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Abstract

Let G be a semi-simple algebraic group over an algebraically closed field k, whose characteristic is positive and does not divide the order of the Weyl group of G, and let $\breve{G}$ be its Langlands dual group over k. Let C be a smooth projective curve over k of genus at least two. Denote by Bun_G the moduli stack of G-bundles on C and $\mathrm{LocSys}_{\breve{G}}$ the moduli stack of $\breve{G}$-local systems on C. Let D_{Bun_G} be the sheaf of crystalline differential operators on Bun_G. In this paper we construct an equivalence between the bounded derived category $D^b(\mathrm{QCoh}(\mathrm{LocSys}^0_{\breve{G}}))$ of quasi-coherent sheaves on some open subset $\mathrm{LocSys}^0_{\breve{G}} \subset \mathrm{LocSys}_{\breve{G}}$ and bounded derived category $D^b(D^0_{\mathrm{Bun}_G}\text{-mod})$ of modules over some localization $D^0_{\mathrm{Bun}_G}$ of D_{Bun_G}. This generalizes the work of Bezrukavnikov and Braverman in the GL_n case.

Contents

1. Introduction

1.1 Geometric Langlands conjecture in prime characteristic

Let G be a reductive algebraic group over $\mathbb{C}$ and let $\breve{G}$ be its Langlands dual group. Let C be a smooth projective curve over $\mathbb{C}$. Let Bun_G be the stack of G-bundles on C and $\mathrm{LocSys}_{\breve{G}}$ be the stack of de Rham $\breve{G}$-local systems on C. The geometric Langlands conjecture (GLC), as proposed by Beilinson and Drinfeld, is a conjectural equivalence between certain appropriately defined category of quasi-coherent sheaves on $\mathrm{LocSys}_{\breve{G}}$ and certain appropriately defined category of $\mathcal{D}$-modules on Bun_G. A precise formulation of this conjecture (over $\mathbb{C}$) can be found in the recent work of Arinkin and Gaitsgory [AG12, Gai13].

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The geometric Langlands duality has a classical limit which amounts to the duality of Hitchin fibrations. The classical duality is established 'generically' by Donagi and Pantev in [DP12] over $\mathbb{C}$.

In this paper, we establish a 'generic' characteristic p version of the geometric Langlands conjecture. Namely, let G be a semi-simple algebraic group over an algebraically closed field k of characteristic p that does not divide the order of the Weyl group of G, and let $\breve{G}$ be its Langlands dual group, defined over k. Let C be a smooth projective curve over k of genus at least two.[1] Then we establish an equivalence of bounded derived category

$$D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0) \simeq D^b(\mathrm{QCoh}(\mathrm{LocSys}_{\breve{G}})^0), \tag{1.1.1}$$

where $\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0$ (respectively $\mathrm{QCoh}(\mathrm{LocSys}_{\breve{G}})^0$) is a certain localization of the category of $\mathcal{D}$-modules on Bun_G (respectively a localization of the category of quasi-coherent sheaves on $\mathrm{LocSys}_{\breve{G}}$). We call (1.1.1) a 'generic' version of the GLC.

One remark is in order. Recall that over a field of positive characteristic, there are different objects that can be called $\mathcal{D}$-modules. In this paper, we use the notion of crystalline $\mathcal{D}$-modules, i.e., $\mathcal{D}$-modules are quasi-coherent sheaves with a flat connection. Likewise, the stack $\mathrm{LocSys}_{\breve{G}}$ is the stack of $\breve{G}$-bundles on C with a flat connection.

1.2 Summary of the construction

The case $G = \mathrm{GL}_n$ has been considered by Bezrukavnikov and Braverman in [BB07] (see [Gro12, Tra11] for various extensions). The main observation is that the geometric Langlands duality in characteristic p formulated in the above form can be thought as a twisted version of its classical limit. Since the classical duality holds 'generically', they proved a 'generic' version of the GLC in the case when $G = \mathrm{GL}_n$.

Our generalization to any semi-simple group G is based on the same observation, but some new ingredients are needed in this general situation.

One of the main difficulties for general G is that the classical duality is more complicated. For $G = \mathrm{GL}_n$, the generic fibers of the Hitchin fibration are the Picard stacks of line bundles on the corresponding spectral curves and the duality of Hitchin fibrations in this case essentially amounts to the self-duality of the Jacobian of an algebraic curve. However, for general G, the fibers of the Hitchin fibration involve more general Picard stacks, such as the Prym varieties etc., and the duality of the Hitchin fibrations for G and $\breve{G}$ over $\mathbb{C}$ are the main theme of [DP12] (see [HT03] for the case $G = \mathrm{SL}_n$). As commented on by the authors, the arguments in [DP12] use transcendental methods in an essential way and therefore cannot be applied to our situation directly.

Our first step is to extend the classical duality to any reductive group G over any algebraically closed field k whose characteristic does not divide the order of the Weyl group of G. Let us first give its statement, and leave the details to § 3. For a reductive group G and a smooth projective curve C over k, and a positive line bundle $\mathcal{L}$ on C, let $\mathrm{Higgs}_{G,\mathcal{L}} \to B$ denote the corresponding Hitchin fibration, on which the Picard stack $\mathscr{P}_{\mathcal{L}} \to B$ acts (see § 2 for a review). There is an open subset $B^0 \subset B$ such that $\mathscr{P}_{\mathcal{L}}|_{B^0}$ is a Beilinson 1-motive (a Picard stack that is essentially an abelian variety, see Appendix A). Fixing a non-degenerate bilinear form on the Lie algebra $\mathfrak{g}$ of G, one can identify the Hitchin base B and the corresponding open subset B^0 for G and $\breve{G}$. The classical duality is the following assertion.

[1] The assumptions on the genus of C and on the semi-simplicity of G should not be essential. We impose them to avoid the DG structure on moduli spaces.

THEOREM 1.2.1. *For a positive line bundle $\mathcal{L}$ on C, there is a canonical isomorphism of Picard stacks*

$$\mathfrak{D}_{\mathrm{cl}} : (\mathscr{P}_{\mathcal{L}}|_{B^0})^\vee \simeq \breve{\mathscr{P}}_{\mathcal{L}}|_{B^0}, \tag{1.2.1}$$

where $(\mathscr{P}_{\mathcal{L}}|_{B^0})^\vee$ is the dual Picard stack of $\mathscr{P}_{\mathcal{L}}|_{B^0}$ (as defined in Appendix A).

Now assume that the characteristic of k is positive. In addition, assume that the genus of C is at least two and that $\mathcal{L} = \omega_C$ is the canonical bundle. We will omit the subscript ω_C and write $\mathscr{P} = \mathscr{P}_{\omega_C}$ etc. The second step then is to construct a twisted version of the above classical duality in this situation. To explain its meaning, let us first introduce a notation: if X is a stack over k, we denote by X' its Frobenius twist, i.e., the pullback of X along the absolute Frobenius endomorphism of k. Let $F_X : X \to X'$ denote the relative Frobenius morphism. We will replace both sides of (1.2.1) by certain torsors under $\mathscr{P}'^\vee$ and $\breve{\mathscr{P}}'$.

We begin to explain the $\breve{\mathscr{P}}'$-torsor $\breve{\mathscr{H}}$, which was introduced in [CZ15]. There is a smooth commutative group scheme $\breve{J}'$ on $C' \times B'$ and $\breve{\mathscr{P}}'$ in fact classifies $\breve{J}'$-torsors. Let us denote by $\breve{J}^p$ the pullback of $\breve{J}'$ along the relative Frobenius $F_{C'\times B'/B'} : C \times B' \to C' \times B'$. This is a group scheme with a canonical connection along C, and therefore it makes sense to talk about $\breve{J}^p$-local systems on $C \times B'$ and their p-curvatures (see [CZ15, Appendix] for generalities). Let $\breve{\mathscr{H}}$ be the stack of $\breve{J}^p$-local systems with some specific p-curvature $\breve{\tau}'$. This is a $\breve{\mathscr{P}}'$-torsor.

Next we explain the $\mathscr{P}'^\vee$-torsor $\mathscr{T}_{\mathscr{D}(\theta_m)}$. According to general nonsense (Appendix A), such a torsor gives a multiplicative $\mathbb{G}_m$-gerbe $\mathscr{D}$ on $\mathscr{P}'$ and vice versa. So it is enough to explain this multiplicative $\mathbb{G}_m$-gerbe $\mathscr{D}(\theta_m)$ on $\mathscr{P}'$. First recall that the sheaf of crystalline differential operators on $\mathscr{P}$ can be regarded as a $\mathbb{G}_m$-gerbe $\mathscr{D}_{\mathscr{P}}$ on the cotangent bundle $T^*\mathscr{P}'$. We will construct a 1-form θ_m on $\mathscr{P}'$, which is multiplicative (in the sense of § C.2). Now, $\mathscr{D} = \mathscr{D}(\theta_m)$ is the gerbe on $\mathscr{P}'$ obtained via pullback of $\mathscr{D}_{\mathscr{P}}$ along the map $\theta_m : \mathscr{P}' \to T^*\mathscr{P}'$.

The twisted version of the classical duality is the following assertion.

THEOREM 1.2.2. *Over B'^0, there is a canonical isomorphism of $\mathscr{P}'^\vee \simeq \breve{\mathscr{P}}'$-torsors*

$$\mathfrak{D} : \mathscr{T}_{\mathscr{D}(\theta_m)}|_{B'^0} \simeq \breve{\mathscr{H}}|_{B'^0}.$$

The final step towards (1.1.1) is to establish two abelianization theorems. Another difference between the geometric Langlands correspondence for GL_n and for a general group G is that in the latter case, there is no canonical equivalence in general. As is widely known to experts (e.g. see [FW08]), the geometric Langlands correspondence for general G should depend on a choice of theta characteristic of the curve C.

Let us fix a square root κ of ω_C. Then the Kostant section of $\mathrm{Higgs}'_G \to B'$ induces a map $\epsilon_{\kappa'} : \mathscr{P}' \to \mathrm{Higgs}'_G$. The first abelianization theorem asserts a canonical isomorphism

$$\epsilon_{\kappa'}^* \mathscr{D}_{\mathrm{Bun}_G} \simeq \mathscr{D}(\theta_m),$$

where $\mathscr{D}_{\mathrm{Bun}_G}$ is the $\mathbb{G}_m$-gerbe (on $\mathrm{Higgs}'_G = T^*\mathrm{Bun}'_G$) of crystalline differential operators on Bun_G and $\mathscr{D}(\theta_m)$ is the $\mathbb{G}_m$-gerbe on $\mathscr{P}'$ mentioned above.

On the dual side, we constructed a canonical morphism in [CZ15]

$$\mathfrak{C} : \breve{\mathscr{H}} \times^{\breve{\mathscr{P}}'} \mathrm{Higgs}'_{\breve{G}} \to \mathrm{LocSys}_{\breve{G}},$$

and the Kostant section of $\mathrm{Higgs}'_{\breve{G}} \to B'$ induces an isomorphism

$$\mathfrak{C}_\kappa : \breve{\mathscr{H}} \simeq \mathrm{LocSys}^{\mathrm{reg}}_{\breve{G}},$$

where $\mathrm{LocSys}^{\mathrm{reg}}_{\breve{G}}$ is a certain open substack of $\mathrm{LocSys}_{\breve{G}}$ (see [CZ15, Remark 3.14]).

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Combining the above three steps and a general version of the Fourier–Mukai transform (Appendix A) will give the desired equivalence (1.1.1).

Let us mention that the morphism $\mathfrak{C}$ was obtained in [CZ15] as a version of Simpson correspondence for smooth projective curves in positive characteristic.

Finally in §§ 5.5 and 5.6, we discuss how the equivalence constructed above depends on the choice of the theta characteristic. This can be regarded as a verification of the predictions of [FW08, § 10] in our settings.

1.3 The Langlands transform

To claim that the above equivalence is the conjectural geometric Langlands transform, one needs to verify several properties that it is supposed to satisfy. We will only briefly discuss these properties (see [Gai13] for more details), and leave the verifications to our next work.

The first property is that the equivalence should intertwine the action of the Hecke operators on the automorphic side and the action of the Wilson operators on the spectral side. Recall that in the case $k = \mathbb{C}$, both categories $D(\mathcal{D}\text{-mod}(\mathrm{Bun}_G))$ and $D(\mathrm{Qcoh}(\mathrm{LocSys}_{\check{G}}))$ admit actions of a family of commuting operators, labeled by points x on the curve and representations V of the group $\check{G}$. Namely, for $x \in C$ and $V \in \mathrm{Rep}(\check{G})$, there is the so-called Wilson operator $W_{V,x}$ acting on $\mathrm{Qcoh}(\mathrm{LocSys}_{\check{G}})$ by tensoring with the locally free sheaf $V_{E_{\mathrm{univ}}}|_{\mathrm{LocSys}_{\check{G}} \times \{x\}}$. On the other side, there is the Hecke operator $H_{V,x}$ acting on $\mathcal{D}\text{-mod}(\mathrm{Bun}_G)$ via certain integral transform (e.g. see [BD91, § 5]). The second property is that the equivalence should satisfy the Whittaker normalization. Namely, the Whittaker $\mathcal{D}$-module $\mathcal{F}_\Psi$ on Bun_G is supposed to transformed to the structure sheaf $\mathcal{O}_{\mathrm{LocSys}_{\check{G}}}$.

In the positive characteristic, it is yet not clear how to define Hecke operators (except those corresponding to minuscule coweights) due to lack of the notion of intersection cohomology $\mathcal{D}$-modules. Our observation is that by the geometric Casselman–Shalika formula [FGV01], the two properties together will imply that the Whittaker coefficients of $\mathcal{D}$-modules on Bun_G can be calculated by applying the Wilson operators on their Langlands transforms and then taking the global sections. This is a well formulated statement in characteristic p and we will verify in the future work that this is satisfied by the equivalence constructed here.

The third property is that the equivalence should be compatible with Beilinson and Drinfeld's construction of automorphic $\mathcal{D}$-modules via opers [BD91]. In the case $G = \mathrm{GL}_n$ this property has been verified in [BT16]. We plan to return to this in the future work.

1.4 Structure of the article

Let us now describe the contents of this paper in more detail.

In § 2 we collect some facts about Hitchin fibrations that are used in this paper. Main references are [Ngô06, Ngô10].

In § 3 we prove the classical duality, i.e., the duality of Hitchin fibrations. This extends the work of [DP12] (over $\mathbb{C}$) to any algebraically closed field whose characteristic does not divide the order of the Weyl group of G. In § 3.7, we discuss the compatibility of the classical duality with twisting by $Z(\check{G})$-torsors. This is used to study the dependence of the equivalence (1.1.1) on the choice of the theta characteristic in §§ 5.5 and 5.6.

In § 4 we construct a canonical multiplicative 1-form θ_m on $\mathscr{P}'$.

In § 5 we deduce our main Theorem 5.0.1 from the twisted duality (see § 5.2) and the two abelianization theorems (see § 5.3).

There are three appendices at the end of the paper.

In Appendix A we collect some basic facts about Beilinson 1-motives and duality on Beilinson 1-motives. In particular, we state a general version of Fourier–Mukai transforms for Beilinson 1-motives.

In Appendix B we recall the basic theory of $\mathcal{D}$-modules over varieties and stacks in positive characteristic, following [BMR08, BB07, OV07, Tra11].

In Appendix C we prove the abelian duality for good Beilinson 1-motives. It asserts that the derived category of $\mathcal{D}$-modules on a 'good' Beilinson 1-motive $\mathscr{A}$ is equivalent to the derived category of quasi-coherent sheaves on the universal extension $\mathscr{A}^\natural$ by vector groups of its dual $\mathscr{A}^\vee$.

1.5 Notations

1.5.1 *Notations related to algebraic stacks.* Our terminology of algebraic stacks follows the book [LB00]. Let k be an algebraically closed field and let p be the characteristic component of k. Let S be a Noetherian scheme over k. In this paper, an algebraic stack $\mathscr{X}$ over S is a stack such that the diagonal morphism

$$\Delta_S : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$$

is representable and quasi-compact and such that there exists a smooth presentation, i.e., a smooth, surjective morphism $X \to \mathscr{X}$ from a scheme X.

An algebraic stack $\mathscr{X}$ is called smooth over S if for every S-scheme U mapping smoothly to $\mathscr{X}$, the structure morphism $U \to S$ is smooth.

For any algebraic stack $\mathscr{X}$, we denote by $\mathscr{X}_{\text{Ét}}$ the big étale site of $\mathscr{X}$. We denote by $\mathscr{X}_{\text{sm}}$ the smooth site on $\mathscr{X}$, i.e., the site for which the underling category has objects consisting of S-schemes U together with a smooth morphism $U \to \mathscr{X}$ and has morphisms $V \to U$ smooth 2-morphisms over $\mathscr{X}$ and for which covering maps are smooth surjective maps of schemes. If $\mathscr{X}$ is a Deligne–Mumford stack, we denote by $\mathscr{X}_{\text{ét}}$ the small étale site of $\mathscr{X}$.

Let $\mathscr{Y} \to \mathscr{X}$ be a quasi-projective morphism of algebraic stacks, with $\mathscr{X}$ smooth and proper over S. We denote by $\mathrm{Sect}_S(\mathscr{X}, \mathscr{Y})$ the stack of 'sections' of $\mathscr{Y}$ over $\mathscr{X}$, i.e., for any $u : U \to S$ we have

$$\mathrm{Sect}_S(\mathscr{X}, \mathscr{Y})(U) = \mathrm{Hom}_{\mathscr{X}}(\mathscr{X} \times_S U, \mathscr{Y}).$$

If the base scheme $S = \mathrm{Spec}(k)$, we write $\mathrm{Sect}(\mathscr{X}, \mathscr{Y}) = \mathrm{Sect}_S(\mathscr{X}, \mathscr{Y})$.

If $\mathscr{X}$ is a smooth algebraic stack over S, we define the relative tangent stack $T(\mathscr{X}/S)$ as the stack that assigns every $\mathrm{Spec}\, R \to S$, the groupoid

$$T(\mathscr{X}/S)(R) := \mathscr{X}(R[\epsilon]/\epsilon^2).$$

It is algebraic and the natural inclusion $R \to R[\epsilon]/\epsilon^2$ induces a morphism

$$\tau_{\mathscr{X}} : T(\mathscr{X}/S) \to \mathscr{X}.$$

It is known that $T(\mathscr{X}/S)$ is a relative Picard stack over $\mathscr{X}$. Therefore, one can associate to it a complex in $D^{[-1,0]}(\mathscr{X}, \mathbb{Z})$, called the relative tangent complex:

$$T^\bullet_{\mathscr{X}/S} = \{T_{\mathscr{X}/S} \to T_{\mathscr{X}}\}.$$

The relative cotangent stack is then defined as

$$T^*(\mathscr{X}/S) := \mathrm{Spec}_{\mathscr{X}}(\mathrm{Sym}_{\mathcal{O}_{\mathscr{X}}} H^0(T^\bullet_{\mathscr{X}/S})).$$

Let $f : \mathscr{X} \to \mathscr{Y}$ be a (representable) morphism between two algebraic stacks over S. We denote the cotangent morphism as the following diagram of maps.

$$\begin{tikzcd} T^*(\mathscr{Y}/S) \times_{\mathscr{Y}} \mathscr{X} \arrow[r, "f_d"] \arrow[d, "f_p"] & T^*(\mathscr{X}/S) \\ T^*(\mathscr{Y}/S) & \end{tikzcd} \tag{1.5.1}$$

1.5.2 *Notations related to Frobenius morphism.* Let S be a Noetherian k-scheme and $\mathscr{X} \to S$ be an algebraic stack over S. If $p\mathcal{O}_S = 0$, we denote by $Fr_S : S \to S$ the absolute Frobenius map of S. We have the following commutative diagram

$$\begin{tikzcd} \mathscr{X} \arrow[r, "F_{\mathscr{X}/S}"] \arrow[dr] & \mathscr{X}^{(S)} \arrow[r, "\pi_{\mathscr{X}/S}"] \arrow[d] & \mathscr{X} \arrow[d] \\ & S \arrow[r, "Fr_S"] & S \end{tikzcd}$$

where the square is Cartesian. We call $\mathscr{X}^{(S)}$ the Frobenius twist of $\mathscr{X}$ along S, and $F_{\mathscr{X}/S}$: $\mathscr{X} \to \mathscr{X}^{(S)}$ the relative Frobenius morphism. If the base scheme S is clear, $\mathscr{X}^{(S)}$ is also denoted by $\mathscr{X}'$ for simplicity.

