THE GENUS, REGIONAL NUMBER, AND BETTI NUMBER OF A GRAPH

RICHARD A. DUKE

1. Introduction. Let the genus of an orientable 2-manifold M be denoted by $\gamma(M)$. The genus, $\gamma(G)$, of a graph G is then the smallest of the numbers $\gamma(N)$ for orientable 2-manifolds N in which G can be embedded. An embedding of G in M is called *minimal* if $\gamma(G) = \gamma(M)$. When each component of the complement of G in M is an open 2-cell, the embedding of G in M is called a 2-*cell embedding*. In **(3)**, J. W. T. Youngs has shown that each minimal embedding is a 2-cell embedding. It follows from the results of **(3)** that for each graph G, there is a number $d(G)$, called the *regional number*, such that for any 2-cell embedding of G in an orientable 2-manifold, the number of (2-cell) complementary domains of G is $\leqslant d(G)$, with equality holding if and only if the embedding is minimal.

The purpose of this paper is to study the relationship of the 1-dimensional Betti number $\beta(G)$ to the genus and regional number. The classical Kuratowski skew curves $K_{3,3}$ and K_5 have Betti numbers 4 and 6 respectively. It follows from Kuratowski's characterization of planar graphs that if a graph G is non-planar (i.e. $\gamma(G) \geqslant 1$), then $\beta(G) \geqslant 4$. It is the author's conjecture that this fact can be generalized to the following statement.

Conjecture. *If* $\gamma(G) = n$, *then* $\beta(G) \geqslant 4n$. The irreducible graphs described in **(4)** show that equality may hold in the above.

Simple Euler characteristic considerations yield

$$d(G) = 1 + \beta(G) - 2\gamma(G). \tag{1}$$

Equation (1) and the fact that $d(G)$ is positive imply that if $\gamma(G) = n$, then $\beta(G) \geqslant 2n$. Several results are obtained that improve upon the value $2n$ in the direction of establishing the above conjecture. Note also that (1) implies the equivalence of the above conjecture to the following statement: If $\gamma(G) = n$, then $d(G) \geqslant 2n + 1$.

The primary tool is contained in Theorem 3.1. Here a sufficient condition for the *non-minimality* of a 2-cell embedding is given and a procedure, called a *reduction*, is described for transforming such a non-minimal embedding into a 2-cell embedding of the same graph in an orientable 2-manifold of lower genus.

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All graphs considered will be finite, connected graphs without loops or multiple edges. The term 2-manifold will always mean a compact, closed, orientable 2-manifold. The notation $G(M)$ will denote an embedding of the graph G in the 2-manifold M as well as the geometric realization of G in M.

2. Edmonds' embedding technique. The reduction procedure mentioned in the Introduction is given in terms of a technique for obtaining all 2-cell embeddings of a given graph G due to J. R. Edmonds **(2)** and described in detail in **(3**, p. 313**)**. Briefly, for each vertex x of G one chooses a cyclic permutation T_x on the set $V(x)$ of all vertices of G adjacent to x. (For simplicity, we assume that G has no points of order 1.) Each choice of a set of such permutations, one for each vertex of G, determines a 2-cell embedding of G as follows. A transformation $T(W) = W$ is defined on the set of all ordered pairs of adjacent vertices of G by $T((x, y)) = (y, T_y(x))$. The pairs of a given orbit of T can be identified with the edges of some regular polygon. Performing identifications on the edges of the polygons thus associated with the orbits of T, one obtains an orientable 2-manifold M with the edges of the polygons yielding a 2-cell embedding $G(M)$. It is part of the Edmonds result that for a given 2-cell embedding $G(M)$, each orientation on M induces a set of permutations that determine the embedding in question.

$(a\ b\ c \ldots)$ will denote the cyclic permutation that sends a into b, b into c, etc.

3. Reduction of an embedding. The reduction procedure mentioned in the Introduction is described in the following theorem.

