Notes

99.12 Primitive integer triangles

In a previous article [1], a method was developed for finding all integer triangles containing an angle whose cosine k was known. Some of these triangles turned out to be primitive, in the sense that the gcd of their sides was 1, whereas others did not. It was asserted, without proof, that in both the particular cases where the given angle was 90° or 120° ($k = 0$ or $k = -\frac{1}{2}$), the method always gave rise to primitive triangles. In this note, we find all values of k where this is indeed the case.

We start off by revisiting definitions and quoting results proved in [1].

Definition 1: Let $\kappa = 2(1 - k)$, where k is the cosine of the given angle. The set $\{\alpha, \beta\}$ is called a κ-pair if α, β are both rational with $\alpha > 0, \beta > 0$ and $\alpha\beta = \kappa$.

Theorem 1: Given any κ-pair α, β, write $\alpha = \dfrac{\alpha_1}{\alpha_2}$, $\beta = \dfrac{\beta_1}{\beta_2}$, where α_1, α_2, β_1, β_2 are positive integers with $\gcd(\alpha_1, \alpha_2) = \gcd(\beta_1, \beta_2) = 1$.

Define $T_{\{\alpha,\beta\}}$ to be the triangle with sides

$$a = \alpha_2(\beta_1 + 2\beta_2), \; b = \beta_2(\alpha_1 + 2\alpha_2), \; c = \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_1\beta_1.$$

Then

(i) $T_{\{\alpha,\beta\}}$ is an integer triangle.

(ii) The cosine of the angle opposite the side c is k.

(iii) $T_{\{\alpha,\beta\}}$ is primitive if, and only if, $d = \gcd(\alpha_1 + 2\alpha_2, \beta_1 + 2\beta_2) = 1$.

Before we can arrive at our main result, which appears in Theorem 2, we must first of all prove some preliminary lemmas. In all that follows, we assume that $k = \dfrac{m}{n}$, where n, m are integers with $n > 0$, $-n < m < n$ and $\gcd(m, n) = 1$.

Lemma 1: $d \mid (n + m)$ for all κ-pairs $\{\alpha, \beta\}$.

Proof: We begin by noting that $\dfrac{n + m}{n} = 1 + k = \dfrac{(a + b + c)(a + b - c)}{2ab}$. Rewriting a, b, c in terms of $\alpha_1, \alpha_2, \beta_1, \beta_2$, we obtain $a + b + c = (\alpha_1 + 2\alpha_2)(\beta_1 + 2\beta_2)$ and $a + b - c = 2a + 2b - (a + b + c)$ $= 2\alpha_2(\beta_1 + 2\beta_2) + 2\beta_2(\alpha_1 + 2\alpha_2) - (\alpha_1 + 2\alpha_2)(\beta_1 + 2\beta_2)$, from which it follows that

$$\frac{n + m}{n} = \frac{2\alpha_2(\beta_1 + 2\beta_2) + 2\beta_2(\alpha_1 + 2\alpha_2) - (\alpha_1 + 2\alpha_2)(\beta_1 + 2\beta_2)}{2\alpha_2\beta_2}.$$

If we now put $\alpha_1 + 2\alpha_2 = \alpha_3 d$ and $\beta_1 + 2\beta_2 = \beta_3 d$ and then cross multiply, we see that $2\alpha_2\beta_2(n + m) = dn(2\alpha_2\beta_3 + 2\beta_2\alpha_3 - d\alpha_3\beta_3)$. We know, however, that $\gcd(d, \alpha_2) = \gcd(d, \beta_2) = 1$ (see [1, Lemma 3]).

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Thus if d is odd, we must have $d \mid (n + m)$. If d is even, then $2\alpha_2\beta_3 + 2\beta_2\alpha_3 - d\alpha_3\beta_3$ is also even. Thus we can divide both sides of the above equation by 2 and conclude again that $d \mid (n + m)$.

Lemma 2: Let n, m be such that $n + m > 1$. Then there exists a κ-pair $\{\alpha, \beta\}$ for which $d = n + m$.

Proof: First of all, assume that $n + m$ is even. We put

$$\alpha_1 = 2, \qquad \beta_1 = \frac{(n - m)(n + m - 1)}{s},$$

$$\alpha_2 = n + m - 1, \qquad \beta_2 = \frac{n}{s},$$

where $s = \gcd\big((n - m)(n + m - 1), n\big)$.

Then $\alpha > 0$. $\beta > 0$ and $\gcd(\alpha_1, \alpha_2) = \gcd(\beta_1, \beta_2) = 1$.