1.5.3 *Notation related to torsors.* Let $\mathcal{G}$ be a smooth affine group scheme over X, and E be a $\mathcal{G}$-torsor on X. We denote by $\mathrm{Aut}(E) = E \times^{\mathcal{G}} \mathcal{G}$ the adjoint torsor and $\mathrm{ad}(E)$ or $\mathfrak{g}_E = E \times^{\mathcal{G}} \mathrm{Lie}\, \mathcal{G}$ the adjoint bundle.

2. The Hitchin fibration

In this section, we review some basic geometric facts of Hitchin fibrations, following [Ngô06, Ngô10]. Only § 2.7 is probably new.

2.1 Notations related to reductive groups

Let G be a reductive algebraic group over k of rank l. We denote by $\check{G}$ its Langlands dual group over k. We denote by $\mathfrak{g}$ (respectively by $\check{\mathfrak{g}}$) the Lie algebra of G (respectively $\check{G}$). Let T denote the abstract Cartan of G with its Lie algebra $\mathfrak{t}$. The counterparts on the Langlands dual side are denoted by $\check{T}, \check{\mathfrak{t}}$. We denote by W the abstract Weyl group of G, which acts on T and $\check{T}$. We denote by $\mathbb{X}^\bullet(T)$ or simply by $\mathbb{X}^\bullet$ (respectively by $\mathbb{X}_\bullet(T)$ or simply by $\mathbb{X}_\bullet$) the character (respectively the cocharacter) group of T. Let $\Phi \subset \mathbb{X}^\bullet(T)$ be the set of roots. Sometimes, we also fix a set of simple roots $\{\alpha_1, \ldots, \alpha_l\}$ and an embedding $\mathfrak{t} \subset \mathfrak{g}$. Then for $\alpha \in \Phi$, let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ denote the corresponding root subspace.

From now on, we assume that the char $k = p$ is zero or $p \nmid |\mathrm{W}|$. We fix a W-invariant non-degenerate bilinear form $(,) : \mathfrak{t} \times \mathfrak{t} \to k$ and identify $\mathfrak{t}$ with $\check{\mathfrak{t}}$ using $(,)$. This invariant form also determines a unique G-invariant non-degenerate bilinear form $\mathfrak{g} \times \mathfrak{g} \to k$, still denoted by $(,)$. Let $\mathfrak{g} \simeq \mathfrak{g}^*$ be the resulting G-equivariant isomorphism.

2.2 Hitchin map

Let $k[\mathfrak{g}]$ and $k[\mathfrak{t}]$ be the algebras of polynomial functions on $\mathfrak{g}$ and on $\mathfrak{t}$ respectively. By Chevalley's theorem, we have an isomorphism $k[\mathfrak{g}]^G \simeq k[\mathfrak{t}]^{\mathrm{W}}$. Moreover, $k[\mathfrak{t}]^{\mathrm{W}}$ is isomorphic

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to a polynomial ring of l variables $u_1,\ldots,u_l$ and each u_i is homogeneous in degree e_i. Let $\mathfrak{c}=\mathrm{Spec}(k[\mathfrak{t}]^{\mathrm{W}})$. The natural $\mathbb{G}_m$ action on $\mathfrak{g}$ induces a $\mathbb{G}_m$-action on $\mathfrak{c}$ and under the isomorphism $\mathfrak{c}\simeq\mathrm{Spec}(k[u_1,\ldots,u_l])\simeq\mathbb{A}^l$ the action is given by

$$h\cdot(a_1,\ldots,a_l)=(h^{e_1}a_1,\ldots,h^{e_l}a_l).$$

Let $\chi:\mathfrak{g}\to\mathfrak{c}$ be the map induced by $k[\mathfrak{c}]\simeq k[\mathfrak{g}]^G\hookrightarrow k[\mathfrak{g}]$. It is a $G\times\mathbb{G}_m$-equivariant map where G acts trivially on $\mathfrak{c}$. Similarly, let $\boldsymbol{\pi}:\mathfrak{t}\to\mathfrak{c}$ be the map induced by $k[\mathfrak{c}]\hookrightarrow k[\mathfrak{t}]$, which is also $\mathbb{G}_m$-equivariant. Let $\mathcal{L}$ be an invertible sheaf on C and $\mathcal{L}^\times$ be the corresponding $\mathbb{G}_m$-torsor. We denote by $\mathfrak{g}_{\mathcal{L}}=\mathfrak{g}\times^{\mathbb{G}_m}\mathcal{L}^\times$, $\mathfrak{t}_{\mathcal{L}}=\mathfrak{t}\times^{\mathbb{G}_m}\mathcal{L}^\times$, and $\mathfrak{c}_{\mathcal{L}}=\mathfrak{c}\times^{\mathbb{G}_m}\mathcal{L}^\times$ the $\mathbb{G}_m$-twist of $\mathfrak{g}$, $\mathfrak{t}$, and $\mathfrak{c}$ with respect to the natural $\mathbb{G}_m$-action.

Let $\mathrm{Higgs}_{G,\mathcal{L}}=\mathrm{Sect}(C,[\mathfrak{g}_{\mathcal{L}}/G])$ be the stack of sections of $[\mathfrak{g}_{\mathcal{L}}/G]$ over C. That is, for each k-scheme S the groupoid $\mathrm{Higgs}_{G,\mathcal{L}}(S)$ consist of maps

$$h_{E,\phi}:C\times S\to[\mathfrak{g}_{\mathcal{L}}/G],$$

or, equivalently, those maps

$$h_{E,\phi}:C\times S\to[\mathfrak{g}/G\times\mathbb{G}_m]$$

such that the composition of $h_{E,\phi}$ with the projection $[\mathfrak{g}/G\times\mathbb{G}_m]\to B\mathbb{G}_m$ is given by the $\mathbb{G}_m$-torsor $\mathcal{L}^\times$. Explicitly, $\mathrm{Higgs}_{G,\mathcal{L}}(S)$ consist of pairs (E,ϕ) (called Higgs bundles), where E is an G-torsor over $C\times S$ and ϕ is an element in $\Gamma(C\times S,\mathrm{ad}(E)\otimes\mathcal{L})$ known as the Higgs field. If the group G is clear from the context, we simply write $\mathrm{Higgs}_{\mathcal{L}}$ for $\mathrm{Higgs}_{G,\mathcal{L}}$.

Let $B_{\mathcal{L}}=\mathrm{Sect}_{\mathrm{Spec}\,k}(C,\mathfrak{c}_{\mathcal{L}})$ be the scheme of sections of $\mathfrak{c}_{\mathcal{L}}$ over C. That is, for each k-scheme S, $B_{\mathcal{L}}(S)$ is the set of sections

$$b:C\times S\to\mathfrak{c}_{\mathcal{L}},$$

or, equivalently, those maps

$$b:C\times S\to[\mathfrak{c}/\mathbb{G}_m]$$

such that the composition of b with the projection $[\mathfrak{c}/\mathbb{G}_m]\to B\mathbb{G}_m$ is given by $\mathcal{L}^\times$. It is called the Hitchin base of G.

The natural G-invariant projection $\chi:\mathfrak{g}\to\mathfrak{c}$ induces a map

$$[\chi_{\mathcal{L}}]:[\mathfrak{g}_{\mathcal{L}}/G]\to\mathfrak{c}_{\mathcal{L}},$$

or more generally

$$[\chi/G\times\mathbb{G}_m]:[\mathfrak{g}/G\times\mathbb{G}_m]\to[\mathfrak{c}/\mathbb{G}_m].\tag{2.2.1}$$

The map $[\chi_{\mathcal{L}}]$ induces a natural map

$$h_{\mathcal{L}}:\mathrm{Higgs}_{\mathcal{L}}=\mathrm{Sect}(C,[\mathfrak{g}_{\mathcal{L}}/G])\to\mathrm{Sect}(C,\mathfrak{c}_{\mathcal{L}})=B_{\mathcal{L}}.$$

Definition 2.2.1. We call $h_{\mathcal{L}}:\mathrm{Higgs}_{\mathcal{L}}\to B_{\mathcal{L}}$ the Hitchin map associated to $\mathcal{L}$.

For any $b\in B_{\mathcal{L}}(S)$ we denote by $\mathrm{Higgs}_{\mathcal{L},b}$ the fiber product $S\times_{B_{\mathcal{L}}}\mathrm{Higgs}_{\mathcal{L}}$.

Observe that the invariant bilinear form $\mathfrak{t}\times\mathfrak{t}\to k$ induces a canonical isomorphism $\mathfrak{t}\simeq\mathfrak{t}^*=:\check{\mathfrak{t}}$, compatible with the W-action. Therefore, there is a canonical isomorphism $\mathfrak{c}\simeq\check{\mathfrak{c}}$ and $B_{\mathcal{L}}\simeq\check{B}_{\mathcal{L}}$. In what follows, we will identify them.

Let $\omega=\omega_C$ be the canonical line bundle of C. We are mostly interested in the case $\mathcal{L}=\omega$. For simplicity, from now on we denote $B=B_\omega$, $\mathrm{Higgs}=\mathrm{Higgs}_\omega$, $h=h_\omega:\mathrm{Higgs}\to B$, and $\mathrm{Higgs}_b=\mathrm{Higgs}_{\omega_C,b}$. We sometimes also write Higgs_G for Higgs to emphasize the group G. Observe that the bilinear form as in § 2.1 together with the Serre duality induces an isomorphism $\mathrm{Higgs}\simeq T^*\,\mathrm{Bun}_G$ (cf. [Hit87]).

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2.3 The Kostant section

In this section, we recall the construction of the Kostant section of the Hitchin map $h_{\mathcal{L}}$. For each simple root α_i we choose a non-zero vector $f_i \in \mathfrak{g}_{-\alpha_i}$. Let $f = \bigoplus_{i=1}^{l} f_i \in \mathfrak{g}$. We complete f into an $\mathfrak{sl}_2$ triple $\{f, h, e\}$ and denote by $\mathfrak{g}^e$ the centralizer of e in $\mathfrak{g}$. A theorem of Kostant says that $f + \mathfrak{g}^e$ consist of regular elements in $\mathfrak{g}$ and the restriction of $\chi : \mathfrak{g} \to \mathfrak{c}$ to $f + \mathfrak{g}^e$ is an isomorphism onto $\mathfrak{c}$. We denote by

$$\mathrm{kos} : \mathfrak{c} \simeq f + \mathfrak{g}^e$$

the inverse of $\chi|_{f+\mathfrak{g}^e}$. Let $\rho(\mathbb{G}_m)$ denote the following $\mathbb{G}_m$-action on $\mathfrak{g}$: it acts trivially on $\mathfrak{t}$, and on $\mathfrak{g}_\alpha$ by $\rho(t)x = t^{\mathrm{ht}(\alpha)}x$ where $\mathrm{ht}(\alpha) = \sum n_i$ if $\alpha = \sum n_i \alpha_i$. We have $\rho(t)f = t^{-1}f$ and $\rho(t)e = te$, in particular $\mathfrak{g}^e$ is invariant under $\rho(\mathbb{G}_m)$. We define a new $\mathbb{G}_m$-action on $\mathfrak{g}$ by $\rho^+(t) = t\rho(t)$. Then $\rho^+(t)f = f$ and $\rho^+(\mathbb{G}_m)$ preserves $f + \mathfrak{g}^e$. With respect to this action, the isomorphism $\mathrm{kos} : \mathfrak{c} \simeq f + \mathfrak{g}^e$ is $\mathbb{G}_m$-equivariant.

The diagonal map $\mathbb{G}_m \to \mathbb{G}_m \times \mathbb{G}_m$ induces a map

$$[\mathfrak{g}/\rho^+(\mathbb{G}_m)] \to [\mathfrak{g}/\mathbb{G}_m \times \rho(\mathbb{G}_m)].$$

By precomposing with the map $[\mathfrak{c}/\mathbb{G}_m] \overset{\mathrm{kos}}{\simeq} [f + \mathfrak{g}^e/\rho^+(\mathbb{G}_m)] \to [\mathfrak{g}/\rho^+(\mathbb{G}_m)]$ we obtain

$$[\mathfrak{c}/\mathbb{G}_m] \to [\mathfrak{g}/\mathbb{G}_m \times \rho(\mathbb{G}_m)].$$

If the action of $\rho(\mathbb{G}_m)$ on $\mathfrak{g}$ factors through the adjoint action of G, for example when G is adjoint, then there is a map $[\mathfrak{g}/\mathbb{G}_m \times \rho(\mathbb{G}_m)] \to [\mathfrak{g}/\mathbb{G}_m \times G]$ which defines a section

$$[\mathfrak{c}/\mathbb{G}_m] \to [\mathfrak{g}/\mathbb{G}_m \times \rho(\mathbb{G}_m)] \to [\mathfrak{g}/\mathbb{G}_m \times G]$$

of (2.2.1), and in particular, we get a section of $h_{\mathcal{L}}$. In general, the action $\rho(\mathbb{G}_m)$ does not necessarily factor through G, but its square does since it is given by the cocharacter $2\rho : \mathbb{G}_m \to G$ where 2ρ is the sum of positive coroots. So if we denote $\mathbb{G}_m^{[2]} \to \mathbb{G}_m$ the square map (so $\mathbb{G}_m^{[2]}$ is isomorphic to $\mathbb{G}_m$, but regarded as its the double cover), we get a map

$$\eta^{1/2} : [\mathfrak{c}/\mathbb{G}_m^{[2]}] \to [\mathfrak{g}/\mathbb{G}_m^{[2]} \times \rho(\mathbb{G}_m^{[2]})] \to [\mathfrak{g}/\mathbb{G}_m^{[2]} \times G].$$

Let $\mathcal{L}^{1/2}$ be a square root of $\mathcal{L}$. Then every $b : S \times C \to [\mathfrak{c}/\mathbb{G}_m]$ in $B_{\mathcal{L}}(S)$ factors through a unique map $b^{1/2} : S \times C \to [\mathfrak{c}/\mathbb{G}_m^{[2]}]$. Therefore, by composing with $\eta^{1/2}$, we get a lift of b:

$$\eta^{1/2}(b) : S \times C \xrightarrow{b^{1/2}} [\mathfrak{c}/\mathbb{G}_m^{[2]}] \xrightarrow{\eta^{1/2}} [\mathfrak{g}/\mathbb{G}_m^{[2]} \times G] \to [\mathfrak{g}/\mathbb{G}_m \times G].$$

The assignment $b \to \eta^{1/2}(b)$ defines a section

$$\eta_{\mathcal{L}^{1/2}} : B_{\mathcal{L}} \to \mathrm{Higgs}_{\mathcal{L}}$$

of the Hitchin map $h_{\mathcal{L}}$.

We fix a square root $\kappa = \omega^{1/2}$ (called a theta characteristic) of ω and write $\kappa = \eta_\kappa : B \to \mathrm{Higgs}$.

2.4 Cameral curve

For any $b \in B_{\mathcal{L}}(S)$, the cameral curve $\widetilde{C}_b$ is defined as the fiber product, as follows.

$$\begin{tikzcd} \widetilde{C}_b \arrow[r] \arrow[d, "\pi_b"] & \mathfrak{t}_{\mathcal{L}} \arrow[d] \\ C \times S \arrow[r, "b"] & \mathfrak{c}_{\mathcal{L}} \end{tikzcd}$$

When $b = \mathrm{id} : B_{\mathcal{L}} \to B_{\mathcal{L}}$, the corresponding cameral curve $\widetilde{C}_{\mathcal{L}} := \widetilde{C}_b$ is called the universal cameral curve. For simplicity, we will write $\widetilde{C} = \widetilde{C}_\omega$, $\pi = \pi_b : \widetilde{C} \to C \times B$.

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2.5 The universal centralizer group schemes

Consider the group scheme I over $\mathfrak{g}$ consisting of pairs

$$I = \{(g, x) \in G \times \mathfrak{g} \mid \mathrm{Ad}_g(x) = x\}.$$

We define $J = \mathrm{kos}^* I$, where $\mathrm{kos} : \mathfrak{c} \to \mathfrak{g}$ is the Kostant section. This is called the universal centralizer group scheme of $\mathfrak{g}$ (see Proposition 2.5.1). To study it, it is convenient to introduce two auxiliary group schemes. We define $J^1 = \mathrm{Res}_{\mathfrak{t}/\mathfrak{c}}(T)^{\mathrm{W}}$ and let J^0 to be the neutral component of J^1. All the group schemes J, J^0 and J^1 are smooth commutative group schemes over $\mathfrak{c}$. The following proposition is proved in [Ngô06] (see also [DG02]).

Proposition 2.5.1.

(1) *There is a unique morphism of group schemes $a : \chi^* J \to I \subset G \times \mathfrak{g}$, which extends the canonical isomorphism $\chi^* J|_{\mathfrak{g}^{\mathrm{reg}}} \simeq I|_{\mathfrak{g}^{\mathrm{reg}}}$.*
(2) *There are natural inclusions $J^0 \subset J \subset J^1$.*
(3) *The inclusion $J \subset J^1 = \mathrm{Res}_{\mathfrak{t}/\mathfrak{c}}(T)^{\mathrm{W}}$ in part (2) defines a morphism*

$$\mathrm{j}^1 : \boldsymbol{\pi}^* J \to T \times \mathfrak{t}$$

of group schemes over $\mathfrak{t}$, which is an isomorphism over $\mathfrak{t}^{rs}$.

All the above constructions can be twisted. Namely, there are $\mathbb{G}_m$-actions on I, J, J^1 and J^0. Moreover, the $\mathbb{G}_m$-action on I can be extended to a $G \times \mathbb{G}_m$-action given by $(h, t) \cdot (x, g) = (t \cdot hxh^{-1}, hgh^{-1})$. The natural morphisms $J \to \mathfrak{c}$ and $I \to \mathfrak{g}$ are $\mathbb{G}_m$-equivariant, and therefore we can twist everything by the $\mathbb{G}_m$-torsor $\mathcal{L}^\times$ to get $J_{\mathcal{L}} \to \mathfrak{c}_{\mathcal{L}}$, $I_{\mathcal{L}} \to \mathfrak{g}_{\mathcal{L}}$ where $J_{\mathcal{L}} = J \times^{\mathbb{G}_m} \mathcal{L}^\times$ and $I_{\mathcal{L}} = I \times^{\mathbb{G}_m} \mathcal{L}^\times$. Similarly, we have $J^0_{\mathcal{L}} \to \mathfrak{c}_{\mathcal{L}}$ and $J^1_{\mathcal{L}} \to \mathfrak{c}_{\mathcal{L}}$, and there are natural inclusions $J^0_{\mathcal{L}} \subset J_{\mathcal{L}} \subset J^1_{\mathcal{L}}$. The group scheme $I_{\mathcal{L}}$ over $\mathfrak{g}_{\mathcal{L}}$ is equivariant under the G-action, hence it descends to a group scheme $[I_{\mathcal{L}}]$ over $[\mathfrak{g}_{\mathcal{L}}/G]$.

2.6 Symmetries of Hitchin fibration

Let $b : S \to B_{\mathcal{L}}$ be an S-point of $B_{\mathcal{L}}$, corresponding to a map $b : C \times S \to \mathfrak{c}_{\mathcal{L}}$. Pulling back $J_{\mathcal{L}} \to \mathfrak{c}_{\mathcal{L}}$ along this map, we obtain a smooth group scheme $J_b = b^* J$ over $C \times S$.

Let $\mathscr{P}_b$ be the Picard category of J_b-torsors over $C \times S$. The assignment $b \to \mathscr{P}_b$ defines a Picard stack over B, denoted by $\mathscr{P}_{\mathcal{L}}$. Let us fix $b \in B_{\mathcal{L}}(S)$, and let $(E, \phi) \in \mathrm{Higgs}_{\mathcal{L},b}$ corresponding to the map $h_{E,\phi} : C \times S \to [\mathfrak{g}_{\mathcal{L}}/G]$. Observe that the morphism $\chi^* J \to I$ in Proposition 2.5.1 induces $[\chi_{\mathcal{L}}]^* J_{\mathcal{L}} \to [I_{\mathcal{L}}]$ of group schemes over $[\mathfrak{g}_{\mathcal{L}}/G]$. Pulling back to $C \times S$ using $h_{E,\phi}$, we get a map

$$a_{E,\phi} : J_b \to h^*_{E,\phi}[I] = \mathrm{Aut}(E, \phi) \subset \mathrm{Aut}(E), \tag{2.6.1}$$

which allows us to twist $(E, \phi) \in \mathrm{Higgs}_{\mathcal{L},b}$ by a J_b-torsor. This construction defines an action of $\mathscr{P}_{\mathcal{L}}$ on $\mathrm{Higgs}_{\mathcal{L}}$ over $B_{\mathcal{L}}$.