3.1. THEOREM. *Let $G(M)$ be an arbitrary 2-cell embedding of a finite connected graph G in an orientable 2-manifold M with given orientation τ. Let $T(W) = W$ be the transformation on the set W of all ordered pairs of adjacent vertices of G associated with $G(M)$ and τ. If T has an orbit R of the form*

$$(b, y), \ldots, (a, b), (b, c), \ldots, (d, b)(b, a), \ldots, (x, b),$$

where $c \neq d$, then there exists a 2-cell embedding of G in an orientable 2-manifold of genus $\gamma(M) - 1$.

Proof. The cyclic permutation on $V(b)$ associated with $T(W) = W$ has the form $T_b = (dac\ldots xy\ldots)$. Let $T'_b = (dc\ldots xay\ldots)$ ($c \neq d$, but we may have $c = x$ and $y = d$). Let $T'(W) = W$ be the transformation determined by the permutations associated with T with T_b replaced by T'_b. The orbits of T' are those of T with R replaced by three new orbits of the form

$$(b, y), \ldots, (a, b); \quad (b, c), \ldots, (d, b); \quad \text{and} \quad (b, a), \ldots, (x, b).$$

If $G(N)$ is the 2-cell embedding determined by T', then the number of components of $N - G(N)$ exceeds that of $M - G(M)$ by two and it follows from (1) that $\gamma(N) = \gamma(M) - 1$.

Definition. The procedure of 3.1 for transforming a given 2-cell embedding $G(M)$ by altering the permutation T_b and "splitting" one orbit into three distinct orbits (thus replacing one open 2-cell by three new ones) will be called a *reduction of* $G(M)$ *at* b.

It can be shown that shifting a single symbol in one permutation T_b as in 3.1 will increase or decrease the number of orbits of T by two, or leave this number unchanged, depending on the arrangement of the pairs involving b among the orbits of T. Since the set of permutations associated with a given 2-cell embedding can be transformed into the set of permutations associated with any other 2-cell embedding of the same graph by a finite sequence of such alterations, one has

3.2. THEOREM. *If there exist 2-cell embeddings of the graph G in orientable 2-manifolds of genera m and n, then for any integer k, $m \leqslant k \leqslant n$, there exists a 2-cell embedding of G in the orientable 2-manifold of genus k.*

Proof. Let $G(M)$ be an arbitrary 2-cell embedding of a finite connected graph G in an orientable 2-manifold M with given orientation τ. Let the cyclic permutations T_x and the transformation $T(W) = W$ associated with $G(M)$ and τ be as described in the Introduction. Let q denote the number of orbits of T. Suppose that for some vertex b of G, T_b has the form $(dac\ldots xy\ldots)$ where $c \neq d$. Let V be the subset of W consisting of the ordered pairs

$$(d, b), \quad (b, a), \quad (a, b), \quad (b, c), \quad (x, b), \quad \text{and} \quad (b, y).$$

By the definition of T we have $T((d, b)) = (b, a)$, $T((a, b)) = (b, c)$, and $T((x, b)) = (b, y)$. Thus the pair (d, b) is followed by (b, a) in some orbit of T. Similarly (a, b) is followed by (b, c) in some orbit and (x, b) is followed by (b, y). If $T'(W) = W$ is the transformation obtained by replacing T_b by $T'_b = (dc\ldots xay\ldots)$, any orbit of T not containing a pair of V will also be an orbit of T'.

The possible distributions of the pairs of V among the orbits of T determine the following three cases.

(i) Suppose all of the pairs of V appear in a single orbit R of T. If R has the form of the orbit described in Theorem 3.1, then, as was shown in the proof of that theorem, R is replaced by three new orbits when T is replaced by T'. The other orbits of T' are then precisely the remaining $q - 1$ orbits of T. Therefore T' has $q + 2$ orbits in all. The other possible arrangement of the pairs of V in R is as follows:

$$(b, y), \quad \ldots, \quad (d, b), \quad (b, a), \quad \ldots, \quad (a, b), \quad (b, c), \quad \ldots, \quad (x, b).$$

In this case the orbits of T' are the $q - 1$ orbits of T other than R, and a new orbit R' that contains all of the pairs in R (although in a different order). In this case T' has q orbits.

(ii) Suppose T has two orbits, R and S, with R containing four of the pairs of V and S containing the remaining two. In this case the orbits of T' are those of T with R and S being replaced by two orbits R' and S' where two of the pairs of V are in R' and the remaining pairs of V are in S'. In this case T' has q orbits.

