h \mapsto \alpha(h,a,b,c)$ is multiplicative on the symmetric polynomial hypergroup $(\mathbb{N}_0,*)$ which implies that this mapping is a positive semicharacter. On the other hand, it follows from the theory of polynomial hypergroups (see [8]) that the positive semicharacters on $(\mathbb{N}_0,*)$ have the form $h \mapsto P_h^{(a,b)}(x)$ with some unique $x \in [1,\infty[$. In order to find the correct x, we compare the explicit representation (10.2) of $P_h^{(a,b)}(x)$ with the eigenvalues $\alpha(h,a,b,c)$ in (10.5) for large values of h. This leads readily to

$$x = x_c = \frac{1}{2}\Big(e^c\sqrt{(a-1)(b-1)} + \frac{1}{e^c\sqrt{(a-1)(b-1)}}\Big)$$

and thus $\alpha(h,a,b,c) = P_h^{(a,b)}(x_c)$ as claimed. □

We can now use the results of Section 8 to define a deformed CAS $(\Gamma,\mathbb{N}_0,\tilde{K})$ according to Proposition 8.8 with $\tilde{K}_0(x,\{y\}) = \delta_{x,y}$ and, for $h \geq 1$,

$$\begin{aligned}\tilde{K}_h(x,\{y\}) :=& \frac{e^{c(d(y,B)-d(x,B))}}{P_h^{(a,b)}(x_c)}K_h(x,\{y\})\\ =& \frac{e^{c(d(y,B)-d(x,B))}}{P_h^{(a,b)}(x_c)\cdot a(a-1)^{h-1}(b-1)^h}\end{aligned}$$

for $x,y \in X$. For $b = 2$, that is, for homogeneous trees, this result fits with the results in [44] where the same kernels were obtained in a different, more computational way.

We point out that in particular the case $x_c = 1$ is interesting which appears precisely for

$$c := -\tfrac{1}{2}\ln((a-1)(b-1)).$$

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In this case, the associated deformed polynomial hypergroup $(\mathbb{N}_0, \tilde{*})$ has the functions

$$\mathbb{N}_0 \to \mathbb{R}, \quad n \mapsto \widetilde{P_n}^{(a,b)}(x) := P_n^{(a,b)}(x)/P_n^{(a,b)}(1) \quad (x \in \mathbb{R})$$

as semicharacters. It can be easily derived from (10.2) that here the dual space $(D, \tilde{*})^\wedge$ corresponds to S from (10.3).

We point out that the ideas of this section may be used to study deformations of infinite commutative association schemes associated with affine buildings for example of type $\tilde{A}_n$.

REMARK 10.3. Consider a homogeneous tree Γ of valeny a, that is, we take $b = 2$ above. Fix a point $B \in \partial\Gamma$ in the boundary and a constant $c \in \mathbb{R} \setminus \{0\}$ as before, and consider the associated multiplicative pair and the associated deformed CAS $(\Gamma, \mathbb{N}_0, \tilde{K})$ as above. Let $v_0 \in \Gamma$ as above. Then, by Proposition 8.8(1), an invariant measure ω_Γ of $(\Gamma, \mathbb{N}_0, \tilde{K})$ is given by $\omega_\Gamma(v) = e^{2c\cdot d(v,B)}$ for $v \in \Gamma$. Its push forward $\pi_{v_0}(\omega_\Gamma) \in M^+(\mathbb{N}_0)$ then satisfies

$$\pi_{v_0}(\omega_\Gamma)(0) = 1, \quad \pi_{v_0}(\omega_\Gamma)(1) = e^{-2c} + (a-1)e^{2c},$$

and for $n \geq 2$,

$$\pi_{v_0}(\omega_\Gamma)(n) = e^{-2cn} + \sum_{l=1}^{n-1} e^{2(-n+2l)}(a-2)(a-1)^{l-1} + e^{2cn}(a-1).$$

This measure $\pi_{v_0}(\omega_\Gamma)$ is usually not equal to the Haar measure ω_c of the (deformed) polynomial hypergroup $(\mathbb{N}_0, *_c)$ with normalization $\omega_c(0) = 1$, as we have

$$\omega_c(1) = \frac{1}{g^c_{1,1,0}} = \frac{((a-1)e^{2c} + 1))^2}{ae^{2c}}$$

which is usually different from $\pi_{v_0}(\omega_\Gamma)(1)$ above. We thus in particular conclude from Lemmas 5.8(1) and 5.9(1) that for $c \neq 0$, the deformed CAS $(\Gamma, \mathbb{N}_0, \tilde{K})$ usually do not have the properties (T1) and (T2).

11. Further constructions of continuous association schemes

In this section we present further constructions of CAS from given ones. We start with the direct product.

DIRECT PRODUCTS 11.1. Let (X_1, D_1, K^1) and (X_2, D_2, K^2) be CAS with associated hypergroups $(D_1, *_1)$ and $(D_2, *_2)$. We then form $X := X_1 \times X_2$ and $D := D_1 \times D_2$. We define the convolution of point measures via the direct product of measures by

$$\delta_{(x_1,x_2)} * \delta_{(y_1,y_2)} := (\delta_{x_1} *_1 \delta_{y_1}) \times (\delta_{x_2} *_2 \delta_{y_2}) \quad (x_1, y_1 \in D_1, x_2, y_2 \in D_2).$$

It is well known that the unique bilinear, weakly continuous extension of this convolution leads to the so-called direct product hypergroup $(D, *)$; see [22, Section 10.5] or [8, Section 1.5.28]. We now put

$$K_{(h_1,h_2)}((x_1, x_2), A_1 \times A_2) := K^1_{h_1}(x_1, A_1) \cdot K^2_{h_2}(x_2, A_2) \tag{11.1}$$

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for $h_i \in D_i$, $x_i \in X_i$, and Borel sets $A_i \subset X_i$ with $i = 1, 2$. It is well known and can be easily checked that (11.1) leads to a unique Markov-kernel $K_{(h_1,h_2)}$ on X for $(h_1, h_2) \in D_1 \times D_2$. Moreover, it can be easily seen that these kernels can be combined to a Markov-kernel K from $X \times D$ to X. We have the following proposition.

