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SPACES OF ORDERINGS IV

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A major goal of this paper is to give a proof of the following isotropy criterion: Let $X = (X,G)$ be a space of orderings in the terminology of [**9**] or [**10**], and let f be a form defined over G. Then f is anisotropic over X if and only if f is anisotropic over some finite subspace of X. This is the content of Theorem 1.4, and generalizes [**1**, Corollary 3.4]. Moreover, in view of the known structure of finite spaces (see [**9**]), this has, essentially, the strength of [**2**, Satz 3.9] or [**12**, Theorem 8.12]. The technique used to prove this criterion is roughly patterned on that of [**6**], and yields some interesting by-products: An interesting invariant of a space of orderings called the *chain length* is introduced (Definition1.1) and spaces of orderings with finite chain length are classified (Theorem 1.6). This extends work in [**4**], [**7**], and [**9**]. It is proved (Theorem 3.2) that every space of orderings X possesses a partition $X = \cup_{\alpha \in M} X(\alpha)$ satisfying: Each $X(\alpha)$ is a fan, and each fan $V \subseteq X$ intersects at most two of the $X(\alpha)$'s. Such a partition is referred to as a *P-structure* on X, and generalizes the partition of the space of orderings of a field induced by the real places of the field. We prove that spaces of finite chain length are just those with finite P-structure (Theorem 3.3). Some properties of the space of orderings of a field which are expressible in terms of real places generalize to P-structures, among them the "exactness" result in [**1**] (see Theorem 3.12).

The paper assumes a good deal of [**9**, **10**, **11**]. On the other hand, the proof of Theorem 1.4 is essentially independent of [**11**], so as remarked in [**8**, Theorem 5.4], the isotropy criterion proved here yields another proof of the representation theorem for $W(X)$ (Theorem 3.5 of [**11**]). This follows along the same lines as that given in [**1**].

The notation is that of [**11**]. In particular, throughout the paper, $X = (X, G)$ will denote a space of orderings in the terminology of [**9**].

1. Chain length and the isotropy theorem. In this section the chain length of a space of orderings is defined, and various properties of this invariant are proved. The structure theorem for spaces of finite chain length and the isotropy theorem are proved modulo the proof of Theorem 1.3. The proof of 1.3 will be given later (see the remark following Theorem 3.3).

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(1.1) *Definition.* The *chain length* of X (denoted cl (X)) is the maximum integer $k \geqq 1$ such that $\exists\, a_0, \ldots, a_k \in G$ satisfying

$$X(a_{i-1}) \subsetneqq X(a_i),\ i = 1, \ldots, k,$$

(or cl $(X) = \infty$ if no such maximum exists).

In view of [9], Lemma 2.1, the condition $X(a_{i-1}) \subsetneqq X(a_i)$ is equivalent to

$$D\langle 1,a_i\rangle \subsetneqq D\langle 1,a_{i-1}\rangle.$$

(1.2) *Remark.* It is easily verified that cl $(X) = 1$ if and only if $|X| = 1$, and $\mathrm{cl}(X) \leqq 2$ if and only if X is a fan (see [**11**], Theorem 4.2 (i)). These results are left as exercises.

Recall [**10**, definition 2.10] that X is said to be *decomposable* if there exist non-empty subspaces X_i of X, $i = 1,2$ such that $X = X_1 \oplus X_2$. Let us denote by $\mathrm{gr}(X)$ the *translation group* of X in the terminology of [**9**], ie,

$$\mathrm{gr}(X) = \{\gamma \in \chi(G) | \gamma X = X\}.$$

Thus $\mathrm{gr}(X)$ is a closed subgroup of $\chi(G)$. Let the *residue space* of X be defined to be $X' = (X', G')$ where $G' = \mathrm{gr}(X)^\perp \subseteq G$, and where X' denotes the image of X in $\chi(G')$ via restriction. Exactly as in the proof of [**9**], Theorem 4.8], X' is a space of orderings. Moreover $\mathrm{gr}(X') = 1$, and X is a group extension of X' (in the terminology of [**10**, definition 3.6]). With this terminology at our disposal, we can state the main theorem concerning spaces of finite chain length.

(1.3) THEOREM. *Suppose* $\mathrm{cl}(X) < \infty$. *Then either* $|X| = 1$, *or* $\mathrm{gr}(X) \neq 1$, *or* X *is decomposable.*

The proof of this key result is found in § 3. For now we concentrate on giving two important applications:

(1.4) ISOTROPY THEOREM. *Suppose a form* f *is anisotropic over a space of orderings* X_0. *Then there exists a finite subspace* $X \subseteq X_0$ *such that* f *is anisotropic over* X.

Proof. Let $X = (X,G)$ be a subspace of X_0 chosen minimal subject to: f is anosotropic over X. (The existence of X is by Zorn's Lemma; see the technique of [**11**, Theorem 5.3].) Let $a_0, \ldots, a_k \in G$ satisfy

$$D\langle 1,a_{i-1}\rangle \subsetneqq D\langle 1, a_i\rangle,\ i = 1, \ldots, k.$$

Thus $\langle 1,a_i\rangle \cong \langle a_{i-1}, a_{i-1}a_i\rangle$ and $a_{i-1} \neq a_i$ for $i = 1, \ldots, k$. We may assume $a_0 = 1$, $a_k = -1$. Let $b_i = a_{i-1}a_i$. Thus $b_i \neq 1$, so $X(b_i)$ is a proper subspace of X. Thus f is isotropic over $X(b_i)$, i.e, there exists a

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form g_i of dimension $n - 2$ (where n denotes the dimension of f) such that $f \sim g_i$ over $X(b_i)$. Thus (comparing signatures)

$$f \otimes \langle 1,b_i\rangle \sim g_i \otimes \langle 1,b_i\rangle \text{ over } X,$$

so, by addition

$$(1)\quad f \otimes \left(\sum_{i=1}^{k} \langle 1, b_i\rangle\right) \sim \sum_{i=1}^{k} g_i \otimes \langle 1, b_i\rangle \quad (\text{over } X).$$

But using the assumptions on $a_0, \ldots, a_k$ we see that (over X)

$$\begin{aligned}\langle b_1, \ldots, b_k\rangle &\cong \langle a_0a_1, a_1a_2, \ldots, a_{k-1}a_k\rangle \cong \langle a_1, a_1a_2, \ldots, a_{k-1}a_k\rangle \\ &\cong \langle 1, a_2, a_2a_3, \ldots, a_{k-1}a_k\rangle \cong \ldots \cong \langle 1, \ldots, 1, a_k\rangle \\ &\cong \langle 1, \ldots, 1, -1\rangle.\end{aligned}$$

Substituting this in (1) yields

$$(2k - 2)f \sim \sum_{i=1}^{k} g_i\langle 1, b_i\rangle.$$

Now f (and hence $(2k - 2)f$, by [**11**, Corollary 3.5 (ii)]) is anisotropic over X, so comparing dimensions, and using [**11**, Lemma 2.4],

$$(2k - 2)n \leqq k(n - 2)(2),$$

ie., $k \leqq \frac{1}{2}n$. This proves $\mathrm{cl}(X) < \infty$.

Now we apply Theorem 1.3. If $|X| = 1$ we are done. Suppose $X = X_1 \oplus X_2$ where $X_i = (X_i, G/\Delta_i)$ is a non-empty subspace of X, $i = 1,2$. Thus there exist elements $a_{i3}, \ldots, a_{in} \in G$ such that

$$f \cong \langle -1, 1, a_{i3}, \ldots, a_{in}\rangle \text{ over } X_i,\ i = 1,2.$$

Since $X = X_1 \oplus X_2$, the natural injection

$$G \rightarrow G/\Delta_1 \times G/\Delta_2$$

is surjective, so there exist $a_3, \ldots, a_n \in G$ such that

$$a_j \equiv a_{ij} (\mathrm{mod}\ \Delta_i),\ 3 \leqq j \leqq n,\ i = 1,2.$$

Then clearly $f \cong \langle 1,-1, a_3, \ldots, a_n\rangle$ over X, a contradiction. Thus X is indecomposible, so $\mathrm{gr}(X) \neq 1$. Let $X' = (X',G')$ denote the residue space of X and decompose f as

$$f \cong \pi_1 f_1 \oplus \ldots \oplus \pi_s f_s$$

where $f_1, \ldots, f_s$ are forms over G', and $\pi_1, \ldots, \pi_s \in G$ are distinct modulo G'. The assertion that f is anisotropic over X is equivalent to the assertion that each $f_1, \ldots, f_s$ is anisotropic over X' (see [**10**, remark 3.7]). There are two cases to consider:

Suppose $s = 1$. Let Δ be any subgroup of G such that G is the direct product $G = \Delta \times G'$, and let $Y = \Delta^\perp \cap X$. Then one verifies easily that $Y = (Y,G/\Delta)$ is a subspace of X and that $(Y,G/\Delta) \sim (X',G')$, this equivalence being induced by the natural isomorphism $G/\Delta \cong G'$. Thus, since f_1 is anisotropic over X', it (and then $f \cong \pi_1 f_1$) is anisotropic over Y. But, on the other hand, gr $X \neq 1$, i.e., $G' \neq G$, i.e., $\Delta \neq 1$, i.e., $Y \subsetneq X$. This contradicts the minimal choice of X.

Thus $s \geqq 2$. It follows that each f_i has strictly lower dimension than f so by induction on the dimension, there exist finite subspaces $Z_1', \ldots, Z_s' \subseteq X'$ such that f_i is anisotropic over Z_i'. Thus $f_1, \ldots, f_s$ are all anisotropic over the subspace of X' generated by $Z_1' \ldots, Z_s'$. Denote this space by $Z' = (Z', G'/\Delta')$. Note Z' is still finite. Let $Z = \Delta'^{\perp} \cap X$. Then $Z = (Z, G/\Delta')$ is a subspace of X, and a group extension of $Z' = (Z', G'/\Delta')$. Moreover, since $\pi_1, \ldots, \pi_s$ are distinct modulo G', f is anisotropic over Z. Thus, by minimal choice of X, $Z = X$, ie., $\Delta' = 1$, i.e., $Z' = X'$ is finite. However, X itself could be infinite (since, a priori, $\mathrm{gr}(X)$ could be infinite). Define G'' to be the subgroup of G generated by G' and $\pi_1, \ldots, \pi_s$, and let X'' denote the restriction of X to G''. Thus (X,G) is a group extension of (X'',G'') which, in turn, is a group extension of (X',G'). Moreover (X'',G'') is finite, and f is anisotropic over X''. Finally, let Δ be subgroup of G so that $G = \Delta \times G''$, and let $Y = \Delta^{\perp} \cap X$. Then $Y = (Y, G/\Delta)$ is a subspace of X naturally equivalent to (X'',G''). Thus Y is finite, and f is anisotropic over Y. Thus $Y = X$ is finite.