(iii) The only remaining possibility is that T has orbits R, S, and U with (d, b) followed by (b, a) in R, (a, b) by (b, c) in S, and (x, b) by (b, y) in U. In this case R, S, and U are replaced in T' by a single orbit of the form

$$(b, y), \quad \ldots, \quad (x, b), \quad (b, a), \quad \ldots, \quad (d, b), \quad (b, c), \quad \ldots, \quad (a, b)$$

containing each of the ordered pairs appearing in R, S, or U. The remaining orbits of T' are the other $q - 3$ orbits of T. In this case T' has $q - 2$ orbits.

In general the number of orbits of T' will be $q - 2$, q, or $q + 2$. If $G(N)$ is the embedding induced by T', we have by (1), $\gamma(N) = \gamma(M) - 1$, $\gamma(M)$, or $\gamma(M) + 1$.

Thus shifting a single symbol in one permutation T_b will change the genus of the induced 2-manifold by at most one. The desired result now follows by observing that the cyclic permutations associated with a given 2-cell embedding can be transformed into the set of permutations associated with any other 2-cell embedding of the same graph by a finite sequence of such alterations.

4. The Betti number. Applying the reduction procedure one obtains

4.1. THEOREM. *For G an arbitrary finite connected graph, $d(G) = 1$ if and only if $\beta(G) = 0$.*

Proof. If $\beta(G) = 0$, it follows that $\gamma(G) = 0$, and hence by (1), that $d(G) = 1$.

Suppose that $d(G) = 1$ and $\beta(G) > 0$. We may then assume that G has no points of order one. Let $G(M)$ be a minimal embedding of G with an orientation τ chosen for M. The transformation $T(W) = W$ associated with $G(M)$ and τ has a single orbit R. Since G has no points of order 1, there exists a pair (a, b) for which

(i) R is of the form $(a, b), T(a, b), \ldots, T^{m-1}(a, b)$ for some positive integer m;

(ii) $T^k(a, b) = (b, a)$ for some integer $k (3 < k < m - 3)$; and

(iii) if $(x, y) = T^i(a, b)$ for $i < k$, then $(y, x) = T^j(a, b)$ for some j, $k < j < m$.

Letting $T(a, b) = (b, c)$ and $T^{k-1}(a, b) = (d, b)$ we have $c \neq d$ by (iii). Also by (iii), we have $(c, b) = T^i(a, b)$ for some i, $k < i < m$. Thus the embedding $G(M)$ can be reduced at b, contradicting $d(G) = 1$. Hence $\beta(G) = 0$.

4.2. LEMMA. *Let $G(M)$ be a minimal embedding of a finite connected graph G with $d(G) = 2$. If each edge of G lies entirely in the boundary of each of the two components of $M - G(M)$, then G is a simple closed curve and M is a 2-sphere.*

Proof. Let $G(M)$ be as above. G has no points of order 1. Let an orientation τ be chosen for M. Denote the orbits of the transformation $T(W) = W$ associated with $G(M)$ and τ by R_1 and R_2. Let (b, a) be a pair of R_1. ((a, b) is then in R_2.) Let

$$T(b, a) = (a, c), \quad T^{-1}(a, b) = (e, a), \quad T^{-1}(b, a) = (y, b),$$

and

$$T(a, b) = (b, x).$$

Suppose that $\beta(G) > 1$. ($d(G) = 2$ implies $\beta(G) \neq 0$.) We may assume, therefore, that $c \neq e$ and $x \neq y$. Let $T^{-1}(b, y) = (w, b)$ in R_2. Since (x, b) is in R_1, $w \neq x$. Replacing $T_a = (debc\ldots)$ by $T'_a = (dbec\ldots)$ and $T_b = (wyaxb\ldots)$ by $T'_b = (wyxab\ldots)$ one obtains a new transformation having two orbits, one of which can be reduced at b. This contradicts $d(G) = 2$. Thus $\beta(G) = 1$. Since G has no points of order 1, G is a simple closed curve and by (1) M is a 2-sphere.

This lemma enables us to take the results of 4.1 one step further.