Proposition 11.2. *$(X := X_1 \times X_2, D := D_1 \times D_2, K)$ is a CAS; it will be called the direct product of (X_1, D_1, K^1) and (X_2, D_2, K^2).*

If (X_1, D_1, K^1) and (X_2, D_2, K^2) are commutative, symmetric, compact, or discrete, then so is (X, D, K). Moreover, the properties (T1) and (T2) are also preserved.

Proof. First of all, one has to check that K is a continuous Markov-kernel in the notion of the beginning of Section 4. For this we first notice that for $g \in C_c(X)$ of the form $g(x_1, x_2) = g_1(x_1)g_2(x_2)$ with $g_i \in C_c(X_i)$, the map $((x_1, x_2), h) \mapsto T_h(g)(x_1, x_2)$ is continuous by the product structure in (11.1). As by the theorem of Stone–Weierstrass, the linear span of functions on X of the form $g(x_1, x_2) = g_1(x_1)g_2(x_2)$ for $g_i \in C_c(X_i)$ is $\|.\|_\infty$-dense in $C_c(X)$, the map above is continuous even for all $g \in C_c(X)$. Taking Lemma 4.8 into account, we see that the map above is continuous even for all $g \in C_b(X)$.

All properties in 4.2(1) and (2) can be checked easily. We only mention that for the projections π_1 and π_2 of the given CAS, the projection $\pi : X \times X \to D$ satisfies $\pi((x_1, x_2), (y_1, y_2)) = (\pi_1(x_1, y_1), \pi_2(x_2, y_2))$ by (11.1). 4.2(3) can also be checked easily by the product structure in (11.1). Moreover, if we define the product measure $\omega_X := \omega_{X_1} \times \omega_{X_2}$, then Equation (4.1) in 4.2(4) can be also checked easily.

The statement about (X, D, K) being commutative, symmetric, compact, or discrete is also trivial.

We next check property (T2). We here first notice that for Haar measures ω_{D_i} of $(D_i, *_i)$ $(i = 1, 2)$, the product $\omega_D := \omega_{D_1} \times \omega_{D_2}$ is a Haar measure of $(D, *)$. Using (T2) for the given schemes, we readily obtain for $f_i \in C_c(D_i)$, $g_i \in C_c(X_i)$, and $x_i \in X_i$ $(i = 1, 2)$ that

$$\int_X f_1(\pi_1(x_1, x_2))f_2(\pi_2(y_1, y_2))\, g(y_1)g(y_2)\, d(\omega_{X_1} \times \omega_{X_2})(y_1, y_2)$$
$$= \int_D \int_X g(y_1)g(y_2)\, K_{h_1,h_2}((x_1, x_2), d(y_1, y_2))\, f_1(h_1)f_2(h_2)d\omega_D(h_1, h_2).$$

Again, the theorem of Stone–Weierstrass shows that $T_f g = T^{f\circ\pi} g$ holds for $f \in C_c(D)$ and $g \in C_c(X)$ as claimed.

Property (T1) can be derived in the same way by a Stone–Weierstrass argument. $\square$

We next turn to joins.

Joins of CAS 11.3. We first recapitulate the join of hypergroups from [22, Section 10.5]. Let $(D_1, *_1)$ be a discrete hypergroup with identity $e_1 \in D_1$, and let $(D_2, *_2)$ be a compact hypergroup with normalized Haar measure ω_{D_2}. We form the disjoint union

$$D := D_1 \vee D_2 := (D_1 \setminus \{e_1\}) \cup D_2$$

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with $D_1 \setminus \{e_1\}$ and D_2 as open subsets. D is locally compact. We define the convolution of point measures on D via

$$\delta_x * \delta_y := \begin{cases} \delta_x *_2 \delta_y & \text{for } x, y \in D_2 \\ (\delta_x *_1 \delta_y)|_{D_1 \setminus \{e_1\}} + (\delta_x *_1 \delta_y)(\{e_1\})\omega_2 & \text{for } x, y \in D_1 \\ \delta_x & \text{for } x \in D_1, y \in D_2 \\ \delta_y & \text{for } y \in D_1, x \in D_2. \end{cases}$$

Assume now that $(D_1, *_1)$ is associated with some discrete CAS (X_1, D_1, K^1) and $(D_2, *_2)$ with some compact CAS (X_2, D_2, K^2). We assume that ω_{D_2} and ω_{X_2} are normalized such that they are both probability measures. We form the join $(D, *)$ as above and put $X := X_1 \times X_2$. Moreover, for $h \in D$, $x_1 \in X_1$, $x_2 \in X_2$, Borel sets $B \subset X_2$ and sets $A \subset X_1$ we put

$$K_h((x_1, x_2), A \times B) := \begin{cases} K_h^2(x_2, B) \cdot \delta_{x_1}(A) & \text{for } h \in D_2 \\ \omega_{X_2}(B) \cdot K_h^1(x_1, A) & \text{for } h \in D_1 \setminus \{e_1\}. \end{cases} \tag{11.2}$$

Clearly, each K_h can be extended uniquely to a Markov-kernel on X, and we can combine the K_h to some Markov-kernel K from $X \times D$ to X.