Let $\mathrm{Higgs}^{\mathrm{reg}}_{\mathcal{L}}$ be the open stack of $\mathrm{Higgs}_{\mathcal{L}}$ consisting of $(E, \phi) : C \to [\mathfrak{g}_{\mathcal{L}}/G]$ that factors through $C \to [(\mathfrak{g}^{\mathrm{reg}})_{\mathcal{L}}/G]$. If $(E, \phi) \in \mathrm{Higgs}^{\mathrm{reg}}_{\mathcal{L}}$, then $a_{E,\phi}$ above is an isomorphism. The Kostant section $\eta_{\mathcal{L}^{1/2}} : B_{\mathcal{L}} \to \mathrm{Higgs}_{\mathcal{L}}$ factors through $\eta_{\mathcal{L}^{1/2}} : B_{\mathcal{L}} \to \mathrm{Higgs}^{\mathrm{reg}}_{\mathcal{L}}$. Following [Ngô06, § 4], we define $B^0_{\mathcal{L}}$ as the open sub-scheme of $B_{\mathcal{L}}$ consisting of $b \in B_{\mathcal{L}}(k)$ such that the image of the map $b : C \to \mathfrak{c}_{\mathcal{L}}$ intersects the discriminant divisor transversally. The following proposition can be extracted from [DG02, DP12, Ngô06].

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PROPOSITION 2.6.1.

(1) *The stack* $\mathrm{Higgs}_{\mathcal{L}}^{\mathrm{reg}}$ *is a* $\mathscr{P}_{\mathcal{L}}$*-torsor, which can be trivialized by a choice of a Kostant section* $\eta_{\mathcal{L}^{1/2}}$.
(2) *One has* $\mathrm{Higgs}_{\mathcal{L}}^{\mathrm{reg}} \times_{B_{\mathcal{L}}} B_{\mathcal{L}}^0 = \mathrm{Higgs}_{\mathcal{L}} \times_{B_{\mathcal{L}}} B_{\mathcal{L}}^0$.
(3) *The restriction of the universal cameral curve* $\widetilde{C}_{\mathcal{L}}|_{B^0} \to B_{\mathcal{L}}^0$ *to* $B_{\mathcal{L}}^0$ *is smooth. The restriction* $\mathscr{P}_{\mathcal{L}}|_{B_{\mathcal{L}}^0}$ *to* $B_{\mathcal{L}}^0$ *is a Beilinson 1-motive.*

Remark 2.6.2. Let $\underline{Disc} : \mathfrak{t} \to k$ be the discriminant function defined by

$$\underline{Disc} = \prod_{\alpha \in \Phi} d\alpha,$$

where Φ is the set of roots of G. The function $\underline{Disc}$ is W-invariant, and thus descends to a function $Disc$ on $\mathfrak{c}$. Moreover, the function $Disc : \mathfrak{c} \to k$ is $\mathbb{G}_m$-equivariant where $\mathbb{G}_m$ acts on k via the character $t \to t^N$ and $N = |\Phi|$. Let $Disc_{\mathcal{L}} : \mathfrak{c}_{\mathcal{L}} \to \mathcal{L}^N$ be the twist of $Disc$. For any $b : C \to \mathfrak{c}_{\mathcal{L}}$, we get a section

$$s_b \in \Gamma(C, \mathcal{L}^N).$$

The zeros of s_b is the branch loci $\mathcal{B}$ of the cameral cover $\pi_b : \widetilde{C}_b \to C$. If $b \in B_{\mathcal{L}}^0(k)$, then $\mathcal{B}$ is multiplicity free. Note that if $\deg \mathcal{L} > 0$ the branch loci $\mathcal{B}$ is non-empty.

2.7 The tautological section $\tau : \mathfrak{c} \to \mathrm{Lie}\, J$

Recall that by Proposition 2.5.1, there is a canonical isomorphism $\chi^* J|_{\mathfrak{g}^{\mathrm{reg}}} \simeq I|_{\mathfrak{g}^{\mathrm{reg}}}$. The sheaf of Lie algebras $\mathrm{Lie}\,(I|_{\mathfrak{g}^{\mathrm{reg}}}) \subset \mathfrak{g}^{\mathrm{reg}} \times \mathfrak{g}$ admits a tautological section $\tilde{\tau} : \mathfrak{g}^{\mathrm{reg}} \to \mathrm{Lie}\,(I|_{\mathfrak{g}^{\mathrm{reg}}})$ given by $x \mapsto x \in \mathrm{Lie}\, I_x$ for $x \in \mathfrak{g}^{\mathrm{reg}}$. This section descends to a tautological section $\tau : \mathfrak{c} \to \mathrm{Lie}\, J$.[2] Recall the following property of τ [CZ15, Lemma 2.2].

LEMMA 2.7.1. *Let* $x \in \mathfrak{g}$*, and* $a_x : J_{\chi(x)} \to I_x \subset G$ *be the homomorphism as in Proposition 2.5.1(1). Then* $da_x(\tau(x)) = x$*, where* da_x *denotes the differential of* a_x.

Let us regard $\mathrm{Lie}\, J$ as a scheme over $\mathfrak{c}$. Besides the section τ, there is a canonical map $c : \mathrm{Lie}\, J \to \mathfrak{c}$ such that $c\tau = \mathrm{id}$. Namely, if we regard $\mathrm{Lie}\,(I|_{\mathfrak{g}^{\mathrm{reg}}})$ as a scheme, then there is a natural map $\mathrm{Lie}\,(I|_{\mathfrak{g}^{\mathrm{reg}}}) \to \mathfrak{c}$ given by

$$\mathrm{Lie}\,(I|_{\mathfrak{g}^{\mathrm{reg}}}) \subset \mathfrak{g} \times \mathfrak{g}^{\mathrm{reg}} \to \mathfrak{c} \times \mathfrak{g}^{\mathrm{reg}} \to \mathfrak{c},$$

which also descends to a morphism $c : \mathrm{Lie}\, J \to \mathfrak{c}$.

The morphisms τ and c have global counterparts (see also [CZ15, § 2.3]). Observe that $\mathbb{G}_m$ acts on $\mathfrak{g} \times \mathfrak{g}^{\mathrm{reg}}$ via natural homotheties on both factors, and therefore on $\chi^* \mathrm{Lie}\, J|_{\mathfrak{g}^{\mathrm{reg}}} \simeq \mathrm{Lie}\,(I|_{\mathfrak{g}^{\mathrm{reg}}}) \subset \mathfrak{g} \times \mathfrak{g}^{\mathrm{reg}}$. This $\mathbb{G}_m$-action on $\chi^* \mathrm{Lie}\, J|_{\mathfrak{g}^{\mathrm{reg}}}$ descends to a $\mathbb{G}_m$-action on $\mathrm{Lie}\, J$, and for any line bundle $\mathcal{L}$ on C the $\mathcal{L}^\times$-twist $(\mathrm{Lie}\, J) \times^{\mathbb{G}_m} \mathcal{L}^\times$ is $\mathrm{Lie}\, J_{\mathcal{L}} \otimes \mathcal{L}$, where $J_{\mathcal{L}}$ is introduced in § 2.5. In addition, both maps τ and c are $\mathbb{G}_m$-equivariant with respect to this $\mathbb{G}_m$-action on $\mathrm{Lie}\, J$ and the natural $\mathbb{G}_m$-action on $\mathfrak{c}$. Therefore, if we define a vector bundle $B_{J,\mathcal{L}}$ over $B_{\mathcal{L}}$, whose fiber over $b \in B_{\mathcal{L}}$ is $\Gamma(C, \mathrm{Lie}\, J_b \otimes \mathcal{L})$, then by twisting τ and c by $\mathcal{L}$, we obtain

$$\tau_{\mathcal{L}} : B_{\mathcal{L}} \to B_{J,\mathcal{L}}, \tag{2.7.1}$$

[2] Indeed, one can check that τ is equal to $\mathrm{kos}^*(\tilde{\tau})$, the pullback of $\tilde{\tau}$ along the Kostant section $\mathrm{kos} : \mathfrak{c} \to \mathfrak{g}^{\mathrm{reg}}$.

which is a canonical section of the projection $\mathrm{pr} : B_{J,\mathcal{L}} \to B_{\mathcal{L}}$, and a canonical map

$$c_{\mathcal{L}} : B_{J,\mathcal{L}} \to B_{\mathcal{L}} \tag{2.7.2}$$

such that $c_{\mathcal{L}}\tau_{\mathcal{L}} = \mathrm{id}$. As before, we omit the subscript $_{\mathcal{L}}$ if $\mathcal{L} = \omega$ for brevity.

Likewise, we introduce the vector bundle $B^*_{J,\mathcal{L}}$ over $B_{\mathcal{L}}$ whose fiber over b is $\Gamma(C, (\mathrm{Lie}\, J_b)^* \otimes \mathcal{L})$. Observe that $B^*_{J,\mathcal{L}}$ is not the dual of $B_{J,\mathcal{L}}$. Rather, when $\mathcal{L} = \omega$, it is the pullback $e^*T^*(\mathscr{P}_{\mathcal{L}}/B_{\mathcal{L}})$ of the cotangent bundle of $\mathscr{P}_{\mathcal{L}} \to B_{\mathcal{L}}$ along the unit section $e : B_{\mathcal{L}} \to \mathscr{P}_{\mathcal{L}}$ and will also be denoted by $\mathbb{T}^*_e(\mathscr{P}_{\mathcal{L}})$ interchangeably later on. We construct a section

$$\tau^*_{\mathcal{L}} : B_{\mathcal{L}} \to B^*_{J,\mathcal{L}} \tag{2.7.3}$$

as follows. The non-degenerate bilinear form $(\,,)$ we fixed in 2.1 induces $\mathfrak{g} \simeq \mathfrak{g}^*$, which restricts to a map $\mathrm{Lie}\, I_x \to (\mathrm{Lie}\, I_x)^*$ for every $x \in \mathfrak{g}^{\mathrm{reg}}$. This map descends to give

$$\iota : \mathrm{Lie}\, J \to (\mathrm{Lie}\, J)^*, \tag{2.7.4}$$

which is $\mathbb{G}_m$-equivariant. We define $\tau^*_{\mathcal{L}}$ as the twist of $\mathfrak{c} \xrightarrow{\tau} \mathrm{Lie}\, J \xrightarrow{\iota} (\mathrm{Lie}\, J)^*$. As before, we omit the subscript $_{\mathcal{L}}$ if $\mathcal{L} = \omega$.

We give another interpretation of this map. Observe that the Kostant section κ induces the map

$$v_\kappa : \mathscr{P} \to \mathrm{Higgs}_G \to \mathrm{Bun}_G \times B$$

over B, and therefore we have the following.

$$\begin{tikzcd} T^*(\mathrm{Bun}_G) \times_{\mathrm{Bun}_G} \mathscr{P} \arrow[r, "(v_\kappa)_d"] \arrow[d, "(v_\kappa)_p"] & T^*(\mathscr{P}/B) \\ T^*\,\mathrm{Bun}_G & \end{tikzcd}$$

Lemma 2.7.2. *The map*

$$\mathscr{P} \xrightarrow{\kappa \times \mathrm{id}} T^*(\mathrm{Bun}_G) \times_{\mathrm{Bun}_G} \mathscr{P} \xrightarrow{(v_\kappa)_d} T^*(\mathscr{P}/B) \simeq \mathbb{T}^*_e\mathscr{P} \times_B \mathscr{P},$$

can be identified with

$$\mathscr{P} \xrightarrow{\mathrm{pr} \times \mathrm{id}} B \times \mathscr{P} \xrightarrow{\tau^* \times \mathrm{id}} \mathbb{T}^*_e\mathscr{P} \times_B \mathscr{P}.$$

Proof. For $b \in B$, we write the restriction of v_κ over b by $v_{\kappa,b} : \mathscr{P}_b \to \mathrm{Bun}_G$. We need to show that for $x \in \mathscr{P}_b$, the image of the point

$$\kappa(x) \in T^*_{v_{\kappa,b}(x)}\,\mathrm{Bun}_G \to T^*_x\mathscr{P}_b \simeq (\mathbb{T}^*_e\mathscr{P})_b$$

coincides with $\tau^*(b)$. Let E denote the G-bundle $v_{k,b}(x)$.

Observe that there is a universal G-torsor E_{univ} over $[\mathfrak{g}/G]$ given by $\mathfrak{g} \to [\mathfrak{g}/G]$, and that $\mathrm{ad}(E_{\mathrm{univ}}) \to [\mathfrak{g}/G]$ is canonically isomorphic to $[\mathfrak{g}/G] \times_{BG} [\mathfrak{g}/G] \xrightarrow{\mathrm{pr}_1} [\mathfrak{g}/G]$. The cotangent map

$$(v_{\kappa,b})_d : T^*_{v_{\kappa,b}(x)}\,\mathrm{Bun}_G \to T^*_x\mathscr{P}_b$$

is induced by twisting

$$\mathrm{kos}^*(\mathrm{ad}(E_{\mathrm{univ}}))^* \to (\mathrm{Lie}\, J)^*$$

by the $(G \times \mathbb{G}_m)$-torsor $(E \times \omega^\times)$. Therefore, it is enough to show that

$$\kappa(x) \in T^*_{v_{\kappa,b}(x)}\,\mathrm{Bun}_G = \Gamma(C, \mathfrak{g}_E \otimes \omega)$$

can be identified with the image of b under

$$\tau(b) \in \Gamma(C, \mathrm{Lie}\, J_b \otimes \omega) \to \Gamma(C, \mathfrak{g}_E \otimes \omega).$$

Let us consider the universal situation. Therefore, we need to show that

$$\mathfrak{c} \xrightarrow{\tau} \mathrm{Lie}\, J \to \mathrm{kos}^*\mathrm{ad}(E_{\mathrm{univ}}) \simeq \mathfrak{c} \times_{BG} [\mathfrak{g}/G]$$

is the same as

$$\mathfrak{c} \xrightarrow{\mathrm{id}\times\mathrm{kos}} \mathfrak{c} \times_{BG} [\mathfrak{g}/G].$$

However, the composition

$$[\mathfrak{g}/G] \xrightarrow{[\chi]^*(\tau)} [\chi]^*\mathrm{Lie}\, J \to \mathrm{ad}(E_{\mathrm{univ}}) \simeq [\mathfrak{g}/G] \times_{BG} [\mathfrak{g}/G]$$

restricts to a map $[\mathfrak{g}^{\mathrm{reg}}/G] \to [\mathfrak{g}^{\mathrm{reg}}/G] \times_{BG} [\mathfrak{g}/G]$, which is easily checked to be the diagonal map using the definition of τ. By pulling back this identification along $\mathrm{kos} : \mathfrak{c} \to [\mathfrak{g}^{\mathrm{reg}}/G]$, we obtain the claim. □

By the similar argument, we have the following lemma, which will be used in § 4. Let $\mathrm{j}^1 : \boldsymbol{\pi}^* J \to T \times \mathfrak{t}$ be the map in Proposition 2.5.1, and let

$$d\mathrm{j}^1 : \boldsymbol{\pi}^*\mathrm{Lie}\, J \to \mathfrak{t} \times \mathfrak{t} \tag{2.7.5}$$

denote its differential. Consider the pullback $\boldsymbol{\pi}^*\tau : \mathfrak{t} \to \pi^*\mathrm{Lie}\, J$ of $\tau : \mathfrak{c} \to \mathrm{Lie}\, J$ along $\boldsymbol{\pi} : \mathfrak{t} \to \mathfrak{c}$.

Lemma 2.7.3. *The composition*

$$\delta : \mathfrak{t} \xrightarrow{\boldsymbol{\pi}^*\tau} \boldsymbol{\pi}^*\mathrm{Lie}\, J \xrightarrow{d\mathrm{j}^1} \mathfrak{t} \times \mathfrak{t}$$

is equal to the diagonal map $\Delta : \mathfrak{t} \to \mathfrak{t} \times \mathfrak{t}$.

Proof. For an embedding $\mathfrak{t} \subset \mathfrak{g}$, the restriction of $d\mathrm{j}^1$ to $\mathfrak{t}^{\mathrm{reg}} = \mathfrak{t} \cap \mathfrak{g}^{\mathrm{reg}}$ is just the restriction to $\mathfrak{t}^{\mathrm{reg}}$ of the isomorphism $\mathrm{Lie}\, J|_{\mathfrak{g}^{\mathrm{reg}}} \simeq \mathrm{Lie}\,(I|_{\mathfrak{g}^{\mathrm{reg}}})$. This follows from the construction of j^1 as in [Ngô10, Proposition 2.4.2]. Therefore, the restriction of δ to $\mathfrak{t}^{\mathrm{reg}}$ is just the diagonal map. The lemma then follows. □

3. Classical duality

In this section, we fix a smooth projective curve C over k and a line bundle $\mathcal{L}$ on C such that $\deg \mathcal{L} > 0$. Except § 3.7, we also fix a connected reductive group G over k. We assume that $p = \mathrm{char}\, k$ does not divide the order of the Weyl group of G. We show that the $\breve{\mathscr{P}}_{\mathcal{L}} \simeq \mathscr{P}_{\mathcal{L}}^{\vee}$ as Picard stacks over B^0. Note that this duality for $k = \mathbb{C}$ is the main theorem of [DP12] (for $G = \mathrm{SL}_n$, see [HT03]). However, as mentioned by the authors, transcendental arguments are used in [DP12] in an essential way, and therefore cannot be applied directly to our situation. Our argument works for any algebraically closed field k of characteristic zero or p with $p \nmid |\mathrm{W}|$.

In fact, it is not hard to construct a canonical isogeny $\mathfrak{D}_{\mathrm{cl}}$ between $\breve{\mathscr{P}}_{\mathcal{L}}$ and $\mathscr{P}_{\mathcal{L}}^{\vee}$. If the adjoint group of G does not contain a simple factor of type B or C, then to show that $\mathfrak{D}_{\mathrm{cl}}$ is an isomorphism is relatively easy. It is to show that $\mathfrak{D}_{\mathrm{cl}}$ is an isomorphism in the remaining cases that some complicated calculations are needed.

Observe in this section, we do not need to assume that $\mathcal{L} = \omega_C$. We only need the assumption that deg $\mathcal{L}$ is positive. However, to simplify the notations, we still omit the subscript ${}_{\mathcal{L}}$.

3.1 Galois description of $\mathscr{P}$

We first introduce several auxiliary Picard stacks.

Let $\widetilde{C} \to B$ be the universal cameral curve. There is a natural action of W on $\widetilde{C}$. For a T-torsor E_T on $\widetilde{C}$, and an element $w \in \mathrm{W}$, there are two ways to produce a new T-torsor. Namely, the first is via the pullback $w^*E_T = \widetilde{C} \times_{w,\widetilde{C}} E_T$, and the second is via the induction $E_T \times^{T,w} T$. We denote

$$w(E_T) = ((w^{-1})^*E_T) \times^{T,w} T.$$

Clearly, the assignment $E_T \mapsto w(E_T)$ defines an action of W on $\mathrm{Bun}_T(\widetilde{C}/B)$, i.e., for every $w, w' \in \mathrm{W}$, there is a canonical isomorphism $w(w'(E_T)) \simeq (ww')(E_T)$ satisfying the usual cocycle conditions.

Example 3.1.1. Let us describe $w(E_T)$ more explicitly in the case $G = \mathrm{SL}_2$. Let s be the unique non-trivial element in the Weyl group, acting on the cameral curve $s : \widetilde{C}_b \to \widetilde{C}_b$. If we identify $T = \mathbb{G}_m$-torsors with invertible sheaves $\mathcal{L}$, then

$$s(\mathcal{L}) = s^*\mathcal{L}^{-1}.$$

Let $\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)$ (or $\mathrm{Bun}_T^{\mathrm{W}}$ for simplicity) denote the Picard stack of strongly W-equivariant T-torsors on $\widetilde{C}/B$. By definition, for a B-scheme S, $\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)(S)$ is the groupoid of $(E_T, \{\gamma_w, w \in W\})$, where E_T is a T-torsor on $\widetilde{C}_S$, and $\gamma_w : w(E_T) \simeq E_T$ is an isomorphism, satisfying the natural compatibility conditions. Another way to formulate these compatibility conditions is provided in [DG02]. Namely, for a T-torsor E_T, let $\mathrm{Aut}_{\mathrm{W}}(E_T)$ be the group consisting of (w, γ_w), where $w \in \mathrm{W}$ and $\gamma_w : w(E_T) \simeq E_T$ is an isomorphism. Then there is a natural projection $\mathrm{Aut}_{\mathrm{W}}(E_T) \to \mathrm{W}$. Then an object of $\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)(S)$ is a pair (E_T, γ), where $\gamma : \mathrm{W} \to \mathrm{Aut}_{\mathrm{W}}(E_T)$ is a splitting of the projection.