Denote by $\mathscr{O}$ the category of all spaces of orderings, and by $\mathscr{C}$ the smallest subcategory of $\mathscr{O}$ such that

(a) $\mathscr{C}$ contains the singleton space,
(b) If $X_1, X_2 \in \mathscr{C}$, then $X_1 \oplus X_2 \in \mathscr{C}$,
(c) If X is a group extension of $X' \in \mathscr{C}$, then $X \in \mathscr{C}$.

Thus elements of $\mathscr{C}$ are spaces obtained from the singleton space using the direct sum and group extension operations a finite number of times. In [**7**] the following characterization of $\mathscr{C}$ is obtained.

(1.5) THEOREM. *The following are equivalent:*
(i) *There exists a pythagorian field K with only finitely many real places such that $X \sim X_K$,*
(ii) $X \in \mathscr{C}$.

Here are some additional characterizations of $\mathscr{C}$:

(1.6) THEOREM. *The following are equivalent:*
(i) $\mathrm{cl}(X) < \infty$.
(ii) *X is generated by finitely many fans.*
(iii) $X \in \mathscr{C}$.

For the proof we require Theorem 1.3 and the following theorem:

(1.7) THEOREM. (i) *Suppose* $X_i = (X_i, G/\Delta_i)$, $i = 1, \ldots, n$ *are subspaces of* X *generating* X. *Then*

$$\mathrm{cl}(X) \leqq \sum_{i=1}^{n} \mathrm{cl}(X_i).$$

(ii) *If, in addition,* $X = X_1 \oplus \ldots \oplus X_n$, *then*

$$\mathrm{cl}(X) = \sum_{i=1}^{n} \mathrm{cl}(X_i).$$

(iii) *If* X *is a group extension of* X', *then* $\mathrm{cl}(X) = \mathrm{cl}(X')$, *except in the case* $|X'| = 1$ (*in which case* X *is a fan*).

Here is a proof of Theorem 1.6 (assuming 1.3 and 1.7). Denote by $\mathscr{D}$ the subcategory of $\mathscr{O}$ consisting of spaces which are generated by finitely many fans. It is clear that $\mathscr{D}$ satisfies (a), (b) and (c). (Hint: If X is a group extension of X', then the inverse image of a fan in X' under the natural projection is a fan in X.) Thus $\mathscr{C} \subseteq \mathscr{D}$, so (iii) $\Rightarrow$ (ii). The implication (ii) $\Rightarrow$ (i) follows from Theorem 1.7(i) and Remark 1.2. The implication (i) $\Rightarrow$ (iii) is proved by induction in the chain length of X. Replacing X by its residue space, we can assume $\mathrm{gr}(X) = 1$ (using Theorem 1.7 (iii) and property (c) of $\mathscr{C}$). Thus, by Theorem 1.3, either $|X| = 1$, or X decomposes. Further, if $X = X_1 \oplus X_2$, $X_i \neq \emptyset$, $i = 1,2$, then $\mathrm{cl}(X_i) < \mathrm{cl}(X)$ by Theorem 1.7 (ii), so by induction, $X_i \in \mathscr{C}$, i.e., $X \in \mathscr{C}$ by (b). Finally, if $|X| = 1$, then $X \in \mathscr{C}$ by (a).

Proof of Theorem 1.7. (i) Suppose $X(a_{j-1}) \subsetneqq X(a_j)$, $j = 1, \ldots, k$. Then for each i, $1 \leqq i \leqq n$, $X_i(a_{j-1}) \subseteq X_i(a_j)$. Moreover, since $X(a_{j-1}) \neq X(a_j)$, there exists i, $1 \leqq i \leqq n$ such that

$$X_i(a_{j-1}) \neq X_i(a_j).$$

(For if $X_i(a_{j-1}) = X_i(a_j)$ for all $i \leqq n$, then

$$a_j a_{j-1} \in \bigcap_{i=1}^{n} \Delta_i = 1, \quad \text{i.e.,} \quad a_j = a_{j-1},$$

a contradiction.) This holds for $j = 1, \ldots, k$. Simple counting yields

$$k \leqq \sum_{i=1}^{n} \mathrm{cl}(X_i), \quad \text{i.e.,}$$

$$\mathrm{cl}(X) \leqq \sum_{i=1}^{n} \mathrm{cl}(X_i).$$

(ii) We are assuming $X = \bigcup_i X_i$ and the natural homomorphism from G into $\prod_i G/\Delta_i$ is an isomorphism. Suppose

$$X_i(a_{i,j-1}) \subsetneqq X_i(a_{i,j}), j = 1, \ldots, k_i, i = 1, \ldots, n.$$

We may as well assume $a_{i,0} = -1$, and $a_{i,k_i} = 1$. Choose elements $b_{ij} \in G$ such that

$$b_{ij} \equiv 1 \ (\text{mod } \Delta_k) \text{ for } k < i,$$
$$b_{ij} \equiv a_{ij} \ (\text{mod } \Delta_i), \text{ and}$$
$$b_{ij} \equiv -1 \ (\text{mod } \Delta_k), \text{ for } k > i.$$

Note that

$$X(b_{ij}) = (\cup_{s<i} X_s) \cup X_i(a_{ij}).$$

It follows that

$$X(b_{10}) \subsetneqq \ldots \subsetneqq X(b_{1k_1}) = X(b_{20}) \subsetneqq \ldots\ldots \subsetneqq X(b_{nk_n}).$$

There are $\sum k_i$ inequalities in this chain, so $\text{cl}(X) \geqq \sum k_i$, and hence

$$\text{cl}(X) \geqq \sum \text{cl}(X_i).$$

The other inequality follows from (i).

(iii) Suppose $|X'| \neq 1$. Suppose

$$X'(a_{i-1}) \subsetneqq X'(a_i),\ i = 1, \ldots, k, \text{ with } a_i \in G'.$$

Then clearly $X(a_{i-1}) \subsetneqq X(a_i)$, $i = 1, \ldots, k$. Thus $\text{cl}(X) \geqq \text{cl}(X')$. Now suppose

$$D\langle 1, a_i\rangle \subsetneqq D\langle 1, a_{i-1}\rangle,\ i = 1, \ldots, k, \text{ with } a_1, \ldots, a_k \in G.$$

We may assume $a_0 = -1$, $a_k = 1$. Then $a_1 \neq -1$. There are two cases to consider.

1*st Case*. Suppose $a_1 \notin G'$. It follows (from the definition of group extension) that $D\langle 1,a_1\rangle = \{1,a_1\}$. Thus $k \leqq 2$ in this case. Thus, since $|X'| \neq 1$, $\text{cl}(X') \geqq 2 \geqq k$.

2*nd Case*. Suppose $a_1 \in G'$. Then $D\langle 1,a_1\rangle \subseteq G'$ (e.g., by [**9**, Lemma 4.9]; note $a_1 \neq -1$). Thus $a_1, \ldots, a_k$ are all in G', and

$$X'(a_{i-1}) \subsetneqq X'(a_i),\ i = 1, \ldots, k.$$

Thus $\text{cl}(X') \geqq k$. Thus, in any case $\text{cl}(X') \geqq k$, so $\text{cl}(X') \geqq \text{cl}(X)$.

We now proceed to prove a deeper property of chain length. This will eventually be used in the proof of Theorem 1.3.

(1.8) Theorem. *Suppose Y is a subspace of X. Then* $\text{cl}(Y) \leqq \text{cl}(X)$.

The proof of this follows easily from the following lemma. The lemma itself doesn't seem to have an easy proof.

(1.9) Lemma. *Suppose $b, a_0, \ldots, a_k \in G$ satisfy*

$$D\langle 1,b\rangle = \{1,b\}, \textit{ and}$$
$$D\langle 1,a_{i-1}\rangle\langle 1,b\rangle \subseteq D\langle 1,a_i\rangle\langle 1, b\rangle,\ i = 1, \ldots, k.$$

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Then there exists $a_i' \in D\langle a_i, a_i b\rangle = \{a_i, a_i b\}$ *such that*

$$D\langle 1, a_{i-1}'\rangle \subseteq D\langle 1, a_i'\rangle,\ i = 1, \dots, k.$$

Proof of 1.8. Suppose, to the contrary, $\mathrm{cl}(Y) > \mathrm{cl}(X)$. Then, in particular, $\mathrm{cl}(X) < \infty$. Choose a subspace $Z \subseteq X$ minimal subject to (1) $Z \supseteq Y$ and (2) $\mathrm{cl}(Z) \leqq \mathrm{cl}(X)$. To show such Z exists, we need only verify that Zorn's Lemma applies. Suppose $\{Z_i\}$ is a collection of subspaces of X satisfying (1) and (2) and linearly ordered by inclusion. Let $Z' = \bigcap_i Z_i$. Then Z' is a subspace of X satisfying (1). To show Z' satisfies (2) suppose $a_0, \dots, a_k \in G$ satisfy

$$Z'(a_j) \subsetneqq Z'(a_{j-1}),\ j = 1, \dots, k.$$

Thus, the set

$$\mathscr{U} = \{\sigma \in X \mid \sigma\langle 1, a_j\rangle = \sigma\langle a_{j-1}, a_{j-1}a_j\rangle,\ j = 1, \dots, k\}$$

is open in X and contains Z'. By compactness, $Z_i \subseteq \mathscr{U}$ for some i, so

$$Z_i(a_j) \subseteq Z_i(a_{j-1}),\ j = 1, \dots, k.$$

These inclusions must be strict, since $Z' \subseteq Z_i$. Thus $k \leqq \mathrm{cl}(Z_i) \leqq \mathrm{cl}(X)$, so $\mathrm{cl}(Z') \leqq \mathrm{cl}(X)$. Thus Zorn's Lemma applies, so Z exists as asserted.

To simplify notation, we may assume $X = Z$. Let $Y = (Y, G/\Delta)$. Since

$$Y \neq X\ (\mathrm{cl}(Y) > \mathrm{cl}(X))$$

it follows that $\Delta \neq 1$, so there exists $a \in \Delta$, $a \neq 1$. Thus $Y \subseteq X(a) \subsetneqq X$. Since $\mathrm{cl}(X) < \infty$, there exists $b \in G$, $b \neq 1$, such that $X(a) \subseteq X(b) \subseteq X$, $X(b)$ maximal. Thus $D\langle 1, b\rangle$ is minimal, ie., $D\langle 1, b\rangle = \{1, b\}$. By the minimal choice of $X(=Z)$, it follows that $\mathrm{cl}(X(b)) > \mathrm{cl}(X)$. On the other hand it follows from Lemma 1.9 that $\mathrm{cl}(X(b)) \leqq \mathrm{cl}(X)$. This is a contradiction.