Proposition 11.4. *(X, D, K) is a CAS which will be called the join of the CAS (X_1, D_1, K^1) and (X_2, D_2, K^2). We shall write the join as $(X_1, D_1, K^1) \vee (X_2, D_2, K^2)$.*

If (X_1, D_1, K^1) and (X_2, D_2, K^2) are commutative or symmetric, then so is (X, D, K). Moreover, if (X_1, D_1, K^1) has property (T2), then so has (X, D, K).

Proof. By the topological structure of X and D it is clear that our kernel K from $X \times D$ to X is continuous. Moreover, the properties 4.2(1) and (2) are obviously satisfied with the projection $\pi : X \times X \to D$ with

$$\pi((x_1, x_2), (y_1, y_2)) := \begin{cases} \pi_2(x_2, y_2) \in D_2 \subset D & \text{for } x_1 = y_1 \\ \pi_1(x_1, y_1) \in D_1 \setminus \{e_1\} \subset D & \text{for } x_1 \neq y_1. \end{cases}$$

To check 4.2(3), we fix $x_i \in X_i$ and Borel sets $A_i \subset X_i$ ($i = 1, 2$). Then for $h_1, h_2 \in D_2$,

$$\begin{aligned} &K_{h_1} \circ K_{h_2}((x_1, x_2), A_1 \times A_2) \\ &\quad = \delta_{x_1}(A_1) \cdot K_{h_1}^2 \circ K_{h_2}^2(x_2, A_2) \\ &\quad = \delta_{x_1}(A_1) \cdot \int_{D_2} K_h^2(x_2, A_2)\, d(\delta_{h_1} *_2 \delta_{h_2})(h) \\ &\quad = \int_{D_2} K_h((x_1, x_2), A_1 \times A_2)\, d(\delta_{h_1} *_2 \delta_{h_2})(h) \end{aligned}$$

as claimed. Moreover, for $h_1 \in D_1$, $h_2 \in D_2$,

$$\begin{aligned} &K_{h_2} \circ K_{h_1}((x_1, x_2), A_1 \times A_2) \\ &\quad = \int_{X_2} K_{h_1}((x_1, y_2), A_1 \times A_2)\, K_{h_2}^2(x_2, dy_2) \\ &\quad = \int_{X_2} \omega_{X_2}(A_2) K_{h_1}^1(x_1, A_1)\, K_{h_2}^2(x_2, dy_2) = K_{h_1}^1(x_1, A_1)\omega_{X_2}(A_2) \\ &\quad = K_{h_1}((x_1, x_2), A_1 \times A_2) \end{aligned}$$

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as claimed. Moreover, for $h_1 \in D_1$, $h_2 \in D_2$ we conclude from Equation (4.4) that

$$\begin{aligned} &K_{h_1} \circ K_{h_2}((x_1, x_2), A_1 \times A_2) \\ &\qquad = \int_{X_1} \int_{X_2} K^2_{h_2}(y_2, A_2)\delta_{y_1}(A_1)\, d\omega_{X_2}(y_2)\, K^1_{h_1}(x_1, dy_1) \\ &\qquad = K^1_{h_1}(x_1, A_1) \cdot \omega_{X_2}(A_2) = K_{h_1}((x_1, x_2), A_1 \times A_2) \end{aligned}$$

as claimed. Finally, for $h_1, h_2 \in D_1$ we conclude from Lemma 4.14(3) with our normalizations that $\int_{D_2} K^2_h(x_2, A_2)\, d\omega_{D_2}(h) = \omega_{X_2}(A_2)$ and thus

$$\begin{aligned} &K_{h_1} \circ K_{h_2}((x_1, x_2), A_1 \times A_2) \\ &\qquad = \int_{X_1} \omega_{X_2}(A_2) K^1_{h_2}(y_1, A_1)\, K^1_{h_1}(x_1, dy_1) \cdot \omega_{X_2}(X_2) \\ &\qquad = \int_{D_1} K^1_h(x_1, A_1)\, d(\delta_{h_1} *_1 \delta_{h_2})(h) \cdot \omega_{X_2}(A_2) \\ &\qquad = \int_{D_1 \setminus \{e_1\}} \omega_{X_2}(A_2) K^1_h(x_1, A_1)\, d(\delta_{h_1} *_1 \delta_{h_2})(h) \\ &\qquad\qquad + (\delta_{h_1} *_1 \delta_{h_2})(\{e_1\})\delta_{x_1}(A_1) \int_{D_2} K^2_h(x_2, A_2)\, d\omega_{D_2}(h) \\ &\qquad = \int_D K_h((x_1, x_2), A_1 \times A_2)\, d(\delta_{h_1} * \delta_{h_2})(h) \end{aligned}$$

which completes the proof of 4.2(3).

In order to check the adjoint relation in Definition 4.2(4), we put $\omega_X := \omega_{X_1} \times \omega_{X_2}$. Let $f_1, g_1 \in C_c(X_1)$, $f_2, g_2 \in C_c(X_2)$ and consider $f, g \in C_c(X)$ with $f(x_1, x_2) := f_1(x_1) f_2(x_2)$ and $g(x_1, x_2) := g_1(x_1) g_2(x_2)$. We conclude from (11.2) that then for all $h \in D$ and $(x_1, x_2) \in X$,

$$T_h f(x_1, x_2) = \begin{cases} f_1(x_1) \cdot T^2_h f_2(x_2) & \text{for } h \in D_2 \\ T^1_h f_1(x_1) \cdot \int_{X_2} f_2\, d\omega_{X_2} & \text{for } h \in D_1 \setminus \{e_1\} \end{cases}$$

and hence

$$\int_X T_h f \cdot g\, d\omega_X = \int_{X_1} f_1 g_1\, d\omega_{X_1} \cdot \int_{X_2} T^2_h f_2 \cdot g_2\, d\omega_{X_2}$$

for $h \in D_2$, and

$$\int_X T_h f \cdot g\, d\omega_X = \int_{X_2} f_2 d\omega_{X_2} \cdot \int_{X_2} g_2 d\omega_{X_2} \cdot \int_{X_1} T^1_h f_1 \cdot g_1\, d\omega_{X_1}$$

for $h \in D_1 \setminus \{e_1\}$. This leads to the adjoint relation 4.2(4) for our particular functions f, g. The assertion for general $f, g \in C_c(X)$ finally follows from a Stone–Weierstrass argument. This completes the proof of the first part of the proposition.