For later purpose, it is worthwhile to give another description of $\mathrm{Bun}_T^{\mathrm{W}}$. Namely, there is a non-constant group scheme $\mathcal{T} = \widetilde{C} \times^{\mathrm{W}} T$ on the stack $[\widetilde{C}/\mathrm{W}]$. Then the pullback functor induces an isomorphism from the stack $\mathrm{Bun}_{\mathcal{T}}$ of $\mathcal{T}$-torsors on $[\widetilde{C}/W]$ to $\mathrm{Bun}_T^{\mathrm{W}}$.

In [DG02], a Galois description of $\mathscr{P}$ in terms of $\mathrm{Bun}_T^{\mathrm{W}}$ is given. We here refine their description.

Let $\mathscr{P}^1$ be the Picard stack over B classifying J^1-torsors on $C \times B$. First, we claim that there is a canonical morphism

$$j^{1,\mathscr{P}} : \mathscr{P}^1 \to \mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B). \tag{3.1.1}$$

To construct $j^{1,\mathscr{P}}$, recall that $J^1 = (\pi_*(T \times \widetilde{C}))^W$, where $\pi : \widetilde{C} \to C \times B$ is the projection, and therefore, for any J^1-torsor E_{J^1} on $C \times S$ (where $b : S \to B$ is a test scheme), one can form a T-torsor on $\widetilde{C}_S$ by

$$E_T := \pi^*E_{J^1} \times^{\pi^*J^1} T. \tag{3.1.2}$$

Clearly, E_T carries on a strongly W-equivariant structure γ, and $j^1(E_{J^1}) = (E_T, \gamma)$ defines the morphism $j^{1,\mathscr{P}}$.

The morphism $j^{1,\mathscr{P}}$, in general, is not an isomorphism. Let us describe the image. Let $\alpha \in \Phi$ be a root and let $i_\alpha : \widetilde{C}_\alpha \to \widetilde{C}$ be the inclusion of the fixed point subscheme of the reflection s_α. Let $T_\alpha = T/(s_\alpha - 1)$ be the torus of coinvariants of the reflection s_α. Then $s_\alpha(E_T)|_{\widetilde{C}_\alpha} \times^T T_\alpha$

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is canonically isomorphic to $E_T|_{\widetilde{C}_\alpha} \times^T T_\alpha$ and therefore $\gamma_{s_\alpha}|_{\widetilde{C}_\alpha}$ induces an automorphism of the T_α-torsor $E_T \times^T T_\alpha$. In other words, there is a natural map

$$r = \prod_{\alpha \in \Phi} r_\alpha : \mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B) \to \left(\prod_{\alpha \in \Phi} \mathrm{Res}_{\widetilde{C}_\alpha/B}(T_\alpha \times \widetilde{C}_\alpha) \right)^{\mathrm{W}}.$$

It is easy to see that $r \circ j^{1,\mathscr{P}}$ is trivial, and one can show the following.

LEMMA 3.1.2. *We have $\mathscr{P}^1 \simeq \ker r$. In other words, $\mathscr{P}^1(S)$ consists of those strongly W-equivariant T-torsors (E_T, γ) such that the induced automorphism of $E_T \times^T T_\alpha|_{\widetilde{C}_\alpha}$ is trivial for every $\alpha \in \Phi$.*

Proof. We shall show that every strongly W-equivariant T-torsor (E_T, γ) such that $r(E_T, \gamma) = 1$ is Zariski locally on $\widetilde{C}$ isomorphic to the trivial one, i.e., the trivial T-torsor together with the canonical W-equivariance structure. If this is the case, then the inverse map from $\ker r \to \mathscr{P}^1$ is given as follows. For every strongly W-equivariant T-torsor (E_T, γ), $\pi_* E_T$ carries on an action of W. Namely, let $x : S \to C$ be a point and $m : S \times_C \widetilde{C}_b \to E_T$ be a point of $\pi_* E_T$ over x. Then $w(m)$ is the point of $\pi_* E_T$ over x given by

$$S \times_C \widetilde{C}_b \overset{1 \times w^{-1}}{\to} S \times_C \widetilde{C}_b \overset{w^{-1}(m)}{\to} (w^{-1})^* E_T \to w(E_T) \overset{s(w)}{\to} E_T.$$

This W-action on $\pi_* E_T$ is compatible with the action of $\pi_*(T \times \widetilde{C})$ in the sense that $w(mt) = w(m)w(t)$. Now let $E_{J^1} = (\pi_* E_T)^W$, then as (E, γ) is locally isomorphic to the trivial one, E_{J^1} is locally isomorphic to J^1, and therefore is a J^1-torsor on C.

To prove the local triviality, we follow the argument as in [DG02, Proposition 16.4]. One reduces to prove the statement for a neighborhood around a point $x \in \cap_\alpha \widetilde{C}_\alpha$. By replacing $\widetilde{C}$ by the local ring around x, one can assume that E_T is trivial. Pick up a trivialization, then the W-equivariance structure on E_T amounts to a 1-cocycle $W \to T(\widetilde{C})$. By evaluating $T(\widetilde{C})$ at the unique closed point x, there is a short exact sequence $1 \to K \to T(\widetilde{C}) \to T(k) \to 1$. The condition $r(E_T, \gamma) = 1$ would mean that the cocycle takes value in K. Since there exists a filtration on K, such that the associated graded is an $\mathbb{F}_p$-vector space and $p \nmid |\mathrm{W}|$, this cocycle is trivial. $\square$

Recall that in [DG02, Ngô06], an open embedding $J \to J^1$ is constructed. To describe the cokernel, we need some notations. Let $\breve{\alpha} \in \breve{\Phi}$ be a coroot. Let

$$\mu_{\breve{\alpha}} := \ker(\breve{\alpha} : \mathbb{G}_m \to T).$$

This is either trivial, or μ_2, depending on whether $\breve{\alpha}$ is primitive or not. Let $\mu_{\breve{\alpha}} \times \widetilde{C}_\alpha$ be the constant group scheme over $\widetilde{C}_\alpha$, regarded as a sheaf of groups over $\widetilde{C}_\alpha$, and let $(i_\alpha)_*(\mu_{\breve{\alpha}} \times \widetilde{C}_\alpha)$ be its push forward to $\widetilde{C}$. Now, the result of [DG02, §§ 11 and 12] can be reformulated as follows: there is a natural exact sequence of sheaves of groups on C.

$$1 \to J \to J^1 \to \pi_* \left(\bigoplus_{\alpha \in \Phi} (i_\alpha)_*(\mu_{\breve{\alpha}} \times \widetilde{C}_\alpha) \right)^{\mathrm{W}} \to 1. \tag{3.1.3}$$

As a result, we obtain a short exact sequence of Picard stacks (see § A.2)

$$1 \to \left(\prod_{\alpha \in \Phi} \mathrm{Res}_{\widetilde{C}_\alpha/B}(\mu_{\breve{\alpha}} \times \widetilde{C}_\alpha) \right)^{\mathrm{W}} \to \mathscr{P} \to \mathscr{P}^1 \to 1. \tag{3.1.4}$$

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To simplify the notation, we will denote $\mathrm{Res}_{\widetilde{C}_\alpha/B}(\mu_{\check{\alpha}}\times\widetilde{C}_\alpha)$ by $\mu_{\check{\alpha}}(\widetilde{C}_\alpha)$ in what follows.

Consider the composition

$$j:\mathscr{P}\to\mathscr{P}^1\to\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B).$$

Combining Lemma 3.1.2 and (3.1.4), we recover a description of $\mathscr{P}$ in terms of $\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)$ as given in [DG02, § 16.3]. Namely, given a strongly W-equivariant T-torsor (E_T,γ), one obtains a canonical trivialization

$$E_T^{\check{\alpha}\circ\alpha}:=(E_T|_{\widetilde{C}_\alpha})\times^{T,\alpha}\mathbb{G}_m\times^{\mathbb{G}_m,\check{\alpha}}T\simeq E_T^0|_{\widetilde{C}_\alpha},\tag{3.1.5}$$

given by $(E_T|_{\widetilde{C}_\alpha})\times^{T,\alpha}\mathbb{G}_m\times^{\mathbb{G}_m,\check{\alpha}}T\simeq E_T|_{\widetilde{C}_\alpha}\otimes s_\alpha(E_T^{-1})|_{\widetilde{C}_\alpha}$. The condition that $r_\alpha(E,\gamma)=1$ is equivalent to the condition that (3.1.5) comes from a trivialization

$$c_\alpha:E_T^\alpha:=(E_T|_{\widetilde{C}_\alpha})\times^{T,\alpha}\mathbb{G}_m\simeq\mathbb{G}_m\times\widetilde{C}_\alpha.\tag{3.1.6}$$

In addition, the set of all such c_α form a $\mu_{\check{\alpha}}$-torsor. Consider the following Picard stack $\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)^+$: for any B-scheme S, its S-points form the Picard groupoid of triples

$$\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}_S)^+:=(E_T,\gamma,c_\alpha,\alpha\in\Phi),\tag{3.1.7}$$

where (E_T,γ) is a strongly W-equivariant T-torsor on $\widetilde{C}_S$, and $c_\alpha:(E_T|_{\widetilde{C}_\alpha})\times^{T,\alpha}\mathbb{G}_m\simeq\mathbb{G}_m\times\widetilde{C}_\alpha$ is a trivialization, which induces (3.1.5) and is compatible with the W-equivariant structure. We call those trivializations $\{c_\alpha\}_{\alpha\in\Phi}$ a $+$-structure on (E_T,γ). Note that, by Lemma 3.1.2, we have the following short exact sequence of Picard stacks:

$$1\to\left(\prod_{\alpha\in\Phi}\mathrm{Res}_{\widetilde{C}_\alpha/B}(\mu_{\check{\alpha}}\times\widetilde{C}_\alpha)\right)^{\mathrm{W}}\to\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)^+\to\mathscr{P}^1\to 1.\tag{3.1.8}$$

Lemma 3.1.3 [DG02, Proposition 16.4]. *We have $\mathscr{P}\simeq\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)^+$.*

Proof. Indeed, the exact sequence (3.1.3) implies that, for any J-torsor $E_J\in\mathscr{P}$ the image $j(E_J)\in\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)$ carries a canonical $+$-structure. This defines a morphism $\mathscr{P}\to\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)^+$ and one can check that it is compatible with the short exact sequences (3.1.4) and (3.1.8). The lemma follows. □

Here is an application of the above discussion. Observe there is the norm map

$$\mathrm{Nm}:\mathrm{Bun}_T(\widetilde{C}/B)\to\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B),\quad E_T\mapsto\left(\bigotimes_{w\in W}w(E_T),\gamma_{\mathrm{can}}\right).$$

We claim that Nm admits a canonical lifting

$$\mathrm{Nm}^{\mathscr{P}}:\mathrm{Bun}_T(\widetilde{C}/B)\to\mathscr{P}.\tag{3.1.9}$$

To show this, we need to exhibit a canonical trivialization

$$c_\alpha:\bigotimes_{w\in\mathrm{W}}w(E_T)|_{\widetilde{C}_\alpha}\times^{T,\alpha}\mathbb{G}_m\simeq\mathbb{G}_m\times\widetilde{C}_\alpha$$

compatible with the strongly W-equivariant structure. However, for any T-torsor E_T, there is a canonical isomorphism $(E_T|_{\widetilde{C}_\alpha}\otimes s_\alpha(E_T)|_{\widetilde{C}_\alpha})\times^{T,\alpha}\mathbb{G}_m\simeq\mathbb{G}_m\times\widetilde{C}_\alpha$, and therefore, we obtain c_α by writing

$$\bigotimes_{w\in W}w(E_T)|_{\widetilde{C}_\alpha}\times^{T,\alpha}\mathbb{G}_m\simeq\bigotimes_{w\in s_\alpha\backslash W}(w(E_T)|_{\widetilde{C}_\alpha}\otimes s_\alpha w(E_T)|_{\widetilde{C}_\alpha})\times^{T,\alpha}\mathbb{G}_m,$$

where $s_\alpha\backslash W$ denotes the quotient of W by the subgroup generated by s_α. The compatibility of the collection $\{c_\alpha\}$ with the W-equivariant structure is clear.

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3.2 Galois description of $\mathscr{P}$-torsors

The above description of $\mathscr{P}$ in terms of $\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)$ can be generalized as follows. Let $\mathscr{D}$ be a J-gerbe on $C \times B$. Similarly to (3.1.2), we define

$$\mathscr{D}_T := (\pi^*\mathscr{D})^{j^1}$$

as the T-gerbe on $\widetilde{C}$ induced from $\mathscr{D}$ using maps $\pi : \widetilde{C} \to C \times B$ and $j^1 : \pi^* J \to T \times \widetilde{C}$ (see § A.5 and § A.6 for the notion of gerbes and functors between them). Since the map j^1 is W-equivariant the gerbe $\mathscr{D}_T$ is strongly W-equivariant. Equivalently, this means that $\mathscr{D}_T$ descends to a $\mathcal{T}$-gerbe on $[\widetilde{C}/\mathrm{W}]$.

Let $\mathscr{T}_{\mathscr{D}}$ be the stack of splittings of $\mathscr{D}$ over B. By definition, for every $S \to B$, $\mathscr{T}_{\mathscr{D}}(S)$ is the groupoid of the splittings of the gerbe $\mathscr{D}|_{C\times S}$. This is a (pseudo) $\mathscr{P}$-torsor. On the other hand, let $\mathscr{T}_{\mathscr{D}_T}^{\mathrm{W}}$ denote the stack of strongly W-equivariant splittings of $\mathscr{D}_T$, i.e., $\mathscr{T}_{\mathscr{D}_T}^{\mathrm{W}}(S)$ is the groupoid of the splittings of $\mathscr{D}_T|_{[\widetilde{C}/\mathrm{W}]\times_B S}$. Our goal is to give a description of $\mathscr{T}_{\mathscr{D}}$ in terms of $\mathscr{T}_{\mathscr{D}_T}^{\mathrm{W}}$.

Let $\alpha \in \Phi$. Similarly to E_T^{α} and $E_T^{\check{\alpha}\circ\alpha}$ as defined in (3.1.5) and (3.1.6), let $\mathscr{D}_T^{\alpha}$, $\mathscr{D}_T^{\check{\alpha}\circ\alpha}$ denote the restrictions to $\widetilde{C}_\alpha$ of the $\mathbb{G}_m$- and T-gerbes on $\widetilde{C}$ induced from $\mathscr{D}_T$ using the maps $\alpha : T \to \mathbb{G}_m$ and $\check{\alpha}\circ\alpha : T \to T$ respectively. The strongly W-equivariant structure on $\mathscr{D}_T$ implies that the T-gerbe $\mathscr{D}_T^{\check{\alpha}\circ\alpha}$ has a canonical splitting F_α^0. Moreover, by a similar argument in § 3.1, one can show that: (i) there is a canonical splitting E_α^0 of the $\mathbb{G}_m$-gerbe $\mathscr{D}_T^{\alpha}$, which induces F_α^0 via the canonical map $\mathscr{D}_T^{\alpha} \to \mathscr{D}_T^{\check{\alpha}\circ\alpha}$ and: (ii) for any strongly W-equivariant splitting (E, γ) of $\mathscr{D}_T$ there is a canonical isomorphism of splittings

$$E^{\check{\alpha}\circ\alpha}|_{\widetilde{C}_\alpha} \simeq F_\alpha^0, \tag{3.2.1}$$

where $E^{\check{\alpha}\circ\alpha}$ is the splitting of $\mathscr{D}_T^{\check{\alpha}\circ\alpha}$ induces by E via the canonical map $\mathscr{D}_T^{\alpha} \to \mathscr{D}_T^{\check{\alpha}\circ\alpha}$. We define $\mathscr{T}_{\mathscr{D}_T}^{\mathrm{W},+}$ as the stack over B whose S-points consist of

$$\mathscr{T}_{\mathscr{D}_T}^{\mathrm{W},+}(S) := (E, \gamma, t_\alpha, \alpha \in \Phi),$$

where (E, γ) is a strongly W-equivariant splittings of $\mathscr{D}_T$ and

$$t_\alpha : E^{\alpha}|_{\widetilde{C}_\alpha} \simeq E_\alpha^0$$

is an isomorphism of splittings of $\mathscr{D}_T^{\alpha}$, which induces (3.2.1) and is compatible with the W-equivariant structure. It is clear that $\mathscr{T}_{\mathscr{D}_T}^{\mathrm{W},+}$ is a $\mathscr{P} = \mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)^+$-torsor.

LEMMA 3.2.1. *There is a canonical isomorphism of $\mathscr{P}$-torsors $\mathscr{T}_{\mathscr{D}} \simeq \mathscr{T}_{\mathscr{D}_T}^{\mathrm{W},+}$.*

Proof. Let $E \in \mathscr{T}_{\mathscr{D}}$ be a splitting of $\mathscr{D}$. Then $E_T := (\pi^*(E))^{j^1}$ defines a splitting of $\mathscr{D}_T$. Since both maps j^1 and π are W-equivariant the splitting E_T has a canonical W-equivariant structure, which we denote by γ. Moreover, by the same reasoning as in § 3.1, there is a canonical isomorphism of splittings $t_\alpha : E_T^{\alpha}|_{\widetilde{C}_\alpha} \simeq E_\alpha^0$ such that the induced isomorphism $E_T^{\check{\alpha}\circ\alpha}|_{\widetilde{C}_\alpha} \simeq (E_\alpha^0)^{\alpha} \simeq F_\alpha^0$ is equal to the one coming from the W-equivariant structure γ. The assignment $E \to (E_T, \gamma, t_\alpha, \alpha \in \Phi)$ defines a morphism $\mathscr{T}_{\mathscr{D}} \to \mathscr{T}_{\mathscr{D}_T}^{\mathrm{W},+}$, which is compatible with their $\mathscr{P}$-torsor structures and hence is an isomorphism. $\square$

3.3 The Abel–Jacobi map

From now on till the end of this section, we restrict to the open subset B^0 of the Hitchin base. To simplify the notations, we use B to denote B^0 unless specified. Recall from Proposition 2.6.1 that the cameral curve $\widetilde{C}$ is smooth over B^0.

Let

$$\mathrm{AJ} : \widetilde{C} \times \mathbb{X}_\bullet(T) \to \mathrm{Bun}_T(\widetilde{C}/B)$$

be the Abel–Jacobi map given by $(x, \check{\lambda}) \mapsto \mathcal{O}(\check{\lambda}x) := \mathcal{O}(x) \times^{\mathbb{G}_m,\check{\lambda}} T$. By composition with $\mathrm{Nm}^{\mathscr{P}}$, we obtain a morphism

$$\mathrm{AJ}^{\mathscr{P}} : \widetilde{C} \times \mathbb{X}_\bullet(T) \to \mathscr{P}.$$

It is W-equivariant, where W acts on $\widetilde{C}\times\mathbb{X}_\bullet(T)$ diagonally and on $\mathscr{P}$ trivially, and is commutative and multiplicative with respect to the group structures on $\mathbb{X}_\bullet(T)$ and on $\mathscr{P}$. Observe that for any $x \in \widetilde{C}_\alpha$, $\mathrm{AJ}^{\mathscr{P}}(x, \check{\alpha})$ is the unit in $\mathscr{P}$. This follows from

$$\bigotimes_{w\in W} w\mathcal{O}(\check{\alpha}x) \simeq \bigotimes_{w\in W/s_\alpha} w\mathcal{O}(\check{\alpha}x + s_\alpha(\check{\alpha})x)$$

being canonically trivialized, and the trivialization is compatible with the W-equivariant structure. Here, as before, W/s_α is the quotient of W by the subgroup generated by s_α.