The proof of 1.9 is broken into three cases: $k = 1$, $k = 2$, $k \geqq 3$. The case $k = 2$ is the difficult case.

Suppose $k = 1$: In this case a_0 is represented by

$$\langle 1, a_1\rangle\langle 1, b\rangle \cong \langle 1, b\rangle \oplus a_1\langle 1, b\rangle,$$

so by 0_4, $\exists\, u, v \in D\langle 1, b\rangle$ such that a_0 is represented by $\langle u, a_1 v\rangle$, i.e., ua_0 is represented by $\langle 1, a_1 uv\rangle$. Take $a_0' = ua_0$, $a_1' = a_1 uv$.

Suppose $k = 2$: We may as well assume that modulo $D\langle 1, b\rangle$,

(1) $a_0 \not\equiv 1$, $a_2 \not\equiv -1$ and $a_{i-1} \not\equiv a_i$ $i = 1,2$.

For if $a_0 \equiv 1$, $a_2 \equiv -1$, or $a_{i-1} \equiv a_i$ for $i = 1$ or 2, then by taking $a_0' = 1$ (resp. $a_2' = -1$, resp. $a_{i-1}' = a_i'$) we would be down to the case

$k = 1$. By the case $k = 1$, we may also assume (replacing a_i by a_ib if necessary) that

(2) $D\langle 1,a_0\rangle \subseteq D\langle 1,a_1\rangle$ and $D\langle 1,ba_1\rangle \subseteq D\langle 1,a_2\rangle$.

We may also assume

(3) $D\langle 1,a_1) \not\subseteq D\langle 1,a_2\rangle$, $D\langle 1,a_1\rangle \not\subseteq D\langle 1,ba_2\rangle$, $D\langle 1,a_0\rangle \not\subseteq D\langle 1,ba_1\rangle$,

and

$$D\langle 1,ba_0\rangle \not\subseteq D\langle 1,ba_1\rangle.$$

We complete the proof by showing that (1), (2), and (3) together with $D\langle 1,b\rangle = \{1,b\}$ imply a contradiction.

Denote by G' the subgroup of G generated by b, a_0, a_1, a_2, and -1, and by X' the restriction of X to $\chi(G')$. By (1) and (2), there exist α_1, α_2, α_3, $\alpha_4 \in X'$ defined by

	b	a_0	a_1	a_2
α_1	+	+	+	+
α_2	+	+	+	−
α_3	+	+	−	−
α_4	+	−	−	−

By (2), these are the only characters in X' making b positive. There may also be a character $\alpha_5 \in X'$ satisfying $\alpha_5(b) = -1$, $\alpha_5(a_1) = 1$. By (2) it follows that $\alpha_5(a_0) = 1, \alpha_5(a_2) = -1$, so there is at most one such character. There may also be characters α_6, α_7, α_8, $\alpha_9 \in X'$ defined by

	b	a_0	a_1	a_2
α_6	−	+	−	+
α_7	−	+	−	−
α_8	−	−	−	+
α_9	−	−	−	−

By (3) and (2) certain of these characters do, in fact, exist, namely (*): one of each of the following pairs exist:

$$\alpha_6,\ \alpha_8;\ \alpha_7,\ \alpha_9;\ \alpha_8,\ \alpha_9;\ \alpha_6,\ \alpha_7.$$

It is clear that $\alpha_1, \alpha_2, \ldots, \alpha_9$ exhaust all possible elements of X'. Note that there are 5 independent characters in X' (eg., α_1, α_2, α_3, α_4, and any one of α_6, α_7, α_8, α_9 which exists), so G' is 5-dimensional over $\mathbf{Z}/2\mathbf{Z}$, and the generators b, a_0, a_1, a_2, -1 are a basis of G'. There are two subcases to consider:

Subcase A. α_6 and α_9 both exist. In this case let σ_1, σ_2, σ_3, σ_4, $\sigma_5 \in X'$ be defined by

$$\sigma_1 = \alpha_2,\ \sigma_2 = \alpha_3,\ \sigma_3 = \alpha_1,\ \sigma_4 = \alpha_6,\ \sigma_5 = \alpha_9.$$

Then one verifies

$$\alpha_4 = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5,\ \alpha_5 = \sigma_2\sigma_3\sigma_4,\ \alpha_7 = \sigma_1\sigma_3\sigma_4,\ \alpha_8 = \sigma_1\sigma_3\sigma_5.$$

Choose $p_1, \ldots, p_5 \in G'$ dual to the basis $\sigma_1, \ldots, \sigma_5$ of $\chi(G')$, i.e., $\sigma_i(p_j) = 1$ if $i \neq j$, $\sigma_i(p_j) = -1$, if $i = j$. As in the proof of Lemma 3.1 of [**9**], the forms

$$f = \langle 1, p_1p_2p_3p_4, p_1p_2p_3p_5\rangle \text{ and } g = \langle p_1p_2p_4p_5, p_2p_3, p_3p_1\rangle$$

are isometric over X. Note that p_4p_5 is positive at σ_1, σ_2, σ_3, and negative at σ_4, σ_5, as is b, so $p_4p_5 = b$. Thus $\langle 1,p_4p_5\rangle$ only represents 1, p_4p_5 over X. Thus, we obtain a contradiction exactly as in [**9**, Lemma 3.1]; namely f must represent $p_1p_2p_4p_5$ on the one hand, but on the other hand, this is impossible.

Subcase B. One of α_6 and $\alpha_9 \notin X'$. Then by (*), α_7 and α_8 both exist. In this case consider the basis $\sigma_1, \ldots, \sigma_5$ defined by

$$\sigma_1 = \alpha_1,\ \sigma_2 = \alpha_3,\ \sigma_3 = \alpha_2,\ \sigma_4 = \alpha_7,\ \sigma_5 = \alpha_8.$$

With respect to this basis,

$$\alpha_4 = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5,\ \alpha_5 = \sigma_2\sigma_3\sigma_4,\ \alpha_6 = \sigma_1\sigma_3\sigma_4,\ \alpha_9 = \sigma_1\sigma_3\sigma_5.$$

Let $p_1, \ldots, p_5 \in G'$ be the dual basis to $\sigma_1, \ldots, \sigma_5$ and proceed as in subcase A.

Suppose $k \geqq 3$. By induction (replacing a_i by a_ib wherever necessary) we may assume

$$D\langle 1,a_0\rangle \subseteq D\langle 1, a_1\rangle \subseteq \ldots \subseteq D\langle 1,a_{k-1}\rangle.$$

Also by induction there exist $a_i^* \in \{a_i,a_ib\}$ such that

$$D\langle 1,a_1^*\rangle \subseteq \ldots \subseteq D\langle 1,a_k^*\rangle.$$

Replacing a_k by ba_k if necessary, we may assume $a_k^* = a_k$. If either $a_1^* = a_1$ or $a_{k-1}^* = a_{k-1}$, we are done, so we may assume $a_1^* = ba_1$, $a_{k-1}^* = ba_{k-1}$. Applying induction a third time there exist elements $a_i' \in \{a_i,a_ib\}$ such that

$$D\langle 1,a_0'\rangle \subseteq \ldots \subseteq D\langle 1,a_{k-2}\rangle \subseteq D\langle 1,a_k'\rangle.$$

If $a_1' = a_1^*$, we are done, so we may assume $a_1' = a_1$. We claim now that

$$D\langle 1,a_{k-1}\rangle \subseteq D\langle 1,a_k'\rangle.$$

Once this is proved we are done. For suppose $\sigma \in X$ is such that $\sigma(a_k') = 1$. Since $a_k' \in \{a_k,a_kb\}$ it follows that if $\sigma(b) = 1$, then $\sigma(a_k) = 1$. Since

$$D\langle 1,a_{k-1}\rangle\langle 1,b\rangle \subseteq D\langle 1,a_k\rangle\langle 1,b\rangle,$$

this implies $\sigma(a_{k-1}) = 1$. Now suppose $\sigma(b) = -1$. Since

$$a_1 = a_1' \in D\langle 1,a_1'\rangle \subseteq D\langle 1,a_k'\rangle, \text{ and } \sigma(a_k') = 1,$$

we must have $\sigma(a_1) = 1$, i.e.,

$$\sigma(a_1^*) = \sigma(ba_1) = -1.$$

Since

$$D\langle 1,a_1^*\rangle \subseteq D\langle 1,a_{k-1}^*\rangle,$$

this implies $\sigma(a_{k-1}^*) = -1$. But $a_{k-1}^* = ba_{k-1}$. Thus

$$\sigma(a_{k-1}) = \sigma(b)\sigma(a_{k-1}^*) = (-1)(-1) = 1.$$

Thus, in any case $\sigma(a_k') = 1 \Rightarrow \sigma(a_{k-1}) = 1$, so

$$D\langle 1,a_{k-1}\rangle \subseteq D\langle 1,a_k'\rangle.$$

2. Components. We define *simply connected* as in [**9**], i.e., for σ, $\tau \in X$, $\sigma \sim_s \tau \Leftrightarrow$ either $\sigma = \tau$, or there exists a 4-element fan $V \subseteq X$ such that $\sigma,\tau \in V$. It is mentioned in [**9**] that $\sim_s$ is an equivalence relation on X, but this is never proved explicitly. To give a proof we need some preliminary results.

(2.1) *Remark.* Many results in [**9**] carry over with little or no modification to infinite spaces of orderings. We have already remarked this for Theorem 4.1 of [**9**] (see [**11**, Lemma 4.1]). For S a subset of $\chi(G)$, and $\gamma \in \chi(G)$, denote by γS the translation of S by γ, i.e.,

$$\gamma S = \{\gamma\sigma | \sigma \in S\} \subseteq \chi(G).$$

Lemma 4.2 of [**9**] goes through in the following modified form: "Suppose S is a subset of X which generates X, and suppose $\gamma \in \chi(G)$ satisfies $\gamma S \subseteq X$. Then $\gamma X = X$". The proof is an easy modification of that in [**9**]. For $\alpha \in \chi(G)$ define X_α as in [**9**], ie.,

$$X_\alpha = X \cap \alpha X = \{\sigma \in X | \sigma\alpha \in X\}.$$

Lemmas 4.3, 4.4, 4.5, 4.6 of [**9**] carry over word for word to the infinite case.

(2.2) LEMMA. *Suppose* $\mathrm{gr}(X) \neq 1$, $|X| \geqq 3$. *Then* $\sigma \sim_s \tau$ *holds for all* $\sigma,\tau \in X$.