Clearly, commutativity and symmetry are preserved under joins.

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We next turn to property (T2). Recapitulate that the compact CAS (X, D, K) has (T2) by Proposition 5.6, and that

$$\omega_D := \omega_{D_1}(\{e_1\}) \cdot \omega_{D_2} + \omega_{D_1}|_{D_1 \setminus \{e_1\}} \in M^+(D)$$

is a Haar measure of the join $(D, *)$ with the normalization $\omega_{D_2} \in M^1(D_2)$. We check (T2) via Lemma 5.5. For this fix $(x_1, x_2) \in X$ and Borel sets $A_1 \subset X_1$, $A_2 \subset X_2$. Then, by 5.5,

$$\begin{aligned}
&\int_D K_h((x_1, x_2), A_1 \times A_2)\, d\omega_D(h) \\
&\quad = \int_{D_2} \dots d\omega_D(h) + \int_{D_1 \setminus \{e_1\}} \dots d\omega_D(h) \\
&\quad = \omega_{D_1}(\{e_1\})\, \delta_{x_1}(A_1) \int_{D_2} K_h^2(x_2, A_2)\, d\omega_{D_2}(h) \\
&\qquad\quad + \omega_{X_2}(A_2)\Big[\int_{D_1} K_h^1(x_1, A_1)\, d\omega_{D_1}(h) - \omega_{D_1}(\{e_1\})\, \delta_{x_1}(A_1)\Big] \\
&\quad = \omega_{D_1}(\{e_1\})\, \delta_{x_1}(A_1)\omega_{X_2}(A_2) + \omega_{X_2}(A_2)[\omega_{X_1}(A_1) - \omega_{D_1}(\{e_1\})\, \delta_{x_1}(A_1)] \\
&\quad = \omega_{X_1}(A_1)\omega_{X_2}(A_2) = \omega_X(A_1 \times A_2).
\end{aligned}$$

As the Borel measures $\int_D K_h((x_1, x_2), .)\, d\omega_D(h)$ and ω_X on $X_1 \times X_2$ are determined uniquely by their values on cylinder sets, it follows that both measures are equal independent of (x_1, x_2). (T2) now follows from Lemma 5.5. □

We now consider iterated joins of finite CAS. For this we fix a sequence $(\Lambda_n)_{n\in\mathbb{N}}$ of finite CAS. We form the iterated joins

$$\Lambda_n^i := \Lambda_n \vee (\dots(\Lambda_3 \vee (\Lambda_2 \vee \Lambda_1))\dots) \quad (n \in \mathbb{N})$$

and

$$\Lambda_n^p := (\dots((\Lambda_1 \vee \Lambda_2) \vee \Lambda_3)\dots) \vee \Lambda_n \quad (n \in \mathbb{N}).$$

We now form the inductive limit of the sequence $(\Lambda_n^i)_{n\in\mathbb{N}}$ of discrete CAS as well as the projective limit of the sequence $(\Lambda_n^p)_{n\in\mathbb{N}}$ of compact CAS in an informal way and obtain as limits a discrete CAS Λ^i and a compact CAS Λ^p. We here do not work out any theory of these limits which are well-defined on the level of hypergroups; see [40]. We here just present these limits as examples of CAS. We start with the inductive limit.

Example 11.5. Consider a sequence of finite CAS $(\Lambda_n = (X_n, D_n, K^n))_{n\in\mathbb{N}}$ with the associated hypergroups $(D_n, *_n)$ with the identities $e_n \in D_n$ and with the normalized Haar measures $\omega_{D_n} \in M^1(D_n)$, and with the normalized adjoint measures $\omega_{X_n} \in M^1(X_n)$. We also fix some sequence $(z_n)_{n\in\mathbb{N}}$ with $z_n \in X_n$ for each n. Assume that $|D_n| \geq 2$ for all n. We now define the discrete inductive limit CAS $\Lambda = (X, D, K)$ as follows: D is the discrete, disjoint union

$$D := D_1 \cup \bigcup_{n\geq 2} (D_n \setminus \{e_n\}),$$

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and X is the discrete, countable set

$$X := \{(x_n)_{n\in\mathbb{N}} : x_n \in X_n \quad \text{for all } n, \quad \text{and} \\ x_n = z_n \quad \text{for all except for at most finitely many } n\}.$$

The convolution of point measures on the hypergroup $(D, *)$ is given by

$$\delta_h * \delta_l := \begin{cases} \delta_l & \text{for } l \in D_n, \quad h \in D_1 \cup \bigcup_{k=1}^{n-1} D_k \setminus \{e_k\}, \quad n \geq 2 \\ \delta_h & \text{for } h \in D_n, \quad l \in D_1 \cup \bigcup_{k=1}^{n-1} D_k \setminus \{e_k\}, \quad n \geq 2 \\ \delta_h *_1 \delta_l & \text{for } h, l \in D_1 \end{cases}$$

and, for $h, l \in D_n \setminus \{e_n\}$ with $n \geq 2$, by the probability measure

$$\delta_h * \delta_l := (\delta_h *_n \delta_l)|_{D_n \setminus \{e_n\}} + (\delta_h *_n \delta_l)(\{e_n\}) \\ \times \Big[\sum_{k=2}^{n-1}\Big(\prod_{m=k+1}^{n-1} \omega_{D_m}(\{e_m\})\Big) \cdot \omega_{D_k}|_{D_k \setminus \{e_k\}} + \Big(\prod_{m=2}^{n-1} \omega_{D_m}(\{e_m\})\Big) \cdot \omega_{D_1}\Big].$$