By pulling back the line bundles, we thus obtain

$$(\mathrm{AJ}^{\mathscr{P}})^\vee : \mathscr{P}^\vee \to \mathrm{Pic}^{\,m}(\widetilde{C} \times \mathbb{X}_\bullet(T))^{\mathrm{W}},$$

where $\mathrm{Pic}^{\,m}(\widetilde{C} \times \mathbb{X}_\bullet(T))^{\mathrm{W}}$ denotes the Picard stack over B of W-equivariant line bundles on $\widetilde{C} \times \mathbb{X}_\bullet(T)$ which are multiplicative with respect to $\mathbb{X}_\bullet(T)$. Observe that there is the canonical isomorphism $\mathrm{Bun}_{\check{T}}^{\mathrm{W}}(\widetilde{C}/B) \to \mathrm{Pic}^{\,m}(\widetilde{C} \times \mathbb{X}_\bullet(T))^{\mathrm{W}}$ given by $(E_{\check{T}}, \gamma) \mapsto \mathcal{L}$, where $\mathcal{L}|_{(x,\check{\lambda})} = E_{\check{T}}^{\lambda}|_x$. Therefore, we can regard $(\mathrm{AJ}^{\mathscr{P}})^\vee$ as a morphism

$$(\mathrm{AJ}^{\mathscr{P}})^\vee : \mathscr{P}^\vee \to \mathrm{Bun}_{\check{T}}^{\mathrm{W}}(\widetilde{C}/B).$$

We claim that $(\mathrm{AJ}^{\mathscr{P}})^\vee$ canonically lifts to a morphism

$$\mathfrak{D}_{\mathrm{cl}} : \mathscr{P}^\vee \to \check{\mathscr{P}}.$$

Let $\mathcal{L}$ be a multiplicative line bundle on $\mathscr{P}$. We thus need to show that

$$(\mathrm{AJ}^{\mathscr{P}})^*\mathcal{L}|_{(\widetilde{C}_\alpha,\check{\alpha})}$$

admits a canonical trivialization, which is compatible with the W-equivariance structure. However, this follows from $\mathrm{AJ}^{\mathscr{P}}((x, \check{\alpha}))$ is the unit of $\mathscr{P}$ and a multiplicative line bundle on $\mathscr{P}$ is canonically trivialized over the unit. To summarize, we have constructed the following commutative diagram.

$$\begin{tikzcd} \mathscr{P}^\vee \arrow[rr, "\mathfrak{D}_{\mathrm{cl}}"] \arrow[dr, "(\mathrm{AJ}^{\mathscr{P}})^\vee"'] & & \check{\mathscr{P}} \arrow[dl, "\check{j}"] \\ & \mathrm{Bun}_{\check{T}}^{\mathrm{W}} & \end{tikzcd} \tag{3.3.1}$$

Now, the classical duality theorem reads as follows.

Theorem 3.3.1. *The morphism $\mathfrak{D}_{\mathrm{cl}}$ is an isomorphism.*

The proof of this theorem occupies §§ 3.4–3.6 below.

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3.4 First reductions

We first show that $\mathfrak{D}_{\mathrm{cl}}$ induces an isomorphism

$$\pi_0(\mathfrak{D}_{\mathrm{cl}}):\pi_0(\mathscr{P}^\vee)\to\pi_0(\breve{\mathscr{P}}).$$

For any S-point $b\in B^0$, $\mathscr{P}_b$ is a Beilinson 1-motive (Appendix A). We have

$$\underline{\mathrm{Aut}}(e)\simeq \mathrm{H}^0(C,J_b),\quad \pi_0(\mathscr{P}_b)=\mathscr{P}_b/W_1\mathscr{P}_b.$$

Observe that

$$\mathrm{H}^0(C,J_b)\simeq\ker\biggl(T^{\mathrm{W}}\to\biggl(\prod_{\alpha\in\Phi}\mathrm{Res}_{\widetilde{C}_\alpha/b}(\mu_{\breve{\alpha}}\times\widetilde{C}_\alpha)\biggr)^{\mathrm{W}}\biggr)=Z(G).$$

By Corollary A.4.3

$$\pi_0(\mathscr{P}^\vee)\simeq(\mathrm{Aut}_{\mathscr{P}}(e))^*.$$

Let us also recall the description of $\pi_0(\mathscr{P})$ as given in [Ngô10, §§ 4.10 and 5.5]. As we restrict $\mathscr{P}$ to B^0, the answer is very simple. Namely, the Abel–Jacobi map

$$\mathrm{AJ}^{\mathscr{P}}:\widetilde{C}\times\mathbb{X}_\bullet(T)\to\mathscr{P}$$

induces a surjective map

$$\pi_0(\widetilde{C}\times\mathbb{X}_\bullet(T))\simeq\mathbb{X}_\bullet(T)\twoheadrightarrow\pi_0(\mathscr{P}),$$

which induces

$$\pi_0(\mathscr{P})^*\simeq Z(\breve{G})\subset\breve{T}^{\mathrm{W}}.$$

Therefore, as abstract groups, $\pi_0(\mathscr{P}^\vee)\simeq\pi_0(\breve{\mathscr{P}})$.

Since $\pi_0(\mathscr{P}^\vee)\simeq\pi_0(\breve{\mathscr{P}})$ are finitely generated abelian groups and are isomorphic abstractly, to show that $\pi_0(\mathfrak{D}_{\mathrm{cl}})$ is an isomorphism, it is enough to show the following.

Lemma 3.4.1. *The induced map $\pi_0(\mathfrak{D}_{\mathrm{cl}})$ is surjective.*

Proof. According to the above description, it is enough to construct a morphism $\widetilde{C}\times\mathbb{X}^\bullet(T)\to\mathscr{P}^\vee$ making the following diagram is commutative.

$$\begin{tikzcd} & \widetilde{C}\times\mathbb{X}^\bullet(T) \arrow[dl] \arrow[dr, "\mathrm{AJ}^{\breve{\mathscr{P}}}"] & \\ \mathscr{P}^\vee \arrow[rr, "\mathfrak{D}_{\mathrm{cl}}"] & & \breve{\mathscr{P}} \end{tikzcd}$$

To this goal, observe that there is the universal line bundle $\mathscr{L}_{\mathrm{univ}}$ on $(\widetilde{C}\times\mathbb{X}^\bullet(T))\times\mathrm{Bun}_T$. Then the pullback of this line bundle to $(\widetilde{C}\times\mathbb{X}^\bullet(T))\times\mathscr{P}$ gives rise to the desired map. The commutativity of this diagram is an easy exercise. □

Next, we see that

$$W_0(\mathfrak{D}_{\mathrm{cl}}):W_0\mathscr{P}^\vee\to W_0\breve{\mathscr{P}}$$

is an isomorphism. Indeed, we can construct $\mathrm{AJ}^{\breve{\mathscr{P}}}:\widetilde{C}\times\mathbb{X}^\bullet(T)\to\breve{\mathscr{P}}$, and therefore $\breve{\mathfrak{D}}_{\mathrm{cl}}:\breve{\mathscr{P}}^\vee\to\mathscr{P}$. By the same argument, it induces an isomorphism $\pi_0(\breve{\mathfrak{D}}_{\mathrm{cl}}):\pi_0(\breve{\mathscr{P}}^\vee)\to\pi_0(\mathscr{P})$. It is easy to check that $\breve{\mathfrak{D}}_{\mathrm{cl}}=\mathfrak{D}_{\mathrm{cl}}^\vee$, and therefore $W_0(\mathfrak{D}_{\mathrm{cl}})$ is also an isomorphism.

Therefore, it is enough to show that $D_{\mathrm{cl}}:P^\vee\to\breve{P}$ is an isomorphism, where P (respectively $\breve{P}$) is the neutral connected component of the coarse moduli space of $\mathscr{P}$ (respectively $\breve{\mathscr{P}}$), and D_{cl} is the map induced by $\mathfrak{D}_{\mathrm{cl}}$. We can prove this fiberwise, and therefore we fix $b\in B(k)$. However, to simplify the notation, in the following discussion we write $\widetilde{C},\mathscr{P}$ instead of $\widetilde{C}_b,\mathscr{P}_b$, etc.

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3.5 The calculation of the coarse moduli

We introduce a few more notations. Let $\mathscr{P}^0$ be the Picard stack of J^0-torsors on C, and let P^0 (respectively P^1) be the neutral connected components of the coarse moduli space of $\mathscr{P}^0$ (respectively $\mathscr{P}^1$).

We first understand P^1. Let Jac denote the Jacobi variety of $\widetilde{C}$. Then $\mathrm{Jac}\otimes\mathbb{X}_\bullet$ is the neutral connected component of the coarse moduli space of Bun_T.

Lemma 3.5.1. *The map $P^1 \to \mathrm{Jac}\otimes\mathbb{X}_\bullet$ is an embedding, and P^1 can be identified with $(\mathrm{Jac}\otimes\mathbb{X}_\bullet)^{\mathrm{W},0}$, the neutral connected component of the* W*-fixed point subscheme of* $\mathrm{Jac}\otimes\mathbb{X}_\bullet$.

Proof. We first show that $P^1 \to \mathrm{Jac}\otimes\mathbb{X}_\bullet$ is injective at the level of k-points. Indeed, up to isomorphism, the strongly W-equivariant structures on a trivializable T-torsor on $\widetilde{C}$ are classified by $\mathrm{H}^1(\mathrm{W}, T(k))$. By Lemma 3.1.2, the kernel of $P^1 \to \mathrm{Jac}\otimes\mathbb{X}_\bullet$ can be identified with the kernel of the natural map

$$\mathrm{H}^1(\mathrm{W}, T(k)) \to \bigoplus_{\widetilde{C}_\alpha} T_\alpha(\widetilde{C}_\alpha).$$

Therefore, it is enough to show that this latter map is injective. Over B^0, $\widetilde{C}_\alpha$ is non-empty for every root α.[3] Then the injectivity is a consequence of the following lemma applied to $M = T(k)$.

Lemma 3.5.2. *Let M be a* W*-module satisfying the following condition: for some (and therefore any) choice of a set of simple roots $\{\alpha_1, \dots, \alpha_l\}$, the natural map*

$$M \to \prod_{i=1}^{l} (1 - s_{\alpha_i})M, \quad m \mapsto ((1 - s_{\alpha_1})m, \dots, (1 - s_{\alpha_l})m)$$

is surjective. Then the natural map

$$\mathrm{H}^1(W, M) \to \prod_{1\leqslant i\leqslant s} M/(1 - s_{\beta_i})M, \quad [c] \mapsto \prod_{1\leqslant i\leqslant s} (c(s_{\beta_i}) \bmod (1 - s_{\beta_i})M)$$

is injective for any choice of a set $\{\beta_1, \dots, \beta_s\} \subset \Phi$ of representatives of Φ/W.

Proof. Let $c : W \to M$ be a cocycle. It follows from the cocycle condition that if $c(s_{\beta_i}) \in (1 - s_{\beta_i})M$, then $c(s_{w(\beta_i)}) \in (1 - s_{w(\beta_i)})M$. Therefore, a class $[c]$ is in the kernel of the map in the lemma only if $c(s_{\alpha_i}) \in (1 - s_{\alpha_i})M$ for a set of simple roots $\{\alpha_1, \dots, \alpha_l\}$. However, by our assumption of M, there exists $m \in M$ such that $c(s_{\alpha_i}) = (1 - s_{\alpha_i})m$ for all $1 \leqslant i \leqslant l$. Then using the cocycle condition, one can show by induction on the length of w that $c(w) = (1 - w)m$ for every $w \in \mathrm{W}$. This means, however, that c is a coboundary. □

To complete the proof, observe that the restriction of the norm map

$$\mathrm{Nm} : \mathrm{Jac}\otimes\mathbb{X}_\bullet \to P \to P^1 \to (\mathrm{Jac}\otimes\mathbb{X}_\bullet)^{\mathrm{W}}$$

to $\mathrm{Nm} : (\mathrm{Jac}\otimes\mathbb{X}_\bullet)^{\mathrm{W}} \to (\mathrm{Jac}\otimes\mathbb{X}_\bullet)^{\mathrm{W}}$ is the multiplication by $|\mathrm{W}|$. Therefore, the image of $P^1 \to \mathrm{Jac}\otimes\mathbb{X}_\bullet$ is $(\mathrm{Jac}\otimes\mathbb{X}_\bullet)^{\mathrm{W},0}$. In addition, $P^1 \to (\mathrm{Jac}\otimes\mathbb{X}_\bullet)^{\mathrm{W},0}$ is a prime-to-p isogeny and therefore its kernel is étale. Then this kernel must be trivial since its underlying group of k-points is trivial. □

[3] Indeed, the same argument in Remark 2.6.2, with the discriminant function $\underline{Disc}$ replaced by the W-invariant function $\prod_{\beta\in\mathrm{W}\alpha} d\beta : \mathfrak{t} \to k$, shows that the fixed point $\widetilde{C}_\alpha$ is non-empty.

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As a result, for any prime $\ell \neq p$,

$$T_\ell P^1 \simeq (\mathrm{H}^1(\widetilde{C}, \mathbb{Z}_\ell(1)) \otimes \mathbb{X}_\bullet)^{\mathrm{W}}.$$

In addition, observe that from the definition of D_{cl}, the map $P^1 \subset \mathrm{Jac} \otimes \mathbb{X}_\bullet \stackrel{\mathrm{Nm}}{\to} P^1$ factors as

$$P^1 \subset \mathrm{Jac} \otimes \mathbb{X}_\bullet \simeq (\mathrm{Jac} \otimes \mathbb{X}^\bullet)^\vee \to (\check{P}^1)^\vee \to \check{P}^\vee \stackrel{\check{D}_{\mathrm{cl}}}{\to} P \to P^1.$$

Therefore D_{cl} is a prime-to-p isogeny. In addition, the map

$$T_\ell \mathrm{Nm} : T_\ell(\mathrm{Jac} \otimes \mathbb{X}_\bullet) \twoheadrightarrow T_\ell(\check{P}^1)^\vee \hookrightarrow T_\ell P^1$$

can be identified with

$$\mathrm{Nm} : \mathrm{H}^1(\widetilde{C}, \mathbb{Z}_\ell(1)) \otimes \mathbb{X}_\bullet \twoheadrightarrow (\mathrm{H}^1(\widetilde{C}, \mathbb{Z}_\ell(1)) \otimes \mathbb{X}_\bullet)_{\mathrm{W}}/(\mathrm{torsion}) \hookrightarrow (\mathrm{H}^1(\widetilde{C}, \mathbb{Z}_\ell(1)) \otimes \mathbb{X}_\bullet)^{\mathrm{W}}.$$

On the other hand, as J^0 is connected, the norm map $\mathrm{Nm} : \pi_* T \to J^1 = (\pi_* T)^{\mathrm{W}}$ factors as $\pi_* T \to J^0 \to J^1$. Therefore, $\mathrm{Nm} : \mathrm{Jac} \otimes \mathbb{X}_\bullet \to P^1$ also factors as

$$\mathrm{Nm} : \mathrm{Jac} \otimes \mathbb{X}_\bullet \to P^0 \to P^1.$$

It follows that $P^0 \to P^1$ is also a prime-to-p isogeny, and for $\ell \neq p$ there is a factorization

$$\mathrm{Nm} : \mathrm{H}^1(\widetilde{C}, \mathbb{Z}_\ell(1)) \otimes \mathbb{X}_\bullet \to T_\ell P^0 \hookrightarrow (\mathrm{H}^1(\widetilde{C}, \mathbb{Z}_\ell(1)) \otimes \mathbb{X}_\bullet)^{\mathrm{W}}.$$

We need the following key result.

Proposition 3.5.3. *The two isogenies $(\check{P}^1)^\vee \to P^1 \leftarrow P^0$ induce an isomorphism $(\check{P}^1)^\vee \simeq P^0$.*

Proof. By the above considerations, the lemma is equivalent to saying that the induced map of Tate modules $T_\ell \mathrm{Nm} : T_\ell(\mathrm{Jac} \otimes \mathbb{X}_\bullet) \to T_\ell P^0$ is surjective for every $\ell \neq p$.

Note that we have the following commutative diagram

$$\begin{tikzcd}
(\mathrm{Jac} \otimes \mathbb{X}_\bullet)[\ell^n] = \mathrm{H}^1(C, \pi_*(\mathbb{X}_\bullet \otimes \mu_{\ell^n})) \arrow[r] \arrow[d] & P^0[\ell^n] \arrow[d] \\
\mathrm{H}^1(C, J^0[\ell^n]) \arrow[r, two heads] & \mathrm{H}^1(C, J^0)[\ell^n]
\end{tikzcd}$$

where the left vertical arrow is induced by $\pi_* T[\ell^n] = \pi_*(\mathbb{X}_\bullet \otimes \mu_{\ell^n}) \to J^0[\ell^n]$, and the bottom row is induced by the Kummer sequence for J^0 and therefore is surjective. Since $\pi_0(\mathscr{P}^0) = H^1(C, J^0)/P^0$ is finitely generated, passing to the inverse limit gives

$$\begin{tikzcd}
\varprojlim_n \mathrm{H}^1(C, \pi_*(\mathbb{X}_\bullet \otimes \mu_{\ell^n})) \arrow[r] \arrow[d] & T_\ell P^0 \arrow[d, "\simeq"] \\
\varprojlim_n \mathrm{H}^1(C, J^0[\ell^n]) \arrow[r, two heads] & \varprojlim_n \mathrm{H}^1(C, J^0)[\ell^n]
\end{tikzcd}$$

where the bottom arrow is surjective. So it is enough to show that the left vertical arrow is also surjective.

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Let $y \in C$, and choose a point $\tilde{y} \in \widetilde{C}$ lying over y. Let $\mathrm{W}_{\tilde{y}} \subset \mathrm{W}$ denote the stabilizer of $\tilde{y}$ under the action of W on $\widetilde{C}$. Note that $\mathrm{W}_{\tilde{y}} = \langle s_\alpha \rangle$ if $\tilde{y} \in \widetilde{C}_\alpha$ and is trivial otherwise. Then the inclusion of $J^0[\ell^n] \subset J^1[\ell^n]$ at y can be identified as

$$J^0[\ell^n]_y \simeq T^{\mathrm{W}_{\tilde{y}},0}[\ell^n] = \mathbb{X}_\bullet^{\mathrm{W}_{\tilde{y}}} \otimes \mu_{\ell^n} \subset J^1[\ell^n]_y \simeq T^{\mathrm{W}_{\tilde{y}}}[\ell^n] = (\mathbb{X}_\bullet \otimes \mu_{\ell^n})^{\mathrm{W}_{\tilde{y}}}. \tag{3.5.1}$$

Therefore, the cokernel of the inclusion $J^0[\ell^n] \subset J^1[\ell^n] = \pi_*(\mathbb{X}_\bullet \otimes \mu_{\ell^n})^{\mathrm{W}}$ is a sheaf supported on the ramification loci of $\pi : \widetilde{C} \to C$, and whose stalk at y can be identified with $\mathrm{H}^1(\mathrm{W}_{\tilde{y}}, \mathbb{X}_\bullet)[\ell^n] \otimes \mu_{\ell^n}$. Since $\mathrm{H}^1(\mathrm{W}_{\tilde{y}}, \mathbb{X}_\bullet)$ is a finite group, passing to the inverse limit gives

$$\varprojlim_n \mathrm{H}^1(C, J^0[\ell^n]) \simeq \varprojlim_n \mathrm{H}^1(C, \pi_*(\mathbb{X}_\bullet \otimes \mu_{\ell^n})^{\mathrm{W}}).$$

Therefore, it is enough to show that the inverse limit of the system of maps

$$\mathrm{Nm} : \mathrm{H}^1(C, \pi_*(\mathbb{X}_\bullet \otimes \mu_{\ell^n})) \to \mathrm{H}^1(C, \pi_*(\mathbb{X}_\bullet \otimes \mu_{\ell^n})^{\mathrm{W}})$$

is surjective.

Let $j : U \subset C$ be the complement of the ramification loci of $\pi : \widetilde{C} \to C$ and let $\tilde{j} : \widetilde{U} \to \widetilde{C}$ be its preimage in $\widetilde{C}$. Let $i : C \backslash U \to C$ be the closed embedding of ramification loci. Then $L_n := \pi_*(\mathbb{X}_\bullet \otimes \mu_{\ell^n})|_U$ is a locally free $\mathbb{Z}/\ell^n$-module on U with an action of W, and the norm map $\mathrm{Nm} : L_n \to L_n^{\mathrm{W}}$ is surjective. Let F_n denote its kernel. Note that since $\tilde{j}_*(\mathbb{X}_\bullet \otimes \mu_{\ell^n}) = \mathbb{X}_\bullet \otimes \mu_{\ell^n}$, we have

$$\pi_*(\mathbb{X}_\bullet \otimes \mu_{\ell^n}) = j_* L_n, \quad \pi_*(\mathbb{X}_\bullet \otimes \mu_{\ell^n})^{\mathrm{W}} = j_* L_n^{\mathrm{W}}.$$

Now, let $N_n = i^* j_* L_n^{\mathrm{W}}$ be the restriction of $j_* L_n^{\mathrm{W}}$ over the ramification loci. Taking cohomology of $0 \to j_! L_n^{\mathrm{W}} \to j_* L_n^{\mathrm{W}} \to N_n \to 0$ then induces the following commutative diagram with rows and columns exact.

$$\begin{tikzcd}
 & \mathrm{H}^1_c(U, L_n) \arrow[r] \arrow[d, "\mathrm{Nm}"] & \mathrm{H}^1(C, \pi_*(\mathbb{X}_\bullet \otimes \mu_{\ell^n})) \arrow[r] \arrow[d, "\mathrm{Nm}"] & 0 \\
\mathrm{H}^0(C, N_n) \arrow[r, "\partial_n"] \arrow[d, equal] & \mathrm{H}^1_c(U, L_n^{\mathrm{W}}) \arrow[r] \arrow[d, "q_n"] & \mathrm{H}^1(C, \pi_*(\mathbb{X}_\bullet \otimes \mu_{\ell^n})^{\mathrm{W}}) \arrow[r] \arrow[d] & 0 \\
\mathrm{H}^0(C, N_n) \arrow[r, "\delta_n"] & \Delta_n \arrow[r] \arrow[d] & Q_n \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}$$

Here Δ_n and Q_n denote the cokernels of the norm maps. Recall that we want to show $\varprojlim Q_n = 0$. From this diagram, this is equivalent to the surjectivity of $\varprojlim_n \delta_n$.