Proof. Let $\gamma \in \mathrm{gr}(X)$, $\gamma \neq 1$. Let $\sigma,\tau \in X$, $\sigma \neq \tau$. First suppose $\tau \neq \gamma\sigma$. Then $V = \{\sigma, \tau, \sigma\gamma, \tau\gamma\}$ is a 4-element fan containing σ, τ. Suppose $\tau = \gamma\sigma$. Since $|X| \geqq 3$, there exists $\sigma' \in X$, $\sigma' \neq \sigma,\sigma\gamma$. Then $V = \{\sigma, \sigma', \sigma\gamma, \sigma'\gamma\}$ is a 4-element fan containing σ, τ.

(2.3) THEOREM. $\sim_s$ *is an equivalence relation on* X.

Proof. Let $\sigma_1, \sigma_2, \sigma_3 \in X$ satisfy $\sigma_1 \sim_s \sigma_2$, $\sigma_2 \sim_s \sigma_3$. We wish to show $\sigma_1 \sim_s \sigma_3$. We may assume $\sigma_1, \sigma_2, \sigma_3$ are all distinct. Let V_1, V_2 be 4-element fans in X such that $\sigma_1, \sigma_2 \in V_1$, $\sigma_2, \sigma_3 \in V_2$. Pick $\gamma_i \in \mathrm{gr}(V_i)$, $\gamma_i \neq 1$. This is possible, since $|\mathrm{gr}(V_i)| = 4$. Then it is clear that $V_i \subseteq X_{\gamma_i}$, $i = 1,2$. Moreover $X_{\gamma_1} \cap X_{\gamma_2} \neq \emptyset$ since

$$\sigma_2 \in V_1 \cap V_2 \subseteq X_{\gamma_1} \cap X_{\gamma_2}.$$

Thus, by Lemma 4.6 of [**9**] (and remark 2.1, above) there exists $\gamma \in \chi(G)$, $\gamma \neq 1$ such that $X_{\gamma_1}, X_{\gamma_2} \subseteq X_\gamma$. Thus $\sigma_1, \sigma_3 \in X_\gamma$. Also $|X_\gamma| \geqq 4$ and $\mathrm{gr}(X_\gamma) \neq 1$ (since $\gamma \in \mathrm{gr}(X_\gamma)$), so by Lemma 2.2, $\sigma_1 \sim_s \sigma_3$.

As in [**9**], we refer to the equivalence classes of X with respect to $\sim_s$ as the (*connected*) *components* of X, and we say X is *connected* if it has only one component. It follows from Corollary 7.5(i) of [**11**] that every connected space is indecomposable. The converse is false (see remark 3.15). Recall (in [**10**]) a space of orderings X is called an *elementary indecomposible* space if either $\mathrm{gr}(X) \neq 1$, and $|X| \geqq 4$; or $|X| = 1$. We will refer to this type of space as an EI-*space* for short. Note by Lemma 2.2, every such space is connected and hence is indeed indecomposible. In [**9**] it is proved that for X finite, the components of X are subspaces (in fact EI-subspaces). This is not valid in general (see remark 2.9), but it is true for two special classes of spaces which we discuss now.

(2.4) THEOREM. *Suppose there exists a finite set of fans*

$$V_i = (V_i, G/\Delta_i)\ i = 1, \ldots, n$$

in X which generate X. Then X has only finitely many components $C_1, \ldots, C_s$, each C_i is an EI-*subspace of X, and $X = C_1 \oplus \ldots \oplus C_s$.*

Before beginning the proof note that this generalizes the main structure result in [**9**] (since finite spaces are certainly generated by finitely many fans).

Proof of 2.4. By decomposing two-element fans into two 1-element fans, we may assume $|V_i| \neq 2$, so each V_i is connected. Fix $\sigma_i \in V_i$. Then $V_i = \sigma_i \mathrm{gr}(V_i)$, so $\sigma_i \mathrm{gr}(V_i) \cup \mathrm{gr}(V_i)$ is a closed subgroup of $X(G)$ containing V_i. It follows that

$$\chi(G/\Delta_i) = \sigma_i \mathrm{gr}(V_i) \cup \mathrm{gr}(V_i),$$

i.e., every element of $\chi(G/\Delta_i)$ is a product of 1 or 2 elements of V_i. Since $\bigcup_{i=1}^n V_i$ generates X, it follows that $\bigcap_{i=1}^n \Delta_i = 1$, i.e.,

$$\chi(G) = \prod_{i=1}^{n} \chi(G/\Delta_i).$$

(This product is not necessarily direct.) Thus every element of $\chi(G)$

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(and hence of X) is a product of a finite number of (at most $2n$) elements of $\bigcup_{i=1}^{n} V_1$.

Now let $\sigma \in X$, and represent σ as

$$\sigma = \tau_1 \ldots \tau_t, \tau_v \in \bigcup V_i, v = 1, \ldots, t.$$

We may assume $\tau_1, \ldots, \tau_t$ are independent (taking a shorter expression if necessary) so by Lemma 3.2 of [**9**], $\sigma \sim \tau_1$. Say $\tau_1 \in V_i$. Then $\tau_1 \sim \sigma_i$ (V_i is connected), so $\sigma \sim \sigma_i$. Thus every element $\sigma \in X$ is connected to one of $\sigma_1, \ldots, \sigma_n$. This proves the first assertion.

Note since each V_i is connected, $V_i \cap C_j \neq \emptyset \Rightarrow V_i \subseteq C_j$. Denote by S_j the set $\{i|1 \leqq i \leqq n, V_i \subseteq C_j\}$. Thus $S_j \cap S_{j'} = \emptyset$ for $j \neq j'$, and $\bigcup S_j = \{1, 2, \ldots, n\}$. The claim is that C_j is generated by the set of fans V_i, $i \in S_j$. For let $\sigma \in C_j$, and represent σ as

$$\sigma = \tau_1 \ldots \tau_t, \quad \tau_v \in \bigcup_{i=1}^{n} V_i, \quad v = 1, \ldots, t,$$

with $\tau_1, \ldots, \tau_t$ independent. Consider τ_v; say $\tau_v \in V_i$. Then by Lemma 3.2 of [**9**], $\tau_v \sim \sigma$, and since $\sigma \in C_j$, it follows that $\tau_v \in C_j$, so $V_i \cap C_j \neq \emptyset$, i.e., $i \in S_j$, i.e.,

$$\tau_v \in \bigcup_{i \in S_j} V_i \text{ for } v = 1, \ldots, t.$$

This proves the claim.

Now consider a component C_j. We wish to prove it is an EI-subspace. We may assume $|C_j| \neq 1$. By replacing each V_i, $i \in S_j$ by a larger fan, if necessary, we can assume $|V_i| \geqq 4$. Since $\sigma_i \sim \sigma_{i'}$ for all $i,i' \in S_j$, $i \neq i'$, there exists a four-element fan $V_{ii'} \subseteq X$ such that $\sigma_i, \sigma_{i'} \in V_{ii'}$. Choose elements $\gamma_i \in \operatorname{gr}(V_i)$, $\gamma_{ii'} \in \operatorname{gr}(V_{ii'})$ $\gamma_i \neq 1$, $\gamma_{ii'} \neq 1$ for all $i,i' \in S_j$, $i \neq i'$, and consider the corresponding subspaces X_{γ_i}, $X_{\gamma_{ii'}}$. Since $\sigma_i \in X_{\gamma_i} \cap X_{\gamma_{ii'}}$, it follows by repeated application of Lemma 4.6 of (**9**), there exists $\gamma \in \chi(G)$, $\gamma \neq 1$ such that

$$X_{\gamma_i}, X_{\gamma_{ii'}} \subseteq X_\gamma \ \forall i,i' \in S_j \quad i \neq i'.$$

In particular (since $V_i \subseteq X_{\gamma_i}$) it follows that

$$\bigcup_{i \in S_j} V_i \subseteq X_\gamma.$$

Since the former generates C_j, this implies $C_j \subseteq X_\gamma$. Since X_γ is connected by Lemma 2.2, $C_j = X_\gamma$.

Note that $X = C_1 \oplus \ldots \oplus C_s$ follows immediately from Corollary 7.5, [**11**] (although it can also be proved independently of the representation theorem for $W(X)$ by the techniques of [**9**, Theorem 3.3]).

(2.5) *Remark*. Note if X satisfies the hypothesis of Theorem 2.4, then $\operatorname{cl}(X) < \infty$ (by the part of Theorem 1.6 which is proved). Note also that the conclusion of 2.4 implies, in particular, that either $\operatorname{gr}(X) \neq 1$, or $|X| = 1$, or X decomposes. Thus we have proved an important special case of Theorem 1.3: If X is generated by a finite set of fans, then the conclusion of 1.3 holds.

A reader mainly interested in the proof of Theorem 1.3 should proceed now directly to § 3.

(2.6) THEOREM. *Suppose X has no infinite fans $V \subseteq X$. (For example, this holds if* $\mathrm{st}(X) < \infty$, *by Theorem* 6.4 *of* [**11**].) *Then every component of X is an* EI-*subspace.*

The following characterization of fans is useful in the proof of 2.6.

(2.7) LEMMA. *A subset Y of X is a fan (and a subspace) if and only if Y is closed in X and satisfies $\alpha\beta\gamma \in Y \,\forall \alpha, \beta, \gamma \in Y$.*

Proof of 2.7. ($\Rightarrow$) Y is closed, being a subspace. Since Y is a fan and $(-1)(-1)(-1) = -1$, the second condition holds by [**11**, 4.2(ii)].

($\Leftarrow$) Fix $\alpha \in Y$. Then $YY = 1YY = \alpha^2 YY = \alpha(\alpha YY) \subseteq \alpha Y$. It follows easily from this that $Y \cup \alpha Y$ is a (closed) subgroup of $\chi(G)$, so, by Pontryagin Duality, $(Y \cup \alpha Y)^{\perp\perp} = Y \cup \alpha Y$. Since $\alpha \in Y$, $(Y \cup \alpha Y)^{\perp} = Y^{\perp}$. Thus $Y^{\perp\perp} = Y \cup \alpha Y$. Thus, if $\gamma \in Y^{\perp\perp}$, $\gamma(-1) = -1$, then, since all elements $\tau \in \alpha Y$ satisfy $\tau(-1) = 1$, $\gamma \in Y$. This shows Y is a fan, using [**11**, 4.2(ii)], (and also a subspace, since it implies in particular that $Y^{\perp\perp} \cap X \subseteq Y$).