A Haar measure on $(D, *)$ is given by the measure

$$\omega_D := \omega_{D_1} + \sum_{n=2}^{\infty}\Big(\prod_{m=1}^{n} \omega_{D_m}(\{e_m\})\Big)^{-1} \cdot \omega_{D_n}|_{D_n \setminus \{e_n\}}.$$

The kernels K_h on X are given by

$$K_h((x_n)_{n\in\mathbb{N}}, \{(y_n)_{n\in\mathbb{N}}\}) = K_h^1(x_1, \{y_1\}) \cdot \delta_{(x_n)_{n\geq 2}, (y_n)_{n\geq 2}}$$

for $h \in D_1$, and by

$$K_h((x_n)_{n\in\mathbb{N}}, \{(y_n)_{n\in\mathbb{N}}\}) = \prod_{n=1}^{l-1} \omega_{X_n}(\{y_n\})\, K_h^l(x_l, \{y_l\})\, \delta_{(x_n)_{n\geq l+1}, (y_n)_{n\geq l+1}}$$

for $h \in D_l \setminus \{e_l\}$ with $l \geq 2$ where we again use the Kronecker-δ. These K_h are in fact Markov-kernels on X where the properties 4.2(1) and (2) obviously hold with the projection π with

$$\pi((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \pi_1(x_1, y_1) \in D_1 \subset D$$

for $(x_n)_{n\geq 2} = (y_n)_{n\geq 2}$, and otherwise with

$$\pi((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \pi_n(x_n, y_n) \in D_n \setminus \{e_n\} \subset D$$

for $n := \max\{l : x_l \neq y_l\} \geq 2$. Notice that the maximum exists by the definition of X. Property 4.2(3) can now be checked by using the computations in the proof of Proposition 11.4. These computations also show that

$$\omega_X(\{(x_n)_{n\in\mathbb{N}}\}) := \prod_{n=1}^{l} \frac{\omega_{X_n}(\{x_n\})}{\omega_{X_n}(\{z_n\})} \quad \text{for } l \quad \text{with } x_n = z_n \quad \text{for } n > l$$

is an adjoint measure, and that (T2) holds by Lemma 5.5. We omit the details of proofs.

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We next turn to the projective limit.

EXAMPLE 11.6. We start with the same sequence $(\Lambda_n = (X_n, D_n, K^n))_{n\in\mathbb{N}}$ with the notations as before. We define the compact projective limit CAS $\Lambda = (X, D, K)$:

$$D := \bigcup_{n\geq 1}(D_n \setminus \{e_n\}) \cup \{e\}$$

is the one-point compactification of the discrete disjoint union

$$\bigcup_{n\geq 1}(D_n \setminus \{e_n\})$$

where the additional nondiscrete limit point e will be the neutral element e of $(D, *)$. The hypergroup convolution of point measures is given by

$$\delta_h * \delta_l := \begin{cases} \delta_l & \text{for } l \in D_n \setminus \{e_n\}, \quad h \in \{e\} \cup \bigcup_{k>n}(D_k \setminus \{e_k\}) \\ \delta_h & \text{for } h \in D_n \setminus \{e_n\}, \quad l \in \{e\} \cup \bigcup_{k>n}(D_k \setminus \{e_k\}) \end{cases}$$

and, for $h, l \in D_n \setminus \{e_n\}$ by the probability measure

$$\delta_h * \delta_l := (\delta_h *_n \delta_l)|_{D_n \setminus \{e_n\}} + (\delta_h *_n \delta_l)(\{e_n\}) \sum_{k>n}\Big(\prod_{m=n+1}^{k-1} \omega_{D_m}(\{e_m\})\Big) \cdot \omega_{D_k}|_{D_k \setminus \{e_k\}}.$$

The normalized Haar measure on the compact hypergroup $(D, *)$ is

$$\omega_D := \sum_{n=1}^{\infty}\Big(\prod_{m=1}^{n-1} \omega_{D_m}(\{e_m\})\Big) \cdot \omega_{D_n}|_{D_n \setminus \{e_n\}} \in M^1(D).$$

Moreover, X is the compact usual direct product $\prod_{n\geq 1} X_n$, and the adjoint measure is the infinite product measure

$$\omega_X := \prod_{n\geq 0} \omega_{X_n}.$$

The kernel K_e on X will be the identity, and for $h \in D_n \setminus \{e_n\} \subset D$ with $n \in \mathbb{N}$, then

$$\begin{aligned} K_h((x_n)_{n\in\mathbb{N}}, &\{(y_1, \dots y_n)\} \times A) \\ &= \delta_{(x_1,\dots,x_{n-1}),(y_1,\dots,y_{n-1})}\, K_h^n(x_n, \{y_n\}) \prod_{l\geq n+1} \omega_{X_l}(A) \end{aligned}$$

for all $(x_n)_{n\in\mathbb{N}} \in X$, $(y_1, \dots y_n) \in X_1 \times \cdots \times X_n$ and Borel sets $A \subset \prod_{l\geq n+1} X_l$. These formulas clearly determine unique Markov-kernels K_h on X where the properties 4.2(1) and (2) obviously hold with the projection $\pi : X \times X \to D$ with

$$\pi((x_n)_{n\in\mathbb{N}}, (x_n)_{n\in\mathbb{N}}) = e \in D$$

and

$$\pi((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \pi_n(x_n, y_n) \in D_n \setminus \{e_n\} \subset D$$

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for $(x_n)_{n\in\mathbb{N}} \neq (x_n)_{n\in\mathbb{N}}$ and $n := \min\{l : x_l \neq y_l\}$. Properties 4.2(3) and (4) can now be checked by using the computations in the proof of Proposition 11.4. Finally, (T2) is clear by compactness and Proposition 5.6.