It is easier to first describe the Pontrjagin dual of ∂_n and q_n. Note that the distinguished triangle $i^* j_* L_n^{\mathrm{W}} \to j_! L_n^{\mathrm{W}}[1] \to j_* L_n^{\mathrm{W}}[1] \to$ is the Verdier dual of the natural distinguished triangle

$$j_*((L_n^{\mathrm{W}})^* \otimes \mu_{\ell^n})[1] \to R j_*((L_n^{\mathrm{W}})^* \otimes \mu_{\ell^n})[1] \to R^1 j_*((L_n^{\mathrm{W}})^* \otimes \mu_{\ell^n}) \to .$$

Therefore, the dual of ∂_n is the natural restriction map

$$\mathrm{res} : \mathrm{H}^1(U, (L_n^{\mathrm{W}})^* \otimes \mu_{\ell^n}) \to \bigoplus_{y \in C - U} \mathrm{H}^1(\mathrm{Spec}\,\mathcal{O}_{C,y}^h \backslash \{y\}, (L_n^{\mathrm{W}})^* \otimes \mu_{\ell^n}), \tag{3.5.2}$$

where $\mathcal{O}_{C,y}^h$ denotes the henselization of $\mathcal{O}_{C,y}$.

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Let $\bar{\eta}$ denote a geometric generic point of $\widetilde{U}$. Its image in U under π is still denoted by $\bar{\eta}$. Then we have

$$(L_n)_{\bar{\eta}} \simeq \mathbb{Z}[\mathrm{W}] \otimes (\mathbb{X}_\bullet \otimes \mu_{\ell^n}),$$

and the monodromy representation $\rho : \pi_1(U, \bar{\eta}) \to \mathrm{GL}((L_n)_{\bar{\eta}})$ is given by

$$\rho(\gamma)(a \otimes b) = \rho(\gamma)a \otimes b.$$

There is another action of W on $(L_n)_{\bar{\eta}}$ given by

$$w(a \otimes b) = aw^{-1} \otimes wb,$$

which gives rise to the W-action on L_n. Then there is a canonical isomorphism

$$\mathbb{X}_\bullet \otimes \mu_{\ell^n} \simeq (L_n^{\mathrm{W}})_{\bar{\eta}}, \quad \lambda \mapsto \sum_{w \in \mathrm{W}} w \otimes w^{-1}\lambda. \tag{3.5.3}$$

Now we have the following commutative diagram

$$\begin{tikzcd}
((L_n)^*_{\bar{\eta}} \otimes \mu_{\ell^n})^{\mathrm{W}} \arrow[r] \arrow[d, equal] & ((F_n)^*_{\bar{\eta}} \otimes \mu_{\ell^n})^{\mathrm{W}} \arrow[r] \arrow[d, equal] & \mathrm{H}^1(\mathrm{W}, \mathbb{X}^\bullet/\ell^n) \arrow[r] \arrow[d, hook, "\rho^*"] & 0 \arrow[d] \\
((L_n)^*_{\bar{\eta}} \otimes \mu_{\ell^n})^{\pi_1(U,\bar{\eta})} \arrow[r] & ((F_n)^*_{\bar{\eta}} \otimes \mu_{\ell^n})^{\pi_1(U,\bar{\eta})} \arrow[r] & \mathrm{H}^1(\pi_1(U,\bar{\eta}), \mathbb{X}^\bullet/\ell^n) \arrow[r] & \mathrm{H}^1(\pi_1(U,\bar{\eta}), (L_n)^*_{\bar{\eta}} \otimes \mu_{\ell^n})
\end{tikzcd}$$

where the second row is the long exact sequence of étale cohomology for locally free $\mathbb{Z}/\ell^n$-modules $0 \to (L_n^{\mathrm{W}})^* \otimes \mu_{\ell^n} \to L_n^* \otimes \mu_{\ell^n} \to F_n^* \otimes \mu_{\ell^n} \to 0$ on U, and the first row is the long exact sequence of the group cohomology for their stalks at $\bar{\eta}$, regarded as W-modules. Here, we use: (i) $\mathrm{H}^1(\mathrm{W}, (L_n)^*_{\bar{\eta}}) = 0$ by Shapiro's lemma; (ii) under the isomorphism (3.5.3), the $\pi_1(U, \bar{\eta})$-action on $(L_n^{\mathrm{W}})_{\bar{\eta}}$ corresponds to the natural action of $\mathrm{W} = \rho(\pi_1(U, \bar{\eta}))$ on $\mathbb{X}_\bullet \otimes \mu_{\ell^n}$; (iii) ρ^* is injective since it is induced by the surjective map $\pi_1(U, \bar{\eta}) \to \mathrm{W}$. Therefore, it follows from the Poincaré duality on U that the Pontrjagin dual of q_n is ρ^*.

Putting these considerations together, we see that the dual of δ_n is $\mathrm{res} \circ \rho^*$. Now we choose a geometric generic point $\bar{\eta}_{\tilde{y}}$ of $\mathrm{Spec}\mathcal{O}^h_{C,y} \backslash \{y\}$ over $\bar{\eta}$. Then $\rho(\pi_1(\mathrm{Spec}\mathcal{O}^h_{C,y} \backslash \{y\}), \bar{\eta}_{\tilde{y}}) = \langle s_{\alpha_{\tilde{y}}} \rangle \subset \mathrm{W}$ for some root $\alpha_{\tilde{y}}$ (depending on $\bar{\eta}_{\tilde{y}}$), and there is the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathrm{W}, \mathbb{X}^\bullet/\ell^n) \arrow[r, "\mathrm{res}"] \arrow[d, hook, "\rho^*"'] \arrow[dr, "\delta_n^*"] & \mathrm{H}^1(\langle s_{\alpha_{\tilde{y}}} \rangle, \mathbb{X}^\bullet/\ell^n) \arrow[d, hook, "\rho^*"] \\
\mathrm{H}^1(\pi_1(U, \bar{\eta}), \mathbb{X}^\bullet/\ell^n) \arrow[r, "\mathrm{res}"] & \mathrm{H}^1(\pi_1(\mathrm{Spec}\mathcal{O}^h_{C,y} \backslash \{y\}, \bar{\eta}_{\tilde{y}}), \mathbb{X}^\bullet/\ell^n),
\end{tikzcd}$$

with vertical arrows injective. Therefore, it remains to show that

$$\bigoplus_{y \in C-U} \varprojlim \mathrm{H}^1(\langle s_{\alpha_{\tilde{y}}} \rangle, \mathbb{X}^\bullet/\ell^n)^* \to \varprojlim \mathrm{H}^1(\mathrm{W}, \mathbb{X}^\bullet/\ell^n)^*$$

is surjective. Note

$$\varprojlim_n \mathrm{H}^1(\mathrm{W}, \mathbb{X}^\bullet/\ell^n)^* = \mathrm{Hom}\left(\varinjlim_n \mathrm{H}^1\left(\mathrm{W}, \frac{1}{\ell^n}\mathbb{X}^\bullet/\mathbb{X}^\bullet\right), \mathbb{Q}_\ell/\mathbb{Z}_\ell\right). \tag{3.5.4}$$

Using $0 \to \mathrm{H}^1(\mathrm{W}, \mathbb{X}^\bullet)/\ell^n \to \mathrm{H}^1(\mathrm{W}, \mathbb{X}^\bullet/\ell^n) \to \mathrm{H}^2(\mathrm{W}, \mathbb{X}^\bullet)[\ell^n] \to 0$, and the fact that $\mathrm{H}^1(W, \mathbb{X}^\bullet)$ is finite, we have

$$\varinjlim_n \mathrm{H}^1\left(\mathrm{W}, \frac{1}{\ell^n}\mathbb{X}^\bullet/\mathbb{X}^\bullet\right) = \mathrm{H}^2(\mathrm{W}, \mathbb{X}^\bullet)[\ell^\infty] = \mathrm{H}^1(\mathrm{W}, \mathbb{X}^\bullet \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

So it reduces to show that

$$\mathrm{H}^1(\mathrm{W}, \mathbb{X}^\bullet \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \bigoplus_{y \in C-U} \mathrm{H}^1(\langle s_{\alpha_{\tilde{y}}}\rangle, \mathbb{X}^\bullet \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

is injective. As mentioned in the proof of Lemma 3.5.1, $\widetilde{C}_\alpha$ is non-empty for every $\alpha \in \Phi$. Hence $\{\alpha_{\tilde{y}}\}$ contain a set of representatives of Φ/W. Now we can apply Lemma 3.5.2 to $M = \mathbb{X}^\bullet \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ to finish the proof of the proposition. □

Now, let $A' = \ker(P^0 \to P)$, and $A = \ker(P \to P^1)$. Then by the above proposition,

$$\ker D_{\mathrm{cl}} = A'/(\breve{A})^*.$$

As both A' and $\breve{A}$ are finite étale groups, it is enough to show that $|A'| = |\breve{A}|$, where for a finite group Γ, $|\Gamma|$ denotes the number of its elements. This is the subject of the next subsection.

3.6 Calculation of finite groups

Let us understand A. In fact, it is better to pick up $\infty \in C$ *away from* the ramification loci. Let $\mathcal{O}_\infty$ denote the completed local ring of C at ∞. Let J_∞ be the dilatation of J along the unit of the fiber of J at ∞. By definition (see [BLR90, § 2] for details), J_∞ is the unique smooth group scheme over C equipped with a natural map $J_\infty \to J$, which is an isomorphism away from ∞ and induces an isomorphism from $J_\infty(\mathcal{O}_\infty)$ to the first congruence subgroup of $J(\mathcal{O}_\infty)$. Let $\mathscr{P}_\infty$ be the Picard stack of J_∞-torsors on C. One can also interpret $\mathscr{P}_\infty$ as the Picard stack of J-torsors on C together with a trivialization at ∞. Observe that $\mathscr{P}_\infty$ is in fact a scheme. Let P_∞ denote the neutral connected component of $\mathscr{P}_\infty$. Similarly, one can define $J^0_\infty, J^1_\infty, P^0_\infty, P^1_\infty$ etc. Let $A_\infty = \ker(P_\infty \to P^1_\infty)$ and $A'_\infty = \ker(P^0_\infty \to P_\infty)$.

Lemma 3.6.1. *There are the following two exact sequences*

$$1 \to A_\infty \to \Gamma(C, J^1/J) \to \pi_0(\mathscr{P}) \to \pi_0(\mathscr{P}^1) \to 1$$

and

$$1 \to \mathrm{Aut}_{\mathscr{P}}(e) \to \mathrm{Aut}_{\mathscr{P}^1}(e) \to A_\infty \to A \to 1.$$

Similarly,

$$1 \to A'_\infty \to \Gamma(C, J/J^0) \to \pi_0(\mathscr{P}^0) \to \pi_0(\mathscr{P}) \to 1$$

and

$$1 \to \mathrm{Aut}_{\mathscr{P}^0}(e) \to \mathrm{Aut}_{\mathscr{P}}(e) \to A'_\infty \to A' \to 1.$$

Proof. Consider the following commutative diagram.

$$\begin{tikzcd} 1 \arrow[r] & J_\infty \arrow[r] \arrow[d] & J^1_\infty \arrow[r] \arrow[d] & J^1_\infty/J_\infty \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & J \arrow[r] & J^1 \arrow[r] & J^1/J \arrow[r] & 1 \end{tikzcd}$$

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Taking $R\Gamma(C,-)$, we obtain the following.

$$\begin{tikzcd} 1 \arrow[r] & \Gamma(C, J^1_\infty/J_\infty) \arrow[r] \arrow[d] & \mathscr{P}_\infty \arrow[r] \arrow[d] & \mathscr{P}^1_\infty \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \Gamma(C, J^1/J) \arrow[r] & \mathscr{P} \arrow[r] & \mathscr{P}^1 \arrow[r] & 1 \end{tikzcd} \tag{3.6.1}$$

Since $\mathscr{P}_\infty$ and $\mathscr{P}^1_\infty$ are schemes, the first row of (3.7.1) gives

$$1 \to A_\infty \to \Gamma(C, J^1_\infty/J_\infty) \to \pi_0(\mathscr{P}_\infty) \to \pi_0(\mathscr{P}^1_\infty) \to 1.$$

Since $J^1_\infty/J_\infty = J^1/J$ and $\pi_0(\mathscr{P}_\infty) = \pi_0(\mathscr{P}), \pi_0(\mathscr{P}^1_\infty) = \pi_0(\mathscr{P}^1)$, we obtain the first exact sequence of the lemma. In addition, combining with the second row of (3.7.1), we obtain the short exact sequence of Beilinson 1-motives

$$1 \to A_\infty \to W_1\mathscr{P} \to W_1\mathscr{P}^1 \to 1,$$

which in turn gives the second exact sequence of the lemma. The proof of the last two exact sequences of the lemma is similar (by considering $R\Gamma$ of the short exact sequence $1 \to J^0_\infty \to J_\infty \to J_\infty/J^0_\infty \to 1$). $\square$

As a corollary, we can write

$$|A| = \frac{|\Gamma(C, J^1/J)|}{|\mathrm{coker}(\mathrm{Aut}_{\mathscr{P}}(e) \to \mathrm{Aut}_{\mathscr{P}^1}(e))||\ker(\pi_0(\mathscr{P}) \to \pi_0(\mathscr{P}^1)|},$$

and

$$|A'| = \frac{|\Gamma(C, J/J^0)|}{|\mathrm{coker}(\mathrm{Aut}_{\mathscr{P}^0}(e) \to \mathrm{Aut}_{\mathscr{P}}(e))||\ker(\pi_0(\mathscr{P}^0) \to \pi_0(\mathscr{P}))|}.$$

Therefore to show that $|\breve{A}| = |A'|$, it is enough to show that:

(1) $|\Gamma(C, \breve{J}^1/\breve{J})| = |\Gamma(C, J/J^0)|$;

(2) $|\mathrm{coker}(\mathrm{Aut}_{\breve{\mathscr{P}}}(e) \to \mathrm{Aut}_{\breve{\mathscr{P}}^1}(e))| = |\mathrm{coker}(\pi_0(\mathscr{P})^* \to \pi_0(\mathscr{P}^0)^*)|$;

(3) $|\ker(\pi_0(\breve{\mathscr{P}}) \to \pi_0(\breve{\mathscr{P}}^1))| = |\ker(\mathrm{Aut}_{\mathscr{P}}(e)^* \to \mathrm{Aut}_{\mathscr{P}^0}(e)^*)|$.

We first prove (1). By (3.1.3),

$$\Gamma(C, \breve{J}^1/\breve{J}) = \left(\bigoplus_\alpha \mu_\alpha(\widetilde{C}_\alpha)\right)^{\mathrm{W}}. \tag{3.6.2}$$

Observe that $\mu_\alpha \neq 0$ if and only if α is not a primitive root, i.e., $\alpha/2 \in \mathbb{X}^\bullet$. On the other hand, from

$$1 \to J/J_0 \to J_1/J_0 \to J_1/J \to 1,$$

one can see that the character group of $\Gamma(C, J/J_0)$ is $(\bigoplus_{x\in\sqcup\widetilde{C}_\alpha}(\mathbb{Q}\alpha \cap \mathbb{X}^\bullet)/\mathbb{Z}\alpha)^{\mathrm{W}}$. Then (1) follows.

Next, we prove (2). In fact, it follows from § 3.4 and [Ngô10, § 4.10] that both maps can be identified with the natural inclusion $Z(\breve{G}) \to \breve{T}^{\mathrm{W}}$.

Finally, we show (3). Recall that $\mathrm{Aut}_{\mathscr{P}}(e) = \{t \in T \mid \alpha(t) = 1,\ \alpha \in \Phi\}$. On the other hand, from the above description of $\Gamma(C, J/J_0)^*$,

$$\mathrm{Aut}_{\mathscr{P}^0}(e) = \{t \in T \mid \lambda(t) = 1 \text{ if } \lambda \in \mathbb{Q}\alpha \cap \mathbb{X}^\bullet, \alpha \in \Phi)\}.$$

Therefore,

$$\ker(\mathrm{Aut}_{\mathscr{P}}(e)^* \to \mathrm{Aut}_{\mathscr{P}^0}(e)^*) = \frac{\sum_{\alpha\in\Phi}(\mathbb{Q}\alpha \cap \mathbb{X}^\bullet)}{\mathbb{Z}\Phi}.$$

To calculate $\ker(\pi_0(\breve{\mathscr{P}}) \to \pi_0(\breve{\mathscr{P}}^1))$, we choose $\tilde{y} \in \widetilde{C}_{\alpha_{\tilde{y}}}$ above $y \in C - U$ for every point in the ramification loci. The restriction of $1 \to \breve{J} \to \breve{J}^1 \to \breve{J}^1/\breve{J} \to 0$ at y then can be identified with

$$1 \to \ker(\breve{\alpha}_{\tilde{y}}) \to \breve{T}^{s_{\alpha_{\tilde{y}}}} \stackrel{\breve{\alpha}_{\tilde{y}}}{\to} \mu_{\alpha_{\tilde{y}}} \simeq \frac{\mathbb{Q}\alpha_{\tilde{y}} \cap \mathbb{X}^\bullet}{\mathbb{Z}\alpha_{\tilde{y}}} \to 1.$$

It follows that the coboundary map $\Gamma(C, \breve{J}^1/\breve{J}) \to \mathrm{H}^1(C, \breve{J})$ can be identified with

$$\bigoplus_{y\in C-U} \frac{\mathbb{Q}\alpha_{\tilde{y}} \cap \mathbb{X}^\bullet}{\mathbb{Z}\alpha_{\tilde{y}}} \to \mathrm{H}^1(C, \breve{J}), \quad \lambda_{\tilde{y}} \in \frac{\mathbb{Q}\alpha_{\tilde{y}} \cap \mathbb{X}^\bullet}{\mathbb{Z}\alpha_{\tilde{y}}} \mapsto \mathrm{AJ}^{\breve{\mathscr{P}}}(\tilde{y}, \lambda_{\tilde{y}}),$$

where $\mathrm{AJ}^{\breve{\mathscr{P}}}$ is the Abel–Jacobi map introduced before. Of course, this map does not really depend on the choice of liftings of $y \in C - U$ since $\mathrm{AJ}^{\breve{\mathscr{P}}}(\tilde{y}, \lambda_{\tilde{y}}) = \mathrm{AJ}^{\breve{\mathscr{P}}}(w\tilde{y}, w\lambda_{\tilde{y}})$.

Now, as in the proof of Lemma 3.6.1, we have a right exact sequence

$$\Gamma(C, \breve{J}^1/\breve{J}) \to \pi_0(\breve{\mathscr{P}}) \to \pi_0(\breve{\mathscr{P}}^1) \to 0.$$

Since the Abel–Jacobi map induces $\mathbb{X}^\bullet/\mathbb{Z}\Phi \simeq \pi_0(\breve{\mathscr{P}})$, we deduce that

$$\ker(\pi_0(\breve{\mathscr{P}}) \to \pi_0(\breve{\mathscr{P}}^1)) = \mathrm{Im}(\Gamma(C, \breve{J}^1/\breve{J}) \to \pi_0(\breve{\mathscr{P}})) = \frac{\sum_{\alpha\in\Phi}(\mathbb{Q}\alpha \cap \mathbb{X}^\bullet)}{\mathbb{Z}\Phi}.$$

Therefore, (3) follows and the proof of Theorem 3.3.1 is complete. □

Remark 3.6.2. As a byproduct of the proof, we obtain

$$\pi_0(\breve{\mathscr{P}}^1) = \frac{\mathbb{X}^\bullet}{\sum_{\alpha\in\Phi}(\mathbb{Q}\alpha \cap \mathbb{X}^\bullet)}.$$

It seems that this expression of $\pi_0(\breve{\mathscr{P}}^1)$ did not appear in literature before.

3.7 A property of $\mathfrak{D}_{\mathrm{cl}}$

In this subsection, we assume that G is semi-simple. We show that the classical duality $\mathfrak{D}_{\mathrm{cl}}$ intertwines certain homomorphisms of Picard stacks over the Hitchin base B^0. As before, we omit the subscript ^{0}.