Proof of 2.6. Let $C \subseteq X$ denote the component generated by $\sigma \in X$. We may assume $|C| \neq 1$. Then, by the proof of 2.2 C is the union of the collection of sets

$$S = \{X_\gamma | \gamma \in \chi(G), \gamma \neq 1, \sigma \in X_\gamma, |X_\gamma| \geqq 4\}.$$

By Lemma 4.6 of [**9**] this system of sets is directed, so it is enough to show S has a maximal element, i.e., we must show there do not exist elements $\gamma_i \in \chi(G)$ such that

$$(*) \quad X_{\gamma_i} \not\subseteq X_{\gamma_{i+1}}, i = 1, 2, 3, \ldots \ldots$$

Suppose such a chain does exist. Fix an element $\sigma \in X_{\gamma_1}$ and consider

$$V_n = \{\sigma, \sigma\gamma_1, \sigma\gamma_2, \sigma\gamma_1\gamma_2, \ldots\ldots, \sigma\gamma_1\gamma_2\ldots\gamma_n\}.$$

Clearly $V_n \subseteq X$ and V_n is a fan (by 2.7). We claim that $|V_n| = 2^n$. For otherwise two of the displayed elements of V_n would be equal, and cancelling σ we would obtain (after taking all the terms to one side and reindexing the γ_i's) a relation of the form

$$\gamma_1\gamma_2\ldots\gamma_s = 1.$$

Thus, $\gamma_1 = \gamma_2 \ldots \gamma_s$. Now let $\tau \in X_{\gamma_2}$. Then $\tau\gamma_1 = \tau\gamma_2 \ldots \gamma_s \in X$, i.e., $\tau \in X_{\gamma_1}$. Thus $X_{\gamma_1} = X_{\gamma_2}$, contradicting (*). Hence $|V_n| = 2^n$. Now let V denote the closure of $\bigcup_{i=1}^{\infty} V_i$. Thus V is closed in X, and

$$\alpha,\beta,\gamma \in V \Rightarrow \alpha\beta\gamma \in V$$

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(this holds for $\cup V_i$ so it holds for V by the continuity of multiplication). It follows by Lemma 2.7 that V is a fan (and a subspace of X). Since $|V| = \infty$, this contradicts the assumption, and proves the theorem.

(2.7) *Remark.* Suppose X has finite stability index and components X_i, $i \in I$. Each non-trivial fan in X is connected, and hence lies in a component. Thus, by Theorem 5.5 of [**11**], $W(X)$ consists of all $f \in C(X,\mathbf{Z})$ which satisfy

$$f|X_i \in W(X_i), \text{ and}$$

$$f(\sigma) \equiv f(\tau) (\text{mod } 2) \text{ for all } \sigma \in X_i, \tau \in X_j, i,j \in I, i \neq j.$$

Further, if $\operatorname{st}(X) \neq 1$, then by Theorem 6.4 of [**11**],

$$\operatorname{st}(X) = \max\{\operatorname{st}(X_i)|i \in I\}.$$

By Theorem 2.6, either $|X_i| = 1$, or $\operatorname{gr}(X_i) \neq 1$. In the former case

$$\operatorname{st}(X_i) = 0, \text{ and } W(X_i) \cong \mathbf{Z}.$$

In the latter case,

$$W(X_i) \cong W(X_i')[G/G_i']$$

with $X_i' =$ the residue space of X_i, and $G_i' = \operatorname{gr}(X_i)^\perp \subseteq G$ (by [**10**, remark 3.8]). Also, in this case, $\operatorname{st}(X_i) = n_i + \operatorname{st}(X_i')$, where $2^{n_i} = (G:G_i')$. (If V is a fan in X_i, its image V' in X_i' is a fan, and the inverse image of V' in X_i is a fan in X_i containing V.) Since $\operatorname{st}(X_i') < \operatorname{st}(X_i)$, this gives a rough inductive description of the Witt ring of a space of orderings with finite stability.

For the remainder of this section, let X denote the space of orderings of some formally real field K. For v a valuation on K denote by A_v (resp. U_v) the valuation ring (resp. the unit group) of (K,v). Let X_v denote the subspace of X consisting of orderings compatible with v in the sense of [**3**] and let $\bar{X}_v$ denote the order space of the residue field of (K,v). Thus

$$X_v \neq \emptyset \Leftrightarrow \bar{X}_v \neq \emptyset,$$

and X_v is a group extension of $\bar{X}_v$. Let $\sigma, \tau \in X$, $\sigma \sim_s \tau$, $\sigma \neq \tau$. Thus there exists a 4-element fan $V \subseteq X$ such that $\sigma,\tau \in V$. By [**3**, Theorem 2.7] there is a real valuation v of K such that $V \subseteq X_v$ and the image of V in $\bar{X}_v$ is a trivial fan. Since V itself is not trivial this implies $K^\bullet \neq U_v K^{\bullet 2}$, so $\operatorname{gr}(X_v) \neq 1$. Thus X_v is connected by Lemma 2.2. It follows that the (non-trivial) component generated by σ is $\cup_{v \in S_\sigma} X_v$ where S_σ denotes the set of valuations v on K satisfying

$$|X_v| \geqq 4, K^\bullet \neq K^{\bullet 2} U_v, \sigma \in X_v.$$

Note the valuations in S_σ are comparable [**12**, Theorem 7.18(1)]. Now suppose $\mathrm{st}(X) < \infty$. Then $K^{\bullet}/U_v K^{\bullet 2}$ is finite for each $v \in S_\sigma$ [**2**, (3.18)]. Choose $v_0 \in S_\sigma$ such that the index is minimal, and define a valuation w on K by

$$A_w = \cup_{v \in S_\sigma} A_v.$$

It follows that w is non-trivial, and $w \in S_\sigma$, in fact,

$$U_w K^{\bullet 2} = U_{v_0} K^{\bullet 2}.$$

By [**12**, Theorem 7.18(2)]

$$\cup_{v \in S_\sigma} X_v = X_w.$$

Thus we have proved:

(2.8) THEOREM. *Suppose X is the space of orderings of a formally real field K, $\mathrm{st}(X) < \infty$. Then every non-trivial component of X has the form X_v for some real valuation v on K satisfying*

$$U_v K^{\bullet 2} \neq K^{\bullet}, \ |X_v| \geqq 4.$$

(2.9) *Remark.* It is known, in the infinite stability case, that the components of X need not be subspaces. One such example can be obtained by taking K to be the rational function field in countably many variables over $\mathbf{Q}$ and using a construction of A. Prestel. This yields a component C which is dense in X but is not all of X. In particular, C is not closed, so is not a subspace. On the other hand, it is known (Corollary 3.18) that a closed component of X is a subspace. (The corresponding question is still open for abstract spaces of orderings.) Here are two problems that appear to be open:

(i) Suppose a non-trivial component of X is closed (and hence a subspace). Then is this subspace an EI-subspace?

(ii) Supposing it is an EI-subspace, does this mean it has the form X_v for some real valuation v satisfying $U_v K^{\bullet 2} \neq K^{\bullet}$? (We have seen in 2.8 that the answer to both questions is "yes" if X has finite stability.)

3. P-structures. Suppose, for the moment, X is the space of orderings of some formally real field K. Denote by M the set of "real" places K, i.e., places $\alpha: K \to \mathbf{R} \cup \{\infty\}$. Then M induces a partition of X: $X = \cup_{\alpha \in M} X(\alpha)$, which satisfies the following properties:

P_1. Each $X(\alpha)$, $\alpha \in M$, is a fan;

P_2. Each fan $V \subseteq X$ intersects at most 2 of the sets $X(\alpha)$, $\alpha \in M$;

P_3. The induced (quotient) topology on M is Hausdorff;

P_4. For each non-empty closed $C \subseteq M$, the set $\cup_{\alpha \in C} X(\alpha)$ is a subspace of X.

(These results are fairly well known. All are collected in [**6**, § 2]. Specifically P_1 is 2.2, P_3 is 2.1, P_4 is 2.8, and P_2 follows from the first claim of 2.3.)

For most of this section we concentrate on the following partial abstraction of this situation.

(3.1) *Definition*. Let X be an arbitrary space of orderings. A partition $X = \bigcup_{\alpha \in M} X(\alpha)$ will be called a *P-structure* on X if it satisfies P_1 and P_2.

P-structures satisfying P_3 and P_4 are discussed briefly at the end of this section. The immediate goal is to prove the following two theorems.

(3.2) THEOREM. *Every space of orderings admits a P-structure.*

(3.3) THEOREM. *Let $X = \bigcup_{\alpha \in M} X(\alpha)$ be a given P-structure on X. Then*

$$\mathrm{cl}(X) < \infty \Leftrightarrow |M| < \infty.$$

In fact, if either is finite, then

$$|M| \leqq \mathrm{cl}(X) \leqq 2|M|.$$

This will complete the proof of Theorem 1.3. For suppose $\mathrm{cl}(X) < \infty$. By Theorem 3.2, there exists a P-structure $X = \bigcup_{\alpha \in M} X(\alpha)$ on X. By Theorem 3.3, $|M| < \infty$. Thus X is generated by the finite set of fans $X(\alpha)$, $\alpha \in M$, so by Remark 2.5, either $\mathrm{gr}(X) \neq 1$, or $|X| = 1$, or X decomposes.

Before proving 3.2 and 3.3, several preliminary results are required.

(3.4) *Definition*. (i) Suppose $X = \bigcup_{\alpha \in M} X(\alpha)$ is a P-structure on X, and Y is a subspace of X. The *induced* P-structure on Y is

$$Y = \bigcup_{\alpha \in N} Y(\alpha) \text{ where}$$
$$N = \{\alpha \in M | X(\alpha) \cap Y \neq \emptyset\} \text{ and } Y(\alpha) = X(\alpha) \cap Y \ \forall\, \alpha \in N.$$

(ii) Suppose $X = X_1 \oplus \ldots \oplus X_n$ and that $X_i = \bigcup_{\alpha \in M_i} X_i(\alpha)$ is a P-structure on X_i, $1 \leqq i \leqq n$. The *induced* P-structure on X is

$$X = \bigcup_{\alpha \in M} X(\alpha) \text{ where}$$

$$M = \bigcup_{i=1}^{n} M_i \quad \text{and} \quad X(\alpha) = X_i(\alpha) \text{ if } \alpha \in M_i.$$

(iii) Suppose $X' = \bigcup_{\alpha \in M} X'(\alpha)$ is a P-structure on the residue space of X' of X. The *induced* P-structure on X is $X = \bigcup_{\alpha \in M} X(\alpha)$ where $X(\alpha)$ is the inverse image of $X'(\alpha)$ under the natural projection.

It is easily verified that the induced P-structures in (i), (ii), and (iii) above, are indeed P-structures.