We notice that for given sequences $(\Lambda_n)_n$ of finite CAS which are not coming from groups, the construction 11.6 leads to examples of compact, nondiscrete strong CAS which are not associated with groups according to Proposition 4.6.

There exist several generalizations of the join on the level of hypergroups like substitutions of open hypergroups in [41] or [8, Section 8.1] or further constructions used in several papers of Heyer, Kawakami, and others; see, for example, [20, 21], and papers cited there. We expect that these constructions should also have a meaning on the level of CAS.

12. Random walks on continuous association schemes

In this section we introduce and investigate random walks on X associated with some given CAS (X, D, K). Before doing so we briefly recapitulate some facts on random walks on the hypergroup $(D, *)$. For simplicity we restrict our attention to the time-homogeneous case.

RANDOM WALKS ON HYPERGOUPS 12.1. Let $(D, *)$ be a second countable hypergroup with identity e. Let either $T := \mathbb{N}_0$ or $T := [0, \infty[$. A family $(\mu_t)_{t\in T} \subset M^1(D)$ of probability measures is called a (discrete or continuous) convolution semigroup, if $\mu_0 = \delta_e$, and if for all $s, t \in T$, $\mu_{s+t} = \mu_s * \mu_t$, and if in the continuous case, the mapping $[0, \infty[\to M^1(D)$, $t \mapsto \mu_t$ is weakly continuous in addition. It can be easily checked and is well known that for each $t \in T$ we may form the Markov-kernel K_t on D via

$$K_t(h, A) := (\delta_x * \mu_t)(A) \quad (h \in D,\ A \in \mathcal{B}(D),\ t \in T).$$

The K_t are in fact Feller-kernels (see the beginning of Section 4) with K_e as a trivial kernel and $K_s \circ K_t = K_{s+t}$ for all $s, t \in T$; see the beginning of Section 4 for the notations. In particular, $(K_t)_{t\in T}$ is a semigroup of transition kernels. As $(D, *)$ is second countable and locally compact, it is a matter of fact that for the starting distribution δ_e and this semigroup there exists a time-homogeneous Markov process $(Y_t)_{t\in T}$ with the transition probabilities

$$P(Y_{s+t} \in A | Y_s = h) = K_t(h, A) = (\delta_x * \mu_t)(A) \quad (x \in D,\ A \in \mathcal{B}(D),\ s, t \in T).$$

These Markov processes are called random walks on $(D, *)$ associated with $(\mu_t)_{t\in T}$.

Notice that for a locally compact group $D = G$, this means that $(Y_t)_{t\in T}$ just consists of group products of independent and identically distributed G-valued random variables in the discrete case, and that $(Y_t)_{t\in T}$ is a Levy process in the continuous case.

For a detailed study of random walks on hypergroups including limit theorems for special classes we refer to [8] and references therein.

We now use these ideas to define random walks $(Z_t)_{t\in T}$ on X for a CAS (X, D, K).

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Random walks on continuous association schemes 12.2. Let (X, D, K) be a CAS with associated hypergroup $(D, *)$ with identity e. Let $T := \mathbb{N}_0$ or $T := [0, \infty[$, and let $(\mu_t)_{t\in T} \subset M^1(D)$ be a (discrete or continuous) convolution semigroup of probability measures on D as before. For each $t \in T$ we now define the Markov-kernel $K^X_{\mu_t}$ on X via

$$K^X_{\mu_t}(x, A) := \int_D K_h(x, A)\, d\mu_t(h) \quad (x \in X,\ A \in \mathcal{B}(X),\ t \in T)$$

which is associated with the transition operator T_{μ_t} of Lemma 4.11, that is, the $K^X_{\mu_t}$ are in fact Feller kernels by 4.11. Moreover, precisely as in Proposition 4.12 we see that $(K^X_{\mu_t})_{t\in T}$ is a semigroup of transition kernels. If we now fix some starting point $x_0 \in X$, we again find some associated time-homogeneous Markov process $(Z_t)_{t\in T}$. These processes are called random walks on X with start in x_0 associated with $(\mu_t)_{t\in T}$.

For $T := [0, \infty[$, we show that $(Z_t)_{t\in T}$ is a so-called Feller process, that is, that the operators T_{μ_t} associated with the Feller-kernels $K^X_{\mu_t}$ satisfy in addition the condition

$$\|T_{\mu_t} g - g\|_\infty \longrightarrow 0 \quad \text{for } t \to 0 \quad \text{and all} \quad g \in C_0(X). \tag{12.1}$$

Proposition 12.3. *Let (X, D, K) be a CAS and $T := [0, \infty[$. Each random walk $(Z_t)_{t\in T}$ on X with start in any $x_0 \in X$ associated with any convolution semigroup $(\mu_t)_{t\in T}$ on $(D, *)$ is a Feller process.*

In particular, $(Z_t)_{t\in T}$ admits a modification such that all paths of this modification are right continuous with left limits (RCLL). For $T := [0, \infty[$, we thus shall assume in addition that the random walk $(Z_t)_{t\in T}$ on X has the RCLL property.