Let $Z(\breve{G})$-tors(C) denote the Picard stack of $Z(\breve{G})$-torsors on C. We start with the construction of two homomorphisms

$$\mathfrak{l}_J : Z(\breve{G})\text{-tors}(C) \times B \to \mathscr{P}^\vee, \quad \breve{\mathfrak{l}}_J : Z(\breve{G})\text{-tors}(C) \times B \to \breve{\mathscr{P}}. \qquad (3.7.1)$$

The definition of $\breve{\mathfrak{l}}_J$ is easy. It is induced by the natural map of group schemes

$$Z(\breve{G}) \times (C \times S) \to \breve{J}_b,$$

for every $b : S \to B$. For $K \in Z(\breve{G})$-tors(C), let

$$K_J := \breve{\mathfrak{l}}_J(\{K\} \times B) \in \breve{\mathscr{P}}(B).$$

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Next we define $\mathfrak{l}_J$. For the purpose, we need to generalize a construction of [BD91, § 4.1]. Let $\pi : \mathcal{C} \to B$ be a smooth proper relative curve over an affine base B (later on $\mathcal{C} = C \times B$). Let

$$0 \to \Pi(1) \to \tilde{\mathcal{G}} \to \mathcal{G} \to 0$$

be an extension of smooth affine group schemes on $\mathcal{C}$ with Π commutative finite étale. Let $\Pi^\vee = \mathrm{Hom}(\Pi, \mathbb{G}_m)$ be its Cartier dual, which is assumed to be étale as well (in particular, the order of Π is prime to char k), and let $\Pi^\vee\text{-tors}(\mathcal{C}/B)$ denote the Picard stack (over B) of $\Pi^\vee$-torsors on $\mathcal{C}$ relative to B. We construct a Picard functor

$$\mathfrak{l}_{\mathcal{G}} : \Pi^\vee\text{-tors}(\mathcal{C}/B) \to \mathrm{Pic}\,(\mathrm{Bun}_{\mathcal{G}}(\mathcal{C}/B))$$

of Picard stacks over B as follows. First, let $\Pi\text{-gerbes}(\mathcal{C}/B)$ denote the Picard 2-stack of Π-gerbes on $\mathcal{C}$ relative to B, regarded as a Picard stack. Then there is the generalized (or categorical) Chern class map

$$\tilde{c}_{\mathcal{G}} : \mathrm{Bun}_{\mathcal{G}}(\mathcal{C}/B) \to \Pi(1)\text{-gerbes}(\mathcal{C}/B)$$

that assigns every B-scheme S and a $\mathcal{G}$-torsor E on $\mathcal{C}_S$, the Picard groupoid of the lifting of E to a $\tilde{\mathcal{G}}$-torsor. We have the following.

LEMMA 3.7.1. *The dual of the Picard stack* $\Pi\text{-gerbes}(\mathcal{C}/B)$ *(as defined in § A.3) is canonically isomorphic to* $\Pi^\vee\text{-tors}(\mathcal{C}/B)$.

We follow [BD91, § 4.1.5] for a 'scientific interpretation' of this lemma and refer to [BD91, §§ 4.1.2–4.1.4] for the precise construction. As explained in § A.1, the Picard stack $\Pi\text{-gerbes}(\mathcal{C}/B)$ is incarnated by the complex $\tau_{\geqslant -1} R\pi_*\Pi[2](1)$, and $\Pi^\vee\text{-tors}(\mathcal{C}/B)$ is incarnated by the complex $\tau_{\leqslant 0} R\pi_*\Pi^\vee[1]$. Let μ'_∞ denote the group of prime-to-p roots of unit. Note that $\pi^!\mu'_\infty \simeq \mu'_\infty[2](1)$. Then by the Verdier duality,

$$R\mathrm{Hom}(R\pi_*\Pi[2](1), \mu'_\infty) \simeq R\pi_* R\mathrm{Hom}(\Pi[2](1), \pi^!\mu'_\infty) \simeq R\pi_*\Pi^\vee.$$

By shifting by [1] and truncating $\tau_{\leqslant 0}$, one obtains the lemma. As explained in [BD91, § 4.1.5], working in the framework of derived categories is not enough to turn the above heuristics into a proof. One can either give a concrete construction as in [BD91, §§ 4.1.2–4.1.4] or understand the above argument in the framework of stable ∞-categories.

Therefore, each $K \in \Pi^\vee\text{-tors}(\mathcal{C}/B)$ defines a morphism

$$\mathfrak{l}_{\mathcal{G},K} : \mathrm{Bun}_{\mathcal{G}}(\mathcal{C}/B) \xrightarrow{\tilde{c}_{\mathcal{G}}} \Pi(1)\text{-gerbes}(\mathcal{C}/B) \xrightarrow{\langle , K\rangle} B\mathbb{G}_m$$

or equivalently a line bundle $\mathcal{L}_{\mathcal{G},K}$ on $\mathrm{Bun}_{\mathcal{G}}(\mathcal{C}/B)$ and the assignment $K \to \mathcal{L}_{\mathcal{G},K}$ defines a homomorphism of Picard stacks

$$\mathfrak{l}_{\mathcal{G}} : \Pi^\vee\text{-tors}(\mathcal{C}/B) \to \mathrm{Pic}(\mathrm{Bun}_{\mathcal{G}}(\mathcal{C}/B)),$$

which factors through the n-torsion of $\mathrm{Pic}(\mathrm{Bun}_{\mathcal{G}})(\mathcal{C}/B)$ where n is the order of $\Pi^\vee$.

Note that in the above discussion we do not assume that $\mathcal{G}$ is commutative. But if $\mathcal{G}$ is commutative, $\mathrm{Bun}_{\mathcal{G}}(\mathcal{C}/B)$ has a natural structure of Picard stack over B and one can check that $\mathfrak{l}_{\mathcal{G}}$ factors through a homomorphism $\mathfrak{l}_{\mathcal{G}} : \Pi^\vee\text{-tors}(\mathcal{C}/B) \to (\mathrm{Bun}_{\mathcal{G}}(\mathcal{C}/B))^\vee$.

Now let $\mathcal{C} = C \times B$, where B is the Hitchin base as before. Let $\mathcal{G} = J_b$ and $\tilde{\mathcal{G}} = (J_{\mathrm{sc}})_b$, where $b : B \to B$ is the identity map, and J_{sc} is the universal regular centralizer for G_{sc}, the simply connected cover of G. Then $\Pi(1) = \Pi_G(1)$ is the fundamental group and $\Pi_G^\vee$ is canonical

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isomorphic to the center $Z(\check{G})$ of $\check{G}$. Therefore, the above construction gives $\mathfrak{l}_J$ as promised in (3.7.1).

Note that similarly we can set $\mathcal{G} = G \times \mathcal{C}$ and $\mathcal{G} = T \times \mathcal{C}$ in the above construction so we obtain

$$\mathfrak{l}_G : Z(\check{G})\text{-tors}(C) \to \mathrm{Pic}(\mathrm{Bun}_G), \quad \mathfrak{l}_T : Z(\check{G})\text{-tors}(C) \to (\mathrm{Bun}_T)^\vee.$$

For $K \in Z(\check{G})$-tors(C), let $\mathcal{L}_{G,K} := \mathfrak{l}_G(K) \in \mathrm{Pic}(\mathrm{Bun}_G)$, $\mathcal{L}_{J,K} := \mathfrak{l}_J(\{K\} \times B) \in (\mathscr{P})^\vee(B)$. The following lemma will be used in § 5.6.

LEMMA 3.7.2. *Let κ be a square root of ω. Then the pullback of $\mathcal{L}_{G,K}$ along the map $\mathscr{P} \xrightarrow{\epsilon_\kappa} \mathrm{Higgs} \xrightarrow{\mathrm{pr}} \mathrm{Bun}_G$ is isomorphic to $\mathcal{L}_{J,K}$, i.e., we have $\mathcal{L}_{J,K} \simeq \epsilon_\kappa^* \circ \mathrm{pr}^* \mathcal{L}_{G,K}$.*

Proof. It is enough to show that the composition

$$\mathscr{P} \xrightarrow{\epsilon_\kappa} \mathrm{Higgs} \xrightarrow{\mathrm{pr}} \mathrm{Bun}_G \xrightarrow{\tilde{c}_G} \Pi_G(1)\text{-gerbes}(C)$$

is isomorphic to

$$\mathscr{P} \xrightarrow{\tilde{c}_J} \Pi_G\text{-gerbes}(C) \times B \to \Pi_G(1)\text{-gerbes}(C).$$

Let $P \in \mathscr{P}$ and $(E, \phi) := \epsilon_\kappa(P)$. We need to construct a functorial isomorphism between $\tilde{c}_J(P)$ and $\tilde{c}_G(E)$ where $\tilde{c}_J(P)$ (respectively $\tilde{c}_G(E)$) is the $\Pi_G(1)$-gerbe of liftings of P to J_{sc}-torsors (respectively G_{sc}-torsors).

Note that the G-torsor E_κ given by the Kostant section has a natural lifting $\tilde{E}_\kappa \in \mathrm{Bun}_{G_{\mathrm{sc}}}$, since the cocharacter $2\rho : \mathbb{G}_m \to G$ has a natural lifting to G_{sc}. Thus any lifting $\tilde{P} \in \tilde{c}_J(P)$ defines a lifting $\tilde{E} := \tilde{P} \times^{J_{\mathrm{sc}}} \tilde{E}_\kappa \in \mathrm{Bun}_{G_{\mathrm{sc}}}$ of $E = P \times^J E_\kappa$ and the assignment $\tilde{P} \to \tilde{E}$ defines a functorial isomorphism between $\tilde{c}_J(P)$ and $\tilde{c}_G(E)$. The lemma follows. □

We write l_G, l_T, l_J for the induced maps between the corresponding coarse moduli spaces. The following lemma is a specialization of our construction of the duality given in Lemma 3.7.1.

LEMMA 3.7.3. *Let n be a positive integer such that $p \nmid n$. Let*

$$\mathfrak{l} : \check{T}[n]\text{-tors}(C) \to (\mathrm{Bun}_T)^\vee[n]$$

be the tensor functor given by the extension $0 \to T[n] \to T \xrightarrow{n} T \to 0$.[4] Then the induced map $l : H^1(C, \check{T}[n]) \to H^1(C, T[n])^\vee$ between the coarse moduli spaces is the same the as map given by the Poincare duality.

Now we are ready to state the result in this subsection.

PROPOSITION 3.7.4. *There is a natural isomorphism of functors $\mathfrak{D}_{\mathrm{cl}} \circ \mathfrak{l}_J \simeq \check{\mathfrak{l}}_J$. In particular, we have $\mathfrak{D}_{\mathrm{cl}}(\mathcal{L}_{J,K}) \simeq K_J$.*

Proof. Let $\check{G}_{\mathrm{ad}}$ denote the adjoint group of $\check{G}$. Note that it is the Langlands dual group of G_{sc}. Let $\check{J}_{\mathrm{ad}}$ be the universal centralizer for $\check{G}_{\mathrm{ad}}$, and $\check{\pi}_{\mathrm{ad}}$ denote the Picard stack of $\check{J}_{\mathrm{ad}}$-torsors. We first claim that the composition

$$Z(\check{G})\text{-tors}(C) \times B \xrightarrow{\mathfrak{l}_J} \mathscr{P}^\vee \overset{\mathfrak{D}_{\mathrm{cl}}}{\simeq} \check{\mathscr{P}} \to \check{\mathscr{P}}_{\mathrm{ad}}$$

[4] Recall that we have a canonical isomorphism $\check{T}[n] \simeq (T[n])^\vee$.

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is trivial. From the construction of $\mathfrak{D}_{\mathrm{cl}}$, we have the following commutative diagram.

$$\begin{tikzcd} \mathscr{P}^{\vee} \arrow[r, "\mathfrak{D}_{\mathrm{cl}}"] \arrow[d] & \breve{\mathscr{P}} \arrow[d] \\ (\mathscr{P}_{\mathrm{sc}})^{\vee} \arrow[r, "\mathfrak{D}_{\mathrm{cl}}"] & \breve{\mathscr{P}}_{\mathrm{ad}} \end{tikzcd}$$

Thus the above composition can be identified with

$$Z(\breve{G})\text{-tors}(C) \times B \xrightarrow{\mathfrak{l}_J} (\mathscr{P})^{\vee} \to (\mathscr{P}_{\mathrm{sc}})^{\vee} \overset{\mathfrak{D}_{\mathrm{cl}}}{\simeq} \breve{\mathscr{P}}_{\mathrm{ad}}.$$

This is trivial since the composition of the first two maps is the dual of

$$\mathscr{P}_{\mathrm{sc}} \to \mathscr{P} \xrightarrow{\tilde{c}_J} \Pi_G(1)\text{-gerbes}(C) \times B,$$

which is trivial by the construction of $\tilde{c}_J$.

On the other hand, the short exact sequence $0 \to Z(\breve{G}) \times B \to \breve{J} \to \breve{J}_{\mathrm{ad}} \to 0$ induces a left exact sequence of Picard stacks

$$0 \to Z(\breve{G})\text{-tors}(C) \times B \xrightarrow{\breve{\mathfrak{l}}_J} \breve{\mathscr{P}} \to \breve{\mathscr{P}}_{\mathrm{ad}}.$$

That is, $Z(\breve{G})$-tors$(C) \times B$ is identified as the kernel of $\breve{\mathscr{P}} \to \breve{\mathscr{P}}_{\mathrm{ad}}$. Therefore, there is a morphism

$$\mathfrak{i} : Z(\breve{G})\text{-tors}(C) \times B \to Z(\breve{G})\text{-tors}(C) \times B$$

such that $\mathfrak{D}_{\mathrm{cl}} \circ \mathfrak{l}_J \simeq \breve{\mathfrak{l}}_J \circ \mathfrak{i}$. We now show that $\mathfrak{i}$ is isomorphic to the identity morphism. As argued in § 3.4, we reduce to show that $\mathfrak{i}$ induced the identity map on the coarse moduli space $H^1(C, Z(\breve{G})) \times B$.

Let $i : H^1(C, Z(\breve{G})) \times B \to H^1(C, Z(\breve{G})) \times B$, $l_J : H^1(C, Z(\breve{G})) \times B \to P^{\vee}$ and $\breve{l}_J : H^1(C, Z(\breve{G})) \times B \to \breve{P}$ be the induced maps on the corresponding coarse moduli spaces. Our goal is to show that $i = \mathrm{id}$. Since $\Gamma(C \times B, \breve{J}_{\mathrm{ad}}) = 0$, $\breve{l}_J$ is injective. Therefore, it suffices to show that

$$\breve{l}_J \circ (i - \mathrm{id}) : H^1(C, Z(\breve{G})) \times B \to \breve{P}$$

is zero. As in § 3.4, we can prove this fiberwise, and therefore we fix $b \in B^0(k)$. Again, to simplify notations, in the following discussion we write $\widetilde{C}, J, P, \breve{P}$ instead of $\widetilde{C}_b, J_b, P_b, \breve{P}_b$, etc.

Let $\breve{j}^1 : \breve{P} \to H^1(\widetilde{C}, \breve{T})$ be the map induced by the morphism $\breve{j}^1 : \pi^*\breve{J} \to \breve{T}$. Then the composition $\breve{j}^1 \circ \breve{l}_J : H^1(C, Z(\breve{G})) \to H^1(\widetilde{C}, \breve{T})$ is also injective (note that $\breve{j}^1 \circ \breve{l}_J$ is induced by the natural map $Z(\breve{G}) \to \breve{T}$). Thus it is enough to show that $\breve{j}^1 \circ \breve{l}_J \circ (i - \mathrm{id}) = 0$. Since $D_{\mathrm{cl}} \circ l_J = \breve{l}_J \circ i$, it is equivalent to show that

$$\breve{j}^1 \circ D_{\mathrm{cl}} \circ l_J - \breve{j}^1 \circ \breve{l}_J = 0. \tag{3.7.2}$$

Let us consider the following diagram

$$\begin{tikzcd} H^1(C, Z(\breve{G})) \arrow[r, "l_J"] \arrow[d, "\mathrm{id}"] & H^1(C, J)^{\vee} \arrow[r, "\mathrm{Nm}^{\vee}"] \arrow[d, "D_{\mathrm{cl}}"] & H^1(\widetilde{C}, T)^{\vee} \arrow[d] \\ H^1(C, Z(\breve{G})) \arrow[r, "\breve{l}_J"] & H^1(C, \breve{J}) \arrow[r, "\breve{j}^1"] & H^1(\widetilde{C}, \breve{T}) \end{tikzcd}$$

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where $\mathrm{Nm}^\vee$ is the dual of (3.1.9), and the right vertical map is $(\mathrm{Jac}\otimes\mathbb{X}_\bullet)^\vee\simeq\mathrm{Jac}\otimes\mathbb{X}^\bullet$. The right rectangle in the above diagram is commutative by the construction of D_{cl} in § 3. Therefore it is enough to show that the outer diagram is also commutative.

Let n be the order of $Z(\check{G})$. Then $\check{j}^1\circ\check{l}_J$ and $\mathrm{Nm}^\vee\circ l_J$ will factor through $H^1(\widetilde{C},\check{T})[n]\simeq H^1(\widetilde{C},\check{T}[n])$ and $H^1(\widetilde{C},T)^\vee[n]\simeq H^1(\widetilde{C},T[n])^\vee$.[5] Thus the outer diagram factors as

$$\begin{tikzcd} H^1(C,Z(\check{G})) \arrow[r, "\mathrm{Nm}^\vee\circ\mathfrak{l}_J"] \arrow[d, "\mathrm{id}"] & H^1(\widetilde{C},T[n])^\vee \arrow[d] \\ H^1(C,Z(\check{G})) \arrow[r, "\check{j}^1\circ\check{l}_J"] & H^1(\widetilde{C},\check{T}[n]) \end{tikzcd}$$

where the right vertical arrow is now given by the Poincare duality. Unraveling the definition of $\mathfrak{l}_J$, one sees that $\mathrm{Nm}^\vee\circ l_J$ can be identified with

$$H^1(C,Z(\check{G}))\to H^1(\widetilde{C},\check{T}[n])\to H^1(\widetilde{C},T[n])^\vee$$

where the first map is induced by the natural morphism $Z(\check{G})\to\check{T}[n]$ and the second map is the map l in Lemma 3.7.3. Then the commutativity of above diagram follows from Lemma 3.7.3. □

4. Multiplicative 1-forms

In this section, we establish a technical result. Namely, we show that the pullback of the canonical 1-form θ_{can} on $T^*\mathrm{Bun}_G$ along $\mathscr{P}\to T^*\mathrm{Bun}_G$ induced by a Kostant section κ is multiplicative in the sense of § C.2.

4.1 Lie algebra valued 1-forms

In this subsection, we restrict everything to B^0 and therefore omit the subscript ^{0} from the notation. Recall that there is a group scheme $\mathcal{T}=\widetilde{C}\times^{\mathrm{W}}T$ over $[\widetilde{C}/\mathrm{W}]$ and Proposition 2.5.1 says that there is a homomorphism $[j^1]:[\pi]^*J\to\mathcal{T}$ where $[\pi]:[\widetilde{C}/\mathrm{W}]\to C\times B$ is the projection. It induces the following commutative diagram.

$$\begin{tikzcd} {[\pi]^*(\Omega_{C\times B}\otimes\mathrm{Lie}\,J)} \arrow[r] \arrow[d] & {[\pi]^*(\Omega_{C\times B/B}\otimes\mathrm{Lie}\,J)} \arrow[d] \\ \Omega_{[\widetilde{C}/W]}\otimes\mathrm{Lie}\,\mathcal{T} \arrow[r] & \Omega_{[\widetilde{C}/W]/B}\otimes\mathrm{Lie}\,\mathcal{T} \end{tikzcd}$$

Note that, due to the product structure on $C\times B$, the arrow in the upper row admits a canonical splitting. Therefore, the tautological section in (2.7.1)

$$(\tau:B\to B_J)\in\Gamma(C\times B,\Omega_{C\times B/B}\otimes\mathrm{Lie}\,J)$$

can be regarded as a section of $[\pi]^*(\Omega_{C\times B}\otimes\mathrm{Lie}\,J)$, which in turn gives

$$\theta_{\widetilde{C}}\in\Gamma([\widetilde{C}/\mathrm{W}],\Omega_{[\widetilde{C}/\mathrm{W}]}\otimes\mathrm{Lie}\,\mathcal{T})=\Gamma(\widetilde{C},\Omega_{\widetilde{C}}\otimes\mathfrak{t})^{\mathrm{W}}. \tag{4.1.1}$$

We denote by $\check{\theta}_{\widetilde{C}}\in\Gamma(\widetilde{C},\Omega_{\widetilde{C}}\otimes\check{\mathfrak{t}})^{\mathrm{W}}$ the corresponding section for the dual group.