(3.5) *Definition*. Suppose X is generated by a finite set of fans. The *canonical* P-structure on X is defined by

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(i) if $|X| = 1$, it is the only possible thing;

(ii) if $\mathrm{gr}(X) \neq 1$, it is that induced by the canonical P-structure on the residue space of X;

(iii) otherwise it is that induced by the canonical P-structures on the components of X.

Note that, by induction on $\mathrm{cl}(X)$, (i), (ii), and (iii) do indeed define a canonical P-structure on X (refer to Theorem 2.4 and its proof, Remark 2.5, and Theorem 1.7). If $X = \bigcup_{\alpha \in M} X(\alpha)$ is the canonical P-structure of such a space X, we will find it useful, in the following, to refer to the fans $X(\alpha)$, $\alpha \in M$ as *X-places*. To clarify a possible misconception, one should note that (unless $|X| = 1$) the canonical P-structure on such a space X is not the only P-structure on X.

(3.6) LEMMA. *Suppose $V \subseteq Y \subseteq X$ where Y, X are each finitely generated by fans, V is a fan which is not a Y-place, and V is maximal in X (in the sense that if $W \subseteq X$ is a fan, $V \subseteq W$, then $V = W$). Then the canonical P-structures on X and Y each induce the same P-structure on the subspace V.*

Proof. We claim that $\mathrm{gr}(X) \subseteq \mathrm{gr}(Y) \subseteq \mathrm{gr}(V)$. For suppose $\gamma \in \mathrm{gr}(X)$. Then $\gamma V \subseteq \gamma X = X$, so $W = V \cup \gamma V \subseteq X$. Since W is a fan, it follows by the maximality of V that $W = V$, i.e., $\gamma V \subseteq V$ i.e., $\gamma V = V$, i.e., $\gamma \in \mathrm{gr}(V)$. Thus $\mathrm{gr}(X) \subseteq \mathrm{gr}(V)$. A similar argument shows $\mathrm{gr}(Y) \subseteq \mathrm{gr}(V)$. With γ as above note $\gamma Y \subseteq \gamma X \subseteq X$. Also, since $\gamma \in \mathrm{gr}(V) \subseteq [V] \subseteq [Y]$, it follows that $\gamma Y \subseteq [Y][Y] = [Y]$. (Recall: $[Y]$ denotes the closed subgroup of $\chi(G)$ generated by Y.) Thus $\gamma Y \subseteq [Y] \cap X = Y$ (Y is a subspace of X). Thus $\gamma Y = Y$, so $\mathrm{gr}(X) \subseteq \mathrm{gr}(Y)$. This proves the claim.

The lemma is proved by induction on $\mathrm{cl}(X)$. Note that by replacing X by its residue space and Y, V by their images in the residue space of X, we reduce to the case $\mathrm{gr}(X) = 1$. Thus by Remark 2.5 either $|X| = 1$ or X has two or more components. But $|X| \neq 1$ ($|X| = 1 \Rightarrow V = Y = X \Rightarrow V$ is a Y-place). Thus X has 2 or more components. Suppose $\mathrm{gr}(Y) \neq 1$. Then $|Y| \neq 2$ (for if $|Y| = 2$, then Y is a fan, so $V = Y$, i.e., V is a Y-place). Thus, by Lemma 2.2, Y is connected. Replacing X by the component of X containing Y we are done, by induction on the chain length. Thus, we may assume $\mathrm{gr}(Y) = 1$. Again $|Y| \neq 1$ (V is not a Y-place), so by Remark 2.5, Y also has two or more components. Now look at V. If $|V| \geqq 4$, then by Lemma 2.2, V is connected, so by replacing Y, X by the components of Y and X respectively that contain V, we are done by induction. Thus we may assume $|V| \leqq 2$. But $|V| \neq 1$ ($|X| \neq 1$, so a maximal fan in X has at least two elements). Thus $|V| = 2$. Being a maximal fan, V cannot lie wholly in a component of X or of Y (since a maximal fan lying in a non-trivial component has at least 4-elements).

Say $V = \{\sigma, \tau\}$. It follows that the induced P-structure on V is $\{\sigma\} \cup \{\tau\}$ in both cases. This completes the proof.

(3.7) *Terminology.* (i) We say X is *P-finite* if X is generated by a finite set of fans.

(ii) Let V be a fan in X. We say a P-finite subspace $Y \subseteq X$ *partitions V stably* if $V \subseteq Y$, and $\forall$ P-finite spaces Z, $Y \subseteq Z \subseteq X$ implies the canonical P-structures on Y and Z each induce the same P-structure on V.

(3.8) LEMMA. *Let X be an arbitrary space of orderings, V an arbitrary fan in X. Then there exists a P-finite subspace $Y \subseteq X$ which partitions V stably.*

Proof. Let W be a maximal fan in X subject to $V \subseteq W$. (The existence of such W is by Zorn's Lemma, and an argument similar to that in the proof of Theorem 2.6.) If W is a Z-place for every P-finite space Z, $W \subseteq Z \subseteq X$, take $Y = W$. Otherwise there exists a P-finite space Y, $W \subseteq Y \subseteq X$ such that W is not a Y-place. By Lemma 3.7, Y partitions W (and hence V) stably.

We now give a proof of Theorem 3.2. Define a relation $\equiv$ on X as follows: For $\sigma,\tau \in X$, let Y be any P-finite subspace of X which partitions the fan $\{\sigma, \tau\}$ stably. Write $\sigma \equiv \tau$ if and only if σ, τ lie in the same Y-place. $\equiv$ is a well-defined equivalence relation on X. This follows from Lemma 3.8 and the fact that the subspace of X generated by a finite number of P-finite subspaces of X is again P-finite.

The claim is that the resulting partition of X is a P-structure. Let $V \subseteq X$ be a fan, and let Y be a P-finite subspace of X which partitions V stably. Let $\sigma_1,\sigma_2 \in V$, $\sigma_1 \not\equiv \sigma_2$, and let $\sigma \in V$. Since V is a fan in Y, V intersects at most two Y-places, so either $\sigma \equiv \sigma_1$, or $\sigma \equiv \sigma_2$. This shows P_2.

It remains to verify P_1. Let T be an equivalence class with respect to $\equiv$, and let σ_1, σ_2, $\sigma_3 \in T$. Thus $\sigma_1 \equiv \sigma_2$, $\sigma_2 \equiv \sigma_3$, so there exists a P-finite space $Y \subseteq X$ which partitions $\{\sigma_1, \sigma_2\}$ and $\{\sigma_2, \sigma_3\}$ stably. Thus σ_1, σ_2, σ_3 all lie in the same Y-place. Since Y-places are fans, it follows that $\sigma_1\sigma_2\sigma_3$ lies in this Y-place so, in particular, $\sigma_1\sigma_2\sigma_3 \in X$. Now let Y' be a P-finite space partitioning the fan $\{\sigma_1, \sigma_2, \sigma_3, \sigma_1\sigma_2\sigma_3\}$ stably. Since $\sigma_1 \equiv \sigma_2 \equiv \sigma_3$, $\sigma_1,\sigma_2,\sigma_3$ all lie in the same Y'-place. Thus $\sigma_1\sigma_2\sigma_3$ lies in this Y'-place, so

$$\sigma_1\sigma_2\sigma_3 \equiv \sigma_1 \equiv \sigma_2 \equiv \sigma_3,$$

i.e., $\sigma_1\sigma_2\sigma_3 \in T$. This proves

$$\sigma_1,\sigma_2,\sigma_3 \in T \Rightarrow \sigma_1\sigma_2\sigma_3 \in T.$$

By continuity of multiplication this will also hold for $\bar{T}$, the closure of T, so $\bar{T}$ is a fan (by 2.7). Let Y'' be a P-finite subspace of X partitioning $\bar{T}$ stably. Thus T is the intersection of some Y''-place with $\bar{T}$. Since $\bar{T}$ is closed and each Y''-place is closed (in Y'' and hence in X), it follows that T is closed. Thus $T = \bar{T}$ is indeed a fan. This proves P_1.

(3.8) LEMMA. *Let $X = \bigcup_{\alpha \in M} X(\alpha)$ be a P-structure on X, let S be any (non-empty) finite subset of M, and let $Y = \bigcup_{\alpha \in S} X(\alpha)$. Then Y is a subspace of X.*

Proof. If $\sigma \in [Y]$, it follows (as in the proof of the first claim of Theorem 2.4) that there exist $\sigma_1, \ldots, \sigma_s \in Y$ such that

$$\sigma = \sigma_1 \sigma_2 \ldots \sigma_s.$$

Now let $\sigma_1, \sigma_2, \sigma_3 \in Y$ and suppose $\sigma = \sigma_1\sigma_2\sigma_3 \in X$. If there is $\alpha \in S$ such that $\sigma_1, \sigma_2, \sigma_3 \in X(\alpha)$, then $\sigma \in X(\alpha)$ by P_1, so $\sigma \in Y$. Otherwise the fan $V = \{\sigma_1, \sigma_2, \sigma_3, \sigma\}$ intersects $X(\alpha)$ and $X(\beta)$ for some $\alpha, \beta \in S$, $\alpha \neq \beta$. By P_2,

$$V \subseteq X(\alpha) \cup X(\beta) \subseteq Y,$$

so $\sigma \in Y$. Thus Y satisfies the following closure property:

(*) $\quad \sigma_1, \sigma_2, \sigma_3 \in Y,\ \sigma_1\sigma_2\sigma_3 \in X \Rightarrow \sigma_1\sigma_2\sigma_3 \in Y.$

That Y is a subspace now follows immediately from the following lemma.

(3.9) LEMMA. *Let Y be any (non-empty) subset of X satisfying (*) and suppose every element of $[Y]$ is a finite product of elements of Y. Then Y is a subspace of X.*

Proof. It is enough to show $[Y] \cap X = Y$. Let $\sigma \in [Y] \cap X$ and express σ as

$$\sigma = \sigma_1 \ldots \sigma_s;\ \sigma_1, \ldots, \sigma_s \in Y.$$

By taking s minimal, we can assume $\sigma_1, \ldots, \sigma_s$ are independent. Note s is odd. If $s = 1$, σ is clearly in Y. If $s = 3$, σ is in Y by (*). Suppose $s \geqq 5$. Then by [**9**, Theorem 3.1] (reindexing if necessary) $\exists\, t$, $3 \leqq t < s$, such that $\tau = \sigma_1 \ldots \sigma_t \in X$. By induction on s, $\tau \in Y$ and $\sigma = \tau\sigma_{t+1} \ldots \sigma_s \in Y$. Thus, in any case, $\sigma \in Y$.