Proof. In order to check (12.1), we fix some $g \in C_0(X)$ and $\varepsilon > 0$. By Lemma 4.9, we find some open neighborhood $U \subset D$ of e such that $|g(x) - g(y)| \le \varepsilon$ holds for all $x, y \in X$ with $\pi(x, y) \in U$. Hence, for each $x \in X$ and $t \ge 0$,

$$\int_U \int_X |g(x) - g(y)|\, K_h(x, dy)\, d\mu_t(h) \le \varepsilon.$$

Thus,

$$\begin{aligned} |T_{\mu_t} g(x) - g(x)| &= \left| \int_D \int_X (g(x) - g(y))\, K_h(x, dy)\, d\mu_t(h) \right| \\ &= \varepsilon + 2\|g\|_\infty\, \mu_t(D \setminus U) \le 2\varepsilon \end{aligned}$$

whenever t is small enough. This proves (12.1). The second statement is well known; see, for example, [24, Section 17]. □

The random walks $(Z_t)_{t\in T}$ on X may be studied by using known results for random walks $(Y_t)_{t\in T}$ on $(D, *)$. This follows from the below result which seems obvious at a first glance and can be seen easily in the group cases $X = G/H$, $D = G//H$.

Theorem 12.4. *Let (X, D, K) be a CAS and $x_0 \in X$ fixed and consider the continuous, open, and closed map $\psi : X \to D$, $\psi(x) := \pi(x_0, x)$.*

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Let $(Z_t)_{t\in T}$ be a random walk on X with start in x_0 associated with some convolution semigroup $(\mu_t)_{t\in T}$ on $(D,*)$ as described above where we assume that all paths are RCLL for $T=[0,\infty[$. Then the process $(\psi(Z_t))_{t\in T}$ is a random walk on $(D,*)$ with start in e associated with $(\mu_t)_{t\in T}$.

For the proof we first consider the case $T=\mathbb{N}_0$ and check that the projected process $(\psi(Z_t))_{t\in T}$ is a Markov process.

For this we fix $n\in\mathbb{N}_0$ as well as Borel sets $A_0,\dots,A_n\in\mathcal{B}(D)$. We consider the subprobability measures $P_{A_0,\dots,A_n}\in M^{(1)}(D)$ and $\tilde{P}_{A_0,\dots,A_n}\in M^{(1)}(X)$ with

$$P_{A_0,\dots,A_n}(A):=P(\psi(X_0)\in A_0,\dots,\psi(X_n)\in A_n,\psi(X_{n+1})\in A)\quad (A\in\mathcal{B}(D))$$

and

$$\tilde{P}_{A_0,\dots,A_n}(B):=P(\psi(X_0)\in A_0,\dots,\psi(X_n)\in A_n,X_{n+1}\in B)\quad (B\in\mathcal{B}(X)).$$

Then clearly, $P_{A_0,\dots,A_n}$ is the image measure of $\tilde{P}_{A_0,\dots,A_n}$ under ψ. We need the following reconstruction of $\tilde{P}_{A_0,\dots,A_n}$ from $P_{A_0,\dots,A_n}$.

Lemma 12.5. *For all $n\in\mathbb{N}_0$, $A_0,\dots,A_n\in\mathcal{B}(D)$, and $B\in\mathcal{B}(X)$,*

$$\tilde{P}_{A_0,\dots,A_n}(B)=\int_D K_h(x_0,B)\,dP_{A_0,\dots,A_n}(h).$$

Proof of Lemma 12.5. We prove the lemma by induction on n.

In fact, for $n=0$ we have the two cases $\psi(x_0)\notin A_0$ and $\psi(x_0)\in A_0$. In the first case the assertion is trivial, and in the second case we get the claim from

$$\begin{aligned}\tilde{P}_{A_0}(B)&=P(Z_1\in B)=K^X_{\mu_1}(x_0,B)\\&=\int_D K_h(x_0,B)\,d\mu_1(h)=\int_D K_h(x_0,B)\,dP_{A_0}(h).\end{aligned}$$

We now turn to the step $n-1\to n$ for $n\geq 1$. As all spaces are second countable and locally compact, we may use the concept of regular conditional probabilities, and obtain from the Markov property of $(Z_t)_{t\in T}$, the assumption of our induction step, and from the axioms of a CAS that

$$\begin{aligned}\tilde{P}_{A_0,\dots,A_n}(B)&=P(\psi(X_0)\in A_0,\dots,\psi(X_n)\in A_n,X_{n+1}\in B)\\&=\int_{\psi^{-1}(A_n)}P(X_{n+1}\in B|\,\psi(X_0)\in A_0,\dots,\psi(X_{n-1})\in A_{n-1},X_n=x_n)\\&\qquad d\tilde{P}_{A_0,\dots,A_{n-1}}(x_n)\\&=\int_{\psi^{-1}(A_n)}K^X_{\mu_1}(x_n,B)\,d\tilde{P}_{A_0,\dots,A_{n-1}}(x_n)\\&=\int_{A_n}\int_X K^X_{\mu_1}(x_n,B)\,K_h(x_0,dx_n)\,dP_{A_0,\dots,A_{n-1}}(h)\\&=\int_{A_n}K_h\circ K^X_{\mu_1}(x_0,B)\,dP_{A_0,\dots,A_{n-1}}(h)\end{aligned}$$

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$$\begin{aligned}
&= \int_{A_n} K_h \circ K^X_{\mu_1}(x_0, B)\, dP_{A_0,\dots,A_{n-1}}(h) \\
&= \int_{A_n} \int_D K_l(x_0, B)\, d(\delta_h * \mu_1)(l)\, dP_{A_0,\dots,A_{n-1}}(h) \\
&= \int_D K_l(x_0, B)\, d(P_{A_0,\dots,A_{n-1}}|_{A_n} * \mu_1)(l) \qquad (12.2)
\end{aligned}$$

where $|_{A_n}$ means the restriction of a measure to A_n. If we take $B = \psi^{-1}(A)$ for $A \in \mathcal{B}(D)$ in (12.2), we obtain from the axioms of a CAS that

$$\begin{aligned}
P_{A_0,\dots,A_n}(A) &= P(\psi(X_0) \in A_0, \dots, \psi(X_n) \in A_n, , \psi(X_{n+1}) \in A) \\
&= \int_D K_l(x_0, \psi^{-1}(A))\, d(P_{A_0,\dots,A_{n-1}}|_{A_n} * \mu_1)(l) \\
&= (P_{A_0,\dots,A_{n-1}}|_{A_n} * \mu_1)(A).
\end{aligned}$$