Since $\dfrac{\alpha_1}{\alpha_2} \times \dfrac{\beta_1}{\beta_2} = \dfrac{2(n - m)}{n} = \kappa$, it follows that $\{\alpha, \beta\}$ is a κ-pair.

Furthermore, $\alpha_1 + 2\alpha_2 = 2(n + m)$ and $\beta_1 + 2\beta_2 = \dfrac{(n + m)(n - m + 1)}{s}$.

If $s > 1$, we note that $\gcd(s, n + m) = 1$, since otherwise $n + m$ and n would have a factor in common. This is impossible since $\gcd(n, m) = 1$. Thus $s \mid (n - m + 1)$ and $\beta_1 + 2\beta_2$ is an integer multiple of $n + m$. The fact that $(n - m + 1)$ is odd means that $\beta_1 + 2\beta_2$ is an odd multiple of $n + m$, which gives us our result.

If, on the other hand, $n + m$ is odd, then we put

$$\alpha_1 = 1, \qquad \beta_1 = \frac{(n - m)(n + m - 1)}{s},$$

$$\alpha_2 = \frac{n + m - 1}{2}, \qquad \beta_2 = \frac{n}{s},$$

where $s = \gcd\big((n - m)(n + m - 1), n\big)$.

It is now left as an exercise to the reader to show that, in this case, it is also true that $\{\alpha, \beta\}$ is a κ-pair for which $d = m + n$.

In order to illustrate the above lemma, we consider two examples. In the first example, we let $n = 15$ and $m = 11$. This implies that $\alpha_1 = 2$, $\alpha_2 = 25$, $(n - m)(n + m - 1) = 100$, $s = \gcd(100, 15) = 5$, $\beta_1 = 20$ and $\beta_2 = 3$ and it follows that $d = \gcd(52, 26) = 26$, which equals $15 + 11$.

In the second example, we let $n = 16$ and $m = -7$. In this case $\alpha_1 = 1$, $\alpha_2 = 4$, $(n - m)(n + m - 1) = 184$, $s = \gcd(184, 16) = 8$, $\beta_1 = 23$ and $\beta_2 = 2$.

It follows that $d = \gcd(9, 27) = 9$ which equals $16 + (-7)$.

We are now able to prove our main result.

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Theorem 2: $T_{\{\alpha,\beta\}}$ is primitive for all possible κ-pairs $\{\alpha, \beta\}$ if, and only if, $k = \dfrac{1 - n}{n}$ for some positive integer n.

Proof: As a simple corollary to Lemma 1, we see that if $m + n = 1$ then d is also equal to 1. On the other hand, Lemma 2 tells us that if $m + n > 1$, there always exists a κ-pair $\{\alpha, \beta\}$ for which $d = n + m$. The fact that $k = \dfrac{m}{n}$ now gives us our result.

Finally, we note that the conditions $\gcd(\alpha_1, \alpha_2) = \gcd(\beta_1, \beta_2) = 1$ are essential for ensuring the validity of the above theorem. In order to show that this is the case, we let $\alpha = \dfrac{1}{1}$, $\beta = \dfrac{2(n - m)}{n}$. Then $\alpha\beta = \kappa$ and the corresponding integer triangle $T_{\{\alpha,\beta\}}$ has sides $4n - 2m$, $3n$ and $5n - 4m$.

This is in itself an interesting triangle since its sides form an arithmetic progression with first term $3n$ and common difference $n - 2m$, thus providing us with a straightforward method for calculating the sides.

However, although $\gcd(\alpha_1, \alpha_2) = 1$, we note that $\gcd(\beta_1, \beta_2) = 1$ when n is odd but 2 when n is even and thus, even if $m + n = 1$, it does not necessarily follow that $T_{\{\alpha,\beta\}}$ is primitive. If we let $n = 2$, $m = -1$, it follows from above that $T_{\{\alpha,\beta\}}$ has sides 6, 10, 14. This is an integer triangle containing an angle of 120° but is not primitive.

References

1. Emrys Read, On integer triangles, *Math. Gaz.*, **98,** (March 2014), pp. 107-112.

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99.13 Solving the quintic in radicals

Introduction

It is well known that the general quintic equation cannot be solved in a finite sequence of radicals. This was proved by Abel and Galois in the 19th century and brought to a close that particular line of research. The important word above is ‘finite’. If we remove that word then the above statement is false, as we shall show in this Note.

The general quintic equation can be transformed into $x^5 + ax + b = 0$ by means of a transformation that only uses square and cube roots. This was proved by the Swedish professor of history E. S. Bring in the 18th century and independently by the English mathematician G. B. Jerrard in the 19th. There are two cases to consider which depend on the sign of a.