(3.10) LEMMA. *If X is not a fan, then any P-structure on X is induced by a P-structure on the residue space (X', G') of X.*

Proof. If $\mathrm{gr}(X) = 1$, then $X' = X$, so the result is clear. Let $X = \bigcup_{\alpha \in M} X(\alpha)$ be a given P-structure on X. For $\alpha \in M$, denote by $X(\alpha)'$ the image of $X(\alpha)$ under the natural projection. The only thing not clear is that the fans $X(\alpha)'$, $\alpha \in M$ are disjoint (for if they are disjoint, then $X' = \bigcup_{\alpha \in M} X(\alpha)'$ is a P-structure on X' inducing the given P-structure). Suppose there exist $\alpha, \beta \in M$, $\alpha \neq \beta$ such that $X(\alpha)' \cap X(\beta)' \neq \emptyset$. Let

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$\sigma \in X(\alpha)$ be such that its image is in $X(\alpha)' \cap X(\beta)$ and let $\delta \in \chi(G/G')$ be such that $\sigma\delta \in X(\beta)$. Suppose $M \neq \{\alpha,\beta\}$. Let $\gamma \in M$, $\gamma \neq \alpha,\beta$, and let $\tau \in X(\gamma)$. Then $W = \{\sigma,\sigma\delta,\tau,\tau\delta\}$ is a fan in X intersecting $X(\alpha)$, $X(\beta)$ and $X(\gamma)$. This contradicts P_2 and proves $M = \{\alpha,\beta\}$. Thus

$$X = X(\alpha) \cup X(\beta) \text{ and } X' = X(\alpha)' \cup X(\beta)'.$$

Now take $\tau \in X(\alpha)$ and W as above. Then $\tau\delta \in X(\beta)$, since

$$\tau\delta \in X(\alpha) \Rightarrow \sigma\delta = (\sigma)(\tau)(\tau\delta) \in X(\alpha).$$

This proves $X(\alpha)' \subseteq X(\beta)'$, i.e., that $X' = X(\beta)'$ is a fan. But then X is a fan, since it is a group extension of a fan. This contradicts the hypothesis and proves the lemma.

We are now in a position to give a proof of Theorem 3.3. Suppose $|M| < \infty$. Then $\mathrm{cl}(X) \leqq 2|M|$ by Remark 1.2 and Theorem 1.7 (i). Still assuming $|M| < \infty$, we now prove, by induction on $\mathrm{cl}(X)$, that $|M| \leqq \mathrm{cl}(X)$. This is clear if $|X| = 1$. Suppose $\mathrm{gr}(X) \neq 1$. If X is a fan, $\mathrm{cl}(X) = 2 \geqq |M|$. If X is not a fan then, by Lemma 3.10, M is induced by a P-structure M' on the residue space X'. Then $|M| = |M'|$ and, by Theorem 1.7 (iii), $\mathrm{cl}(X) = \mathrm{cl}(X')$. Thus, replacing X by X', we may assume $\mathrm{gr}(X) = 1$. Thus by Remarks 2.5, X decomposes, say $X = X_1 \oplus X_2$. Let M_i denote the P-structure on X_i induced by M (see Definition 3.4 (i)). Clearly

$$|M_1| + |M_2| \geqq |M|.$$

Also

$$\mathrm{cl}(X_1) + \mathrm{cl}(X_2) = \mathrm{cl}(X),$$

by Theorem 1.7 (ii). Since X_i has smaller chain length than X, $i = 1,2$, the result follows by induction.

Now suppose only that $\mathrm{cl}(X) < \infty$. Let $S \subseteq M$ be a finite subset. By Lemma 3.8, $Y = \cup_{\alpha \in S} X(\alpha)$ is a subspace of X. By Theorem 1.8, $\mathrm{cl}(Y) \leqq \mathrm{cl}(X)$. But by applying what we have just proved to the space Y with P-structure $Y = \cup_{\alpha \in S} X(\alpha)$, we have $|S| \leqq \mathrm{cl}(Y)$. Thus $|S| \leqq \mathrm{cl}(X)$. It follows, since $S \subseteq M$ is an arbitrary finite set, that $|M| < \infty$.

For the rest of this section, fix a P-structure $X = \cup_{\alpha \in M} X(\alpha)$ on X. For $\alpha \in M$, let $\Delta(\alpha) = X(\alpha)^\perp$. Thus, as a space of orderings,

$$X(\alpha) = (X(\alpha), G/\Delta(\alpha)).$$

Now suppose $\alpha,\beta \in M$, $\alpha \neq \beta$. By Lemma 3.8,

$$X(\alpha) \cup X(\beta) = (X(\alpha) \cup X(\beta), G/\Delta(\alpha) \cap \Delta(\beta))$$

is a subspace of X. Denote by $G_{\alpha\beta}$ the group $G/\Delta(\alpha)\Delta(\beta)$. For $\gamma \in \chi(G_{\alpha\beta}) \subseteq \chi(G)$, we have $\gamma(-1) = 1$ (for $X(\alpha)$, $X(\beta)$ are fans, so if

$\gamma(-1) = -1$, then $\gamma \in X(\alpha) \cap X(\beta)$, a contradiction). It follows that $-1 \in \Delta(\alpha)\Delta(\beta)$.

Theorem 1.6 together with Theorem 3.3 completely classify spaces satisfying $|M| < \infty$. One may also consider spaces satisfying

$$|X(\alpha)| \leqq 2 \; \forall \, \alpha \in M, \text{ and } G_{\alpha\beta} = 1 \; \forall \alpha, \beta \in M, \alpha \neq \beta.$$

These are easily seen to be just the spaces satisfying $\mathrm{st}(X) \leqq 1$. Such spaces are well understood. There is a larger class of spaces including both the above types. This is classified in the following.

(3.11) THEOREM. *The following are equivalent:*

(i) *There exists a finite* $S \subseteq M$ *satisfying* $|X(\alpha)| \leqq 2 \; \forall \, \alpha \notin S$, *and* $G_{\alpha\beta} = 1 \; \forall \, \alpha, \beta, \alpha \neq \beta, \alpha \notin S$.

(ii) *The subspace of* X *generated by the non-trivial fans* [**11**, 7.3] *has finite chain length.*

Proof. (i) $\Rightarrow$ (ii). Define $Y = \bigcup_{\alpha \in S} X(\alpha)$. Y is a subspace of X by Lemma 3.8. It is finitely generated by fans so has finite chain length. Thus it suffices to show that each four-element fan $V \subseteq X$ lies in Y. Let $V = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ be a 4-element fan in X. Suppose $V \cap X(\alpha) \neq \emptyset$. If $V \subseteq X(\alpha)$, then $|X(\alpha)| \geqq 4$ so $\alpha \in S$ and $V \subseteq Y$. Otherwise V intersects some $X(\beta)$, $\beta \neq \alpha$, and $V \subseteq X(\alpha) \cup X(\beta)$. Reindexing the elements of V, we may assume $\sigma_1, \sigma_2 \in X(\alpha)$, $\sigma_3, \sigma_4 \in X(\beta)$. (For if $\sigma_1, \sigma_2, \sigma_3 \in X(\alpha)$, then $\sigma_4 = \sigma_1\sigma_2\sigma_3 \in X(\alpha)$, i.e., $V \subseteq X(\alpha)$. Similarly, if $\sigma_1, \sigma_2, \sigma_3 \in X(\beta)$, then $V \subseteq X(\beta)$.) Thus the non-trivial character $\sigma_1\sigma_2 = \sigma_3\sigma_4$ is trivial on $\Delta(\alpha)\Delta(\beta)$, so $G_{\alpha\beta} \neq 1$. Thus $\alpha, \beta \in S$, so $V \subseteq Y$.

(ii) $\Rightarrow$ (i). Let Y denote the subspace of X generated by the nontrivial fans, and suppose $\mathrm{cl}\,(Y) < \infty$. Thus, all 4-element fans of X lie in Y, and the induced P-structure in Y is finite, i.e., $S := \{\alpha \in M | X(\alpha) \cap Y \neq \emptyset\}$ is finite. Suppose $\alpha \notin S$. Then $X(\alpha) \nsubseteq Y$, so $|X(\alpha)| \geqq 2$. Suppose also $\beta \in M$, $\beta \neq \alpha$. If $G_{\alpha\beta} \neq 1$, then there is a 4-element fan $\{\sigma, \tau, \sigma\gamma, \tau\gamma\}$ not in Y obtained by picking any σ, τ, γ subject to $\gamma \in \chi(G_{\alpha\beta})$, $\gamma \neq 1$, $\sigma \in X(\alpha)$, $\tau \in X(\beta)$. Thus $G_{\alpha\beta} = 1$.

We now present an abstract version of the "exactness" property of fields conjectured in [**5**], and proved in [**1**]. Denote by $\bar{a}$ the canonical image in $G_{\alpha\beta}$ of $a \in G$. Also denote by $\phi_{\alpha\beta}$ the canonical ring homomorphism

$$\phi_{\alpha\beta} : W(X(\alpha)) \rightarrow \mathbf{Z}/2\mathbf{Z}[G_{\alpha\beta}]$$

given by $\phi_{\alpha\beta}\langle a_1, \ldots, a_n\rangle = \bar{a}_1 + \ldots + \bar{a}_n$. It may not be apparent at first glance that $\phi_{\alpha\beta}$ is well-defined. However, since $-1 \in \Delta(\alpha)\Delta(\beta)$ and $W(X(\alpha))$ is an integral group ring [**10**, Remark 3.8], this is easily verified.

(3.12) THEOREM. *Let* $g \in C(X, \mathbf{Z})$. *Then* $g \in W(X)$ *if and only if*

(i) $g|X(\alpha) \in W(X(\alpha)) \; \forall \, \alpha \in M$ *and*

(ii) $\phi_{\alpha\beta}(g|X(\alpha)) = \phi_{\beta\alpha}(g|X(\beta)) \; \forall \, \alpha, \beta \in M, \alpha \neq \beta$.