If we insert this identity in the end of (12.2), we get the claim

$$\tilde{P}_{A_0,\dots,A_n}(B) = \int_D K_h(x_0, B)\, dP_{A_0,\dots,A_n}(h). \qquad \square$$

Proof of Theorem 12.4. We first consider the case $T = \mathbb{N}_0$ and check that $(\psi(Z_t))_{t\in T}$ is a Markov process. For this consider $n \in \mathbb{N}$ and $A_0, \dots, A_n, A \in \mathcal{B}(D)$. Let $(\mathcal{F}_t)_{t\in T}$ be the canonical filtration of $(\psi(Z_t))_{t\in T}$ on the associated probability space $(\Omega, \mathcal{A}, P)$. Then, by Lemma 12.5 and the first lines of (12.2),

$$\begin{aligned}
&\int_{\{\psi(X_0)\in A_0,\dots,\psi(X_n)\in A_n\}} \mathbf{1}_{\{\psi(X_{n+1})\in A\}}\, dP \\
&\quad = P(\psi(X_0) \in A_0, \dots, \psi(X_n) \in A_n, \psi(X_{n+1}) \in A) \\
&\quad = \int_{A_n} K_h \circ K^X_{\mu_1}(x_0, \psi^{-1}(A))\, dP_{A_0,\dots,A_{n-1}}(h) \\
&\quad = \int_{\{\psi(X_0)\in A_0,\dots,\psi(X_n)\in A_n\}} K_{\psi(X_n)} \circ K^X_{\mu_1}(x_0, \psi^{-1}(A))\, dP. \qquad (12.3)
\end{aligned}$$

As the last integrand is measurable with respect to the σ-algebra $\sigma(\psi(X_n)) \subset \mathcal{F}_n$, we obtain from (12.3) that a.s.

$$P(\psi(X_{n+1}) \in A | \mathcal{F}_n) = K_{\psi(X_n)} \circ K^X_{\mu_1}(x_0, \psi^{-1}(A))$$

and thus a.s.

$$\begin{aligned}
P(\psi(X_{n+1}) \in A | \sigma(\psi(X_n))) &= K_{\psi(X_n)} \circ K^X_{\mu_1}(x_0, \psi^{-1}(A)) \\
&= P(\psi(X_{n+1}) \in A | \mathcal{F}_n). \qquad (12.4)
\end{aligned}$$

Therefore, $(\psi(Z_t))_{t\in T}$ is a Markov process. Moreover, a comparison of (12.4) with the transition probabilities of random walks on $(D, *)$ shows immediately that $(\psi(Z_t))_{t\in T}$ is a random walk on $(D, *)$ associated with $(\mu_t)_{t\in T}$ and start in e as claimed.

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Let us now turn to the case $T = [0, \infty[$. A change of the notations of the preceding proof shows readily that for all $n \in \mathbb{N}$, $0 < t_1 < t_2 < \cdots < t_{n+1}$, and $A \in \mathcal{B}(D)$,

$$\begin{aligned} P(\psi(X_{t_{n+1}}) \in A |\, \sigma(\psi(X_{t_n}))) &= K_{\psi(X_{t_n})} \circ K^X_{\mu_{t_{n+1}-t_n}}(x_0, \psi^{-1}(A)) \\ &= P(\psi(X_{t_{n+1}}) \in A |\, \sigma(\psi(X_{t_0}), \ldots \psi(X_{t_n}))) \quad \text{a.s.} \end{aligned}$$

Standard arguments from the theory of Markov processes with RCLL paths (see [24, Section 17]) now show that

$$P(\psi(X_{t_{n+1}}) \in A |\, \sigma(\psi(X_{t_n}))) = P(\psi(X_{t_{n+1}}) \in A |\, \sigma(\psi(X_t); t \in [0, t_n])) \quad \text{a.s.}$$

This is the Markov property, and the proof can be completed as for $T = \mathbb{N}_0$. □

For many classes of commutative hypergroups $(D, *)$ there exist limit theorems for random walks on $(D, *)$ like (strong) laws of large numbers, central limit theorems, and so on; see [8, Ch. 7] and references therein. Theorem 12.4 may now be used to transfer these results to random walks on X for suitable commutative CAS (X, D, K). This seems to be interesting in particular for examples which appear as deformations of group CAS, as here random walks on X may be seen as 'radial random walks with additional drift' on the homogeneous space X. This seems to be interesting in particular for such random walks on affine buildings and on noncompact Grassmann manifolds. Some results in this direction can be found in [9].

13. Open problems

We finally collect some open problems which appeared in the preceding course on CAS.

(1) Do there exist (commutative) CAS with (T1), but without (T2)?

(2) Does (T2) always imply (T1)? Notice that this is the case in the discrete and in the commutative case, and that it is likely that it can be shown in the compact case.

(3) Is each discrete CAS a generalized association scheme? The answer is positive in the finite case. The general problem is part of:

(4) Does each (commutative) CAS (X, D, K) admit a further associated (commutative) CAS $(X, D, \tilde{K})$ with the same spaces X, D, and the same projection π such that $(X, D, \tilde{K})$ has property (T2).

(5) Give examples of (commutative) CAS (X, D, K) with X,D connected, which are not of the form $X = G/H$, $D = G//H$ for a locally compact group G with a compact subgroup H or deformations of such group cases.

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