4.3. THEOREM. *For an arbitrary finite connected graph G, $d(G) = 2$ if and only if $\beta(G) = 1$.*

Proof. If $\beta(G) = 1$, it follows that $\gamma(G) = 0$, and hence by (1), that $d(G) = 2$.

Suppose $d(G) = 2$. It follows that $\beta(G) > 0$. Assume G has no points of order 1. Let $G(M)$ be a minimal embedding and $T(W) = W$ the transformation associated with $G(M)$ and some orientation for M. Denote the orbits of T by R_1 and R_2. Assume there exists a pair (a, b) such that (a, b) and (b, a) are both in R_2. We may then choose an (a, b) for which $T(b, a) = (a, x)$ is in R_2 and (x, a) is in R_1. Let N denote the numbers of ordered pairs in R_2. We may choose a pair (r, s) in R_2 such that

(i) $T^k(a, b) = (b, a)$ for some integer $k < N$;

(ii) $T^m(a, b) = (r, s)$ and $T^n(a, b) = (s, r)$ for some integers m and n with $0 \leqslant m < n \leqslant k$; and

(iii) if $(e, f) = T^i(a, b)$ for some i $(m < i < n)$, then either (f, e) is in R_1 or $(f, e) = T^j(a, b)$ for some j $(0 < j < m$ or $n < j < N)$.

Letting $T(r, s) = (s, t)$ and $T^{-1}(s, r) = (u, s)$, we have $u \neq t$ by (iii). s is not an element of any pair of the form $T^i(a, b)$ where $n < i < N$ or $0 < i < m$, for otherwise we could reduce $G(M)$ at s. By (iii) the ordered pair (t, s) does not occur "between" (r, s) and (s, r). Thus (t, s) is in R_1.

Replacing $T_a = (dbxc\ldots)$ by $T'_a = (dxbc\ldots)$, the orbits R_1 and R_2 are replaced by new orbits R'_1 and R'_2 where R'_2 can be reduced at s, contradicting $d(G) = 2$.

Thus there exists no pair (a, b) with both (a, b) and (b, a) in R_2 (or both in R_1). By (1) we have $\beta(G) = 1$.

The results of 4.1 and 4.3 together with equation (1) yield the following corollaries.

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4.3.1. COROLLARY. *Let G be an arbitrary finite connected graph. If $G(M)$ is an embedding of G in an orientable 2-manifold M such that $M - G(M)$ has at most two components, then either M is a 2-sphere (and G is a planar graph) or $G(M)$ fails to be a minimal embedding.*

4.3.2. COROLLARY. *If the finite connected graph G has genus $n > 0$, then $\beta(G) \geqslant 2n + 2$.*

As an application of these results we have the irreducibility of the non-planar Kuratowski graph $K_{3,3}$, i.e. the 3×3 bipartite graph; cf. **(4)**. For if G is the graph obtained by deleting any single edge of $K_{3,3}$, then $\beta(G) = 3$. Any 2-cell embedding of G in the torus has just two complementary domains and such an embedding is not minimal.

5. Conclusion.

For any 2-cell embedding $G(M)$ for which there are two or more complementary domains, it is possible to order the complementary domains so that for each one, other than the first, there is a boundary edge that is contained entirely in the boundary of some preceding domain. By deleting one such edge for each complementary domain, one obtains a subgraph K of G whose complement in M consists of exactly one open 2-cell. By (1), $\beta(K) = 2n$. Thus each minimal embedding $G(M)$ of a non-planar graph G may be thought of as an embedding of a subgraph K of G with $\beta(K) = 2n$ and such that the complement of K in M is connected, together with at least two additional edges of G, each having its end points in K. The relationship of $\beta(G)$ to $\gamma(G)$ conjectured in the Introduction would imply that the number of edges of $G - K$ is actually at least $2n$.

It would be sufficient to prove this conjecture for the set of all n-irreducible graphs for each positive integer n. By **(1, 4)**, the conjecture does hold for all of the known irreducible graphs, but the task of finding all of these special graphs is far from complete and a direct attack on the Betti number problem, although difficult, seems to be the most promising.

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University of Washington,
Seattle, Wash.