Proof. The necessity of (i) and (ii) is clear. Conversely, assume (i) and (ii). By [**11**, Theorem 5.3] it suffices to show g is represented over each fan $V \subseteq X$. If $V \subseteq X(\alpha)$, this is true by (i). Otherwise, by P_2, there exist $\alpha, \beta \in M$, $\alpha \neq \beta$ such that $V \subseteq X(\alpha) \cup X(\beta)$. In this case, it is enough to show $g|Y \in W(Y)$ where Y denotes the space $X(\alpha) \cup X(\beta)$. By (i) there exist forms f_1, f_2 representing g over $X(\alpha)$ and $X(\beta)$ respectively. By collecting entries having the same value in $G_{\alpha\beta}$ we can write f_i as

$$f_i \sim \sum_{p \in G_{\alpha\beta}} f_{ip}$$

where, for each p,

$$f_{ip} \cong \langle a_{i1p}, a_{i2p}, \dots \rangle$$

with $\bar{a}_{ijp} = p$ (and f_{ip} is the zero form for all but a finite number of p). By (ii),

$$\phi_{\alpha\beta}(f_1) = \phi_{\beta\alpha}(f_2).$$

This implies that f_{1p} and f_{2p} have the same dimension modulo 2, so by modifying by hyperbolic planes, we may assume

$$\dim f_{1p} = n_p = \dim f_{2p} \ \forall\ p \in G_{\alpha\beta}.$$

For fixed j,p, $\bar{a}_{1jp} = p = \bar{a}_{2jp}$ so $a_{1jp}a_{2jp} \in \Delta(\alpha)\Delta(\beta)$. Thus there exist $c_{1jp} \in \Delta(\alpha)$, $c_{2jp} \in \Delta(\beta)$ such that

$$a_{1jp}a_{2jp} = c_{1jp}c_{2jp}.$$

Define $a_{jp} = a_{1jp}c_{1jp} = a_{2jp}c_{2jp}$, and define f_p by

$$f_p \cong \langle a_{1p}, \dots, a_{n_p p}\rangle.$$

Then clearly $f = \sum_{p \in G_{\alpha\beta}} f_p$ is a form representing g over Y.

For the rest of the section we consider P-structures satisfying P_3 and P_4 (see the first paragraph of § 3).

(3.13) PROPOSITION. *The following are equivalent:*
(i) *M satisfies P_3;*
(ii) *$\bigcup_{a \in \Delta(\alpha)} X(\alpha)$ is open in X $\forall$ $a \in G$;*
(iii) *If M is topologized by declaring the sets*

$$M(a) = \{\alpha \in M | a \in \Delta(\alpha)\},\ a \in G,$$

to be a subbasis for open sets, then the natural map $\lambda: X \to M$ is continuous.

Proof. (i) $\Rightarrow$ (ii). Let $a \in G$. $X(-a)$ is closed in X, and hence compact. Thus $\lambda(X(-a))$ is compact and hence is closed in M, i.e., $\lambda^{-1}(\lambda(X(-a)))$ is closed in X. The result follows by noticing that $\bigcup_{a \in \Delta(\alpha)} X(\alpha)$ is just the compliment of $\lambda^{-1}(\lambda(X(-a)))$.

(ii) $\Rightarrow$ (iii). This is clear from $\lambda^{-1}(M(a)) = \bigcup_{a \in \Delta(\alpha)} X(\alpha)$.

(iii) $\Rightarrow$ (i). Let $\alpha, \beta \in M$, $\alpha \neq \beta$. Since $-1 \in \Delta(\alpha)\Delta(\beta)$, there exists

$a \in G$ such that $a \in \Delta(\alpha)$, $-a \in \Delta(\beta)$. Thus $\alpha \in M(a)$, $\beta \in M(-a)$. Since $M(a) \cap M(-a) = \emptyset$, this proves the topology of M generated by this subbasis is Hausdorff. Thus $C \subseteq M$ is closed in this topology if and only if $\lambda^{-1}(C)$ is closed in X, i.e., this topology is the quotient topology.

(3.14) PROPOSITION. *The following are equivalent:*
(i) *M satisfies P_4;*
(ii) *$\forall$ closed disjoint C_1, $C_2 \subseteq M$, there exists $a \in G$ such that $C_1 \subseteq M(a)$, $C_2 \subseteq M(-a)$;*
(iii) *$\forall$ closed $C \subseteq M$ and $\forall\, \alpha \in M$, $\alpha \notin C$, there exists $a \in G$ such that $C \subseteq M(a)$, $\alpha \in M(-a)$. (Here, the topology on M is the quotient topology.)*

Proof. (i) $\Rightarrow$ (ii). Let $C = C_1 \cup C_2$. Then C is closed, and

$$\cup_{\alpha \in C} X(\alpha) = (\cup_{\alpha \in C_1} X(\alpha)) \cup (\cup_{\alpha \in C_2} X(\alpha)).$$

The sets $\cup_{\alpha \in C_i} X(\alpha)$ $(i = 1,2)$ are closed and disjoint in the space $\cup_{\alpha \in C} X(\alpha)$. Thus the function

$$f : \cup_{\alpha \in C} X(\alpha) \rightarrow \{1, -1\}$$

defined by

$$f = 1 \text{ on } \cup_{\alpha \in C_1} X(\alpha) \text{ and } f = -1 \text{ on } \cup_{\alpha \in C_2} X(\alpha)$$

is continuous. It is easily verified that f satisfies the fan condition of [**11**, Theorem 7.2], i.e., if V is any 4-element fan in X, then

$$\sum_{\sigma \in V} f(\sigma) \equiv 0 \pmod 4.$$

It follows that there exists $a \in G$ representing f. Then $C_1 \subseteq X(a)$, $C_2 \subseteq X(-a)$.

(ii) $\Rightarrow$ (iii) is obvious, since points are closed in M.

(iii) $\Rightarrow$ (i). Let $C \subseteq M$ be closed, and let $Y = (Y, G/\Delta)$ denote the subspace of X generated by $\cup_{\beta \in C} X(\beta)$. If $Y \neq \cup_{\beta \in C} X(\beta)$, then the induced P-decomposition of Y has the form

$$Y = (\cup_{\beta \in C} X(\beta)) \cup (Y \cap X(\alpha)) \cup \ldots, \text{ with}$$
$$Y \cap X(\alpha) \neq \emptyset.$$

By (iii), there exists $a \in G$ such that $C \subseteq M(a)$, $\alpha \in M(-a)$. From $C \subseteq M(a)$, it follows that

$$a \in \cap_{\beta \in C} \Delta(\beta).$$

Since $\cup_{\beta \in C} X(\beta)$ generates Y, this is just Δ, i.e., $a \in \Delta$. But $\exists\, \sigma \in Y \cap X(\alpha)$, and (since $\alpha \in M(-a)$) it follows that $\sigma(a) = -1$. This is a contradiction. Thus $Y = \cup_{\beta \in C} X(\beta)$.

(3.15) *Remark.* Since, in any case, points are closed in M, and M satisfies 3.8, it follows that any finite P-structure does satisfy P_3 and P_4.

On the other hand P-structures exist which violate P_3. In fact, if X is the space of orderings of the rational function field $K = \mathbf{Q}(t)$, one can verify that its canonical P-structure (i.e., the one used in the proof of Theorem 3.2) violates P_3. (This is also an example of an indecomposible space which is not connected.) It is not known if P-structures exist violating P_4. Note, by the proof of Lemma 3.8, that if $S \subseteq M$, then $Y = \bigcup_{\alpha \in S} X(\alpha)$ satisfies condition (*). It is conjectured that a closed subset $Y \subseteq X$ satisfying this closure condition is automatically a subspace. If true, this would imply P_4 holds for any P-structure.

The following theorem is an interesting application of the isotropy theorem (Theorem 1.4). Note its relation with the conjecture mentioned in the above remark.

(3.16) THEOREM. *A non-empty subset $Y \subseteq X$ is a subspace if and only if it is clopen in the subspace it generates and satisfies:*

(*) *If $\sigma_1,\sigma_2,\sigma_3 \in Y$, and $\sigma_1\sigma_2\sigma_3 \in X$, then $\sigma_1\sigma_2\sigma_3 \in Y$.*

Proof. To prove the non-trivial implication, we assume that Y satisfies (*) and (replacing X by the subspace generated by Y if necessary) that Y is clopen in X. Thus, if $g: X \rightarrow \mathbf{Z}$ denotes the characteristic function of Y, then $g \in C(X, \mathbf{Z})$, so by [**11**, Lemma 5.4] there exists $n \geqq 1$ such that $2^n g \in W(X)$. Let $f = \langle a_1, \ldots, a_l \rangle$ be an anisotropic form representing $2^n g$. Thus $\sigma f = 2^n$, if $\sigma \in Y$, and $\sigma f = 0$, if $\sigma \notin Y$. It is enough to show $l = 2^n$, for then $Y = X(a_1, \ldots, a_l)$, a subspace of X. By Theorem 1.4 there exists a finite subspace Z of X over which f is anisotropic. $Y \cap Z$ satisfies (*), so by Lemma 3.9 (applied to the space Z and the subset $Y \cap Z$) $Y \cap Z$ is a subspace of Z. Thus there exist $b_1, \ldots, b_k \in G$ such that

$$Y \cap Z = Z(b_1, \ldots, b_k).$$

If $k > n$, then

$$p: = \langle 1,b_1 \rangle \otimes \ldots \otimes \langle 1,b_k \rangle \cong 2^{k-n} f \text{ (over } Z),$$

so p' represents 1 (over Z), and by [**11**, Lemma 6.3] we can reduce k. Thus we may assume $k \leqq n$, so $f \cong 2^{n-k} p$ (over Z). Thus, comparing dimensions, $l = 2^n$.

It was remarked in 2.9 that if X is the space of orderings of a formally real field, then every closed component of X is a subspace. We are now in a position (after the following lemma) to prove this for any space of orderings which admits a P-structure satisfying P_4.

(3.17) LEMMA. *Let C be a component of X, $|C| \neq 1$, and let $\alpha \in M$. Then $X(\alpha) \cap C \neq \emptyset \Rightarrow X(\alpha) \subseteq C$.*

Proof. If $|X(\alpha)| \neq 2$, $X(\alpha)$ is connected (Lemma 2.2), so $X(\alpha) \subseteq C$. Suppose $|X(\alpha)| = 2$, let $\sigma \in X(\alpha) \cap C$, let $\tau \in C$, $\tau \neq \sigma$, and let V be a 4-element fan $\sigma, \tau \in V$. Since V is connected, $V \subseteq C$. Since $|X(\alpha)| = 2$, $V \nsubseteq X(\alpha)$, so there exists β, $\beta \neq \alpha$, $V \subseteq X(\alpha) \cup X(\beta)$. Since $V \cap X(\alpha)$, $V \cap X(\beta)$ are both (non-empty) fans $|V \cap X(\alpha)| = 2$, so

$$X(\alpha) = X(\alpha) \cap V \subseteq V \subseteq C.$$

(3.18) COROLLARY. *If X admits a P-structure M satisfying P_4, then each closed component of X is a subspace.*

Proof. Let C be a closed component of X. The result is clear if $|C| = 1$. Otherwise, by Lemma 3.17, $C = \lambda^{-1}(\lambda(C))$. This is a subspace by P_4.

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