We shall give an alternative description of $\theta_{\widetilde{C}}$. We denote by

$$\delta_\omega:\mathfrak{t}_\omega\to\mathfrak{t}_\omega\times_C\mathfrak{t}_\omega$$

[5] Note that $p\nmid n$.

the $\mathbb{G}_m$-twist by ω of the map δ as in Lemma 2.7.3. We regard $\mathfrak{t}_\omega$ and $\mathfrak{t}_\omega \times_C \mathfrak{t}_\omega$ (via the first projection) as schemes over $\mathfrak{c}_\omega$ and define

$$\delta_{\widetilde{C}} : \widetilde{C} = e^*\mathfrak{t}_\omega \to e^*(\mathfrak{t}_\omega \times_C \mathfrak{t}_\omega) = \widetilde{C} \times_C (T^*C \otimes \mathfrak{t})$$

to be the base change of δ_ω via the evaluation map $e : C \times B \to \mathfrak{c}_\omega$. By Lemma 2.7.3, $\delta_{\widetilde{C}}$ is just the pullback of the diagonal map $\mathfrak{t}_\omega \to \mathfrak{t}_\omega \times_C \mathfrak{t}_\omega$ along $e : C \times B \to \mathfrak{c}_\omega$.

By construction, the section $\theta_{\widetilde{C}} \in \Gamma(\widetilde{C}, \Omega_{\widetilde{C}} \otimes \mathfrak{t})^{\mathrm{W}}$ is equal to the composition

$$\widetilde{C} \xrightarrow{\delta_{\widetilde{C}}} \widetilde{C} \times_C (T^*C \otimes \mathfrak{t}) \to T^*\widetilde{C} \otimes \mathfrak{t} \tag{4.1.2}$$

where the last map is the cotangent map for the projection $\widetilde{C} \to C$.

The description of $\theta_{\widetilde{C}}$ in (4.1.2) implies the following relation between $\theta_{\widetilde{C}}$ and $\breve{\theta}_{\widetilde{C}}$.

Lemma 4.1.1. *Let $\sigma : \Gamma(\widetilde{C}, \Omega_{\widetilde{C}} \otimes \mathfrak{t})^{\mathrm{W}} \simeq \Gamma(\widetilde{C}, \Omega_{\widetilde{C}} \otimes \breve{\mathfrak{t}})^{\mathrm{W}}$ be the canonical isomorphism induced by the non-degenerate invariant form $(,)$ on $\mathfrak{t}$. We have $\sigma(\theta_{\widetilde{C}}) = \breve{\theta}_{\widetilde{C}}$.*

Remark 4.1.2. The 1-form $\theta_{\widetilde{C}}$ is related to the canonical 1-form ω_C of C in the following way. Let $\tilde{e} : \widetilde{C} \to T^*C \otimes \mathfrak{t}$ $(= \mathfrak{t}_\omega)$ be the natural W-equivariant map (see § 2.4). The natural W-equivariant pairing $\mathbb{X}^\bullet \times \mathfrak{t} \to k$ induces a W-equivariant map

$$\nu : \widetilde{C} \times \mathbb{X}^\bullet \xrightarrow{\tilde{e}\times \mathrm{id}} (T^*C \otimes \mathfrak{t}) \times \mathbb{X}^\bullet \to T^*C, \tag{4.1.3}$$

where W acts diagonally on $\widetilde{C} \times \mathbb{X}^\bullet$. Now the pullback of the canonical 1-form ω_C on T^*C along ν defines a section $\nu^*\omega_C \in \Gamma(\widetilde{C}, \Omega_{\widetilde{C}} \otimes \mathfrak{t})^{\mathrm{W}}$, and using the description of $\theta_{\widetilde{C}}$ in (4.1.2) one can check that $\theta_{\widetilde{C}} = \nu^*\omega_C$.

4.2 Canonical 1-form

Let us denote by $T^* \mathrm{Bun}_G^0$ the maximal smooth open substack of $T^* \mathrm{Bun}_G$. Then there is a tautological section

$$\theta_{\mathrm{can}} : T^* \mathrm{Bun}_G^0 \to T^*(T^* \mathrm{Bun}_G^0).$$

Note that $T^* \mathrm{Bun}_G \times_B B^0 \subset T^* \mathrm{Bun}_G^0$. From now on, we restriction everything to the open part B^0 and therefore will omit ^{0} from the subscript. Note that for a choice of the Kostant section κ, we have an isomorphism $\epsilon_\kappa : \mathscr{P} \simeq T^* \mathrm{Bun}_G$, and therefore we may regard θ_{can} as a section $\mathscr{P} \to T^*\mathscr{P}$, denoted by θ_κ.

Let $\mathrm{AJ}^{\mathscr{P}} : \widetilde{C} \times \mathbb{X}_\bullet \to \mathscr{P}$ be the Abel–Jacobi map. Write the pullback as

$$(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa = \{\theta_{\kappa,\lambda}\}_{\lambda\in\mathbb{X}_\bullet} \in \Gamma(\widetilde{C} \times \mathbb{X}_\bullet, \Omega_{\widetilde{C}})^{\mathrm{W}},$$

where $\theta_{\kappa,\lambda} \in \Gamma(\widetilde{C}, \Omega_{\widetilde{C}})$ is the restriction of $(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa$ to $\widetilde{C} \times \{\lambda\}$. A section $\{\alpha_\lambda\}_{\lambda\in\mathbb{X}_\bullet} \in \Gamma(\widetilde{C} \times \mathbb{X}_\bullet, \Omega_{\widetilde{C}})$ (respectively $\Gamma(\widetilde{C} \times \mathbb{X}_\bullet, \Omega_{\widetilde{C}/B})$) is called $\mathbb{X}_\bullet$-multiplicative if it satisfies $\alpha_{\lambda+\mu} = \alpha_\lambda + \alpha_\mu$, for any $\lambda, \mu \in \mathbb{X}_\bullet$. Clearly, any $\mathbb{X}_\bullet$-multiplicative section $\{\alpha_\lambda\}_{\lambda\in\mathbb{X}_\bullet}$ corresponds to a $\breve{\mathfrak{t}}$-valued section $\alpha \in \Gamma(\widetilde{C}, \Omega_{\widetilde{C}} \otimes \breve{\mathfrak{t}})$ (respectively $\Gamma(\widetilde{C}, \Omega_{\widetilde{C}/B} \otimes \breve{\mathfrak{t}})$). The rest of the section is mainly devoted to the proof of the following result.

Proposition 4.2.1. *The 1-form $(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa$ is $\mathbb{X}_\bullet$-multiplicative. Moreover, if we regard $(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa$ as a section of $\Gamma(\widetilde{C}, \Omega_{\widetilde{C}} \otimes \breve{\mathfrak{t}})^{\mathrm{W}}$, we have*

$$(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa = \breve{\theta}_{\widetilde{C}},$$

where $\breve{\theta}_{\widetilde{C}}$ is the section defined in § 4.1.

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We have the following corollary. Recall the notion of multiplicative sections $\mathscr{P} \to T^*\mathscr{P}$ as defined in § C.2.

Corollary 4.2.2. *The section θ_κ is multiplicative in the sense of § C.2. In addition, it is independent of the choice of Kostant section κ.*

Proof. We first show that θ_κ is multiplicative. Consider the section

$$m^*\theta_\kappa : \mathscr{P} \times_B \mathscr{P} \to T^*\mathscr{P} \times_{\mathscr{P}} (\mathscr{P} \times_B \mathscr{P}) \to T^*(\mathscr{P} \times_B \mathscr{P}),$$

where the first map is the base change of θ_κ along the multiplication $m : \mathscr{P} \times_B \mathscr{P} \to \mathscr{P}$, and the second map is the differential m_d of m. On the other hand, consider

$$(\theta_\kappa, \theta_\kappa) : \mathscr{P} \times_B \mathscr{P} \to (T^*\mathscr{P} \times T^*\mathscr{P})|_{\mathscr{P}\times_B\mathscr{P}} \to T^*(\mathscr{P} \times_B \mathscr{P}).$$

We need to show that $(\theta_\kappa, \theta_\kappa) = m^*\theta_\kappa$.

We first have the following lemma, whose proof is independent of Proposition 4.2.1.

Lemma 4.2.3. *The projection of θ_κ along $T^*\mathscr{P} \to T^*(\mathscr{P}/B)$ is multiplicative. More precisely, the images of $m^*\theta_\kappa$ and $(\theta_\kappa, \theta_\kappa)$ in $T^*(\mathscr{P} \times_B \mathscr{P}/B)$ are the same.*

Proof. Consider the following short exact sequence of vector bundles on $\mathscr{P} \times_B \mathscr{P}$

$$0 \to T^*B \times_B (\mathscr{P} \times_B \mathscr{P}) \to T^*(\mathscr{P} \times_B \mathscr{P}) \to T^*(\mathscr{P} \times_B \mathscr{P}/B) \to 0.$$

As the projection of θ_κ along $T^*\mathscr{P} \to T^*(\mathscr{P}/B)$ is identified with $\tau^* \times \mathrm{id}$ (cf. Lemma 2.7.2), the restriction of θ_κ to each fiber $\mathscr{P}_b$ is given by the 'constant' 1-form $\tau^*|_b \in \Gamma(C, (\mathrm{Lie}\, J_b)^* \otimes \omega)$. Therefore, $(\theta_\kappa, \theta_\kappa) = m^*\theta_\kappa$ in $T^*(\mathscr{P} \times_B \mathscr{P}/B)$. □

Therefore, their difference can be regarded as a section

$$m^*\theta_\kappa - (\theta_\kappa, \theta_\kappa) \in \Gamma(\mathscr{P} \times_B \mathscr{P}, \mathrm{pr}^*\Omega_B) = (\pi_0(\mathscr{P}) \times \pi_0(\mathscr{P})) \times \Gamma(B, \Omega_B).$$

The Abel–Jacobi map $\mathrm{AJ}^{\mathscr{P}} : \widetilde{C} \times \mathbb{X}_\bullet \to \mathscr{P}$ induces a map

$$\mathrm{AJ}^{\mathscr{P},2} : \widetilde{C} \times \mathbb{X}_\bullet \times \mathbb{X}_\bullet \to \mathscr{P} \times_B \mathscr{P}.$$

It is enough to show that the pullback of $m^*\theta_\kappa - (\theta_\kappa, \theta_\kappa)$ in

$$\Gamma((\widetilde{C} \times \mathbb{X}_\bullet \times \mathbb{X}_\bullet), \mathrm{pr}^*\Omega_B) = (\mathbb{X}_\bullet \times \mathbb{X}_\bullet) \times \Gamma(B, \Omega_B)$$

vanishes. By Proposition 4.2.1, the one form $(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa = \{\theta_{\kappa,\lambda}\}_{\lambda \in \mathbb{X}_\bullet}$ is $\mathbb{X}_\bullet$-multiplicative, and thus for any $\lambda, \mu \in \mathbb{X}_\bullet$ we have

$$(\mathrm{AJ}^{\mathscr{P},2})^*(m^*\theta_\kappa - (\theta_\kappa, \theta_\kappa))|_{\widetilde{C}\times\{\lambda\}\times\{\mu\}} = \theta_{\kappa,\lambda+\mu} - (\theta_{\kappa,\lambda} + \theta_{\kappa,\mu}) = 0.$$

This finishes the proof of multiplicative property of θ_κ. The independence of κ follows from $(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa = \breve{\theta}_{\widetilde{C}}$. □

Notation. In what follows, we denote the multiplicative 1-form θ_κ on $\mathscr{P}$ by θ_m.

Remark 4.2.4. Let $a := m \circ (\mathrm{AJ}^{\mathscr{P}} \times \mathrm{id}) : (\widetilde{C} \times \mathbb{X}_\bullet) \times_B \mathscr{P} \to \mathscr{P} \times_B \mathscr{P} \to \mathscr{P}$ be the action map. Then Remark 4.1.2 together with Corollary 4.2.2 implies that

$$a^*\theta_m = \breve{\nu}^*(\omega_C) \boxtimes \theta_m.$$

Here $\breve{\nu}$ is the map in (4.1.3) for the dual group. In the case of $G = \mathrm{GL}_n$, (a variant of) this identity was proved in [BB07, Theorem 4.12].

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4.3 Proof of Proposition 4.2.1: first reductions

Let $\underline{\theta}_\kappa$ and $\underline{\breve{\theta}}_{\widetilde{C}}$ be the projections of $(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa$ and $\breve{\theta}_{\widetilde{C}}$ along

$$\Gamma(\widetilde{C}\times\mathbb{X}_\bullet,\Omega_{\widetilde{C}})\to\Gamma(\widetilde{C}\times\mathbb{X}_\bullet,\Omega_{\widetilde{C}/B}).$$

Lemma 4.2.3 implies that $\underline{\theta}_\kappa$ is $\mathbb{X}_\bullet$-multiplicative and can be regarded as an element in $\Gamma(\widetilde{C},\Omega_{\widetilde{C}/B}\otimes\breve{\mathfrak{t}})^{\mathrm{W}}$. Let us first show that $\underline{\theta}_\kappa=\underline{\breve{\theta}}_{\widetilde{C}}$.

Recall in (2.7.4) we have introduced a morphism $\iota:\mathrm{Lie}\,J\to(\mathrm{Lie}\,J)^*$. It follows from the definition of ι in *loc. cit.* that the following diagram is commutative

$$\begin{tikzcd}
{[\pi]^*\mathrm{Lie}\,J} \arrow[r, "{[\pi]^*\iota}"] \arrow[d] & {[\pi]^*(\mathrm{Lie}\,J)^*} \\
\mathrm{Lie}\,\mathcal{T} \arrow[r] & \mathrm{Lie}\,\breve{\mathcal{T}} \arrow[u]
\end{tikzcd}$$

where the arrow in the bottom row is the morphism $\mathrm{Lie}\,\mathcal{T}\to\mathrm{Lie}\,\breve{\mathcal{T}}$ induced by the invariant from $(\,,)$ on $\mathfrak{t}$. (Recall (cf. § 4.1) that $\mathcal{T}:=\widetilde{C}\times^W T$ is a group scheme over $[\widetilde{C}/W]$ and $[\pi]:[\widetilde{C}/W]\to C\times B$.) It induces the following commutative diagram.

$$\begin{tikzcd}
\Gamma(C\times B,\Omega_{C\times B/B}\otimes\mathrm{Lie}\,J) \arrow[r, "\iota_*"] \arrow[d] & \Gamma(C\times B,\Omega_{C\times B/B}\otimes\mathrm{Lie}\,J^*) \\
\Gamma(\widetilde{C},\Omega_{\widetilde{C}/B}\otimes\mathfrak{t})^{\mathrm{W}} \arrow[r, "\sigma"] & \Gamma(\widetilde{C},\Omega_{\widetilde{C}/B}\otimes\breve{\mathfrak{t}})^{\mathrm{W}} \arrow[u, "\upsilon"']
\end{tikzcd}$$

Recall the sections $\tau\in\Gamma(C\times B,\Omega_{C\times B/B}\otimes\mathrm{Lie}\,J)$ and $\tau^*\in\Gamma(C\times B,\Omega_{C\times B/B}\otimes\mathrm{Lie}\,J^*)$ in § 2.7. Note the map υ in the diagram above is an isomorphism[6] and it identifies $\underline{\theta}_\kappa$ with the section τ^*. On the other hand, Lemma 4.1.1 implies the section $\underline{\breve{\theta}}_{\widetilde{C}}\in\Gamma(\widetilde{C},\Omega_{\widetilde{C}/B}\otimes\breve{\mathfrak{t}})^{\mathrm{W}}$ is equal to the image of τ under the composition of the morphisms in the lower left corner of the above diagram. Thus, $\upsilon(\underline{\breve{\theta}}_{\widetilde{C}})=\iota_*(\tau)=\tau^*$. Therefore both $\underline{\breve{\theta}}_{\widetilde{C}}$ and $\underline{\theta}_\kappa$ map to τ^* under the isomorphism υ, which implies $\underline{\breve{\theta}}_{\widetilde{C}}=\underline{\theta}_\kappa$.

As a consequence, difference $\breve{\theta}_{\widetilde{C}}-(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa$ can be regarded as a section

$$\breve{\theta}_{\widetilde{C}}-(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa\in\Gamma(\widetilde{C}\times\mathbb{X}_\bullet,\mathrm{pr}^*\Omega_B). \tag{4.3.1}$$

We need to show that it is zero. Let $\widetilde{U}\subset\widetilde{C}$ be the largest open subset such that $\widetilde{U}\to C\times B$ is étale. It is enough to show that $\breve{\theta}_{\widetilde{C}}-(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa|_{\widetilde{U}\times\mathbb{X}_\bullet}=0$. Note that for $\tilde{x}\in\widetilde{U}$ we have a canonical decomposition $T_{\tilde{x}}\widetilde{C}=T_xC\oplus T_bB$ and by (4.3.1) it suffices to show that $(\breve{\theta}_{\widetilde{C}}-(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa)|_{T_bB}=0$. As the section $\breve{\theta}_{\widetilde{C}}$ is induced by the canonical splitting $\Omega_{C\times B/B}\otimes\mathrm{Lie}\,\breve{J}\to\Omega_{C\times B}\otimes\mathrm{Lie}\,\breve{J}$, the restriction of $\breve{\theta}_{\widetilde{C}}$ to T_bB is zero, so we reduce to show that $(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa|_{T_bB}=0$, i.e., for any $\lambda\in\mathbb{X}_\bullet$ and $v\in T_bB$ we have

$$\langle\theta_{\kappa,\lambda},v\rangle=\langle(\mathrm{AJ}^{\mathscr{P}})^*\theta_\kappa|_{\widetilde{C}\times\{\lambda\}},v\rangle=0. \tag{4.3.2}$$

For the later purpose, we introduce some notations. Let (E_κ,ϕ_κ) be the Higgs field on $C\times B$ obtained by the pullback along the Kostant section κ. For every $\lambda\in\mathbb{X}_\bullet$ let $\mathrm{AJ}^{\mathscr{P},\lambda}:\widetilde{C}\to\mathscr{P}$ denote the corresponding component of the Abel–Jacobi map and let

$$(E_{\tilde{x}},\phi_{\tilde{x}}):=\mathrm{AJ}^{\mathscr{P},\lambda}(\tilde{x})\times^{J_b}(E_\kappa,\phi_\kappa)|_{C\times\{b\}},$$

[6] It is the relative cotangent map of the isogeny $\mathscr{P}\to\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)$.

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be the image of $\tilde{x}$ under the map $\widetilde{C} \stackrel{\mathrm{AJ}^{\mathscr{P},\lambda}}{\to} \mathscr{P} \stackrel{\epsilon_\kappa}{\simeq} T^* \operatorname{Bun}_G$. We also define

$$a_\lambda : \widetilde{C} \stackrel{\mathrm{AJ}^{\mathscr{P},\lambda}}{\to} \mathscr{P} \stackrel{\epsilon_\kappa}{\simeq} T^* \operatorname{Bun}_G \to \operatorname{Bun}_G.$$

Since $\theta_\kappa = \epsilon_\kappa^* \theta_{\mathrm{can}}$, we have

$$\langle \theta_{\kappa,\lambda}, v \rangle = \langle (\mathrm{AJ}^{\mathscr{P},\lambda})^* \epsilon_\kappa^* \theta_{\mathrm{can}}, v \rangle = \langle \theta_{\mathrm{can}}, (\epsilon_\kappa)_* (\mathrm{AJ}^{\mathscr{P},\lambda})_* v \rangle = \langle \phi_{\tilde{x}}, a_{\lambda *} v \rangle,$$

where $a_{\lambda *} : T_{\tilde{x}} \widetilde{C} \to T_{E_{\tilde{x}}} \operatorname{Bun}_G \simeq H^1(C, \mathrm{ad} E_{\tilde{x}})$ is the differential of a_λ and the last pairing is induced by the Serre duality $H^0(C, \mathrm{ad} E_{\tilde{x}} \otimes \Omega_C) \simeq H^1(C, \mathrm{ad} E_{\tilde{x}})^*$.

Therefore we reduce to show the following.

Proposition 4.3.1. *For any $v \in T_b B \subset T_{\tilde{x}} \widetilde{C} = T_x C \oplus T_b B$, the pairing $\langle \phi_{\tilde{x}}, a_{\lambda *} v \rangle$ is zero.*

4.4 Proof of Proposition 4.3.1: calculations of differentials

We shall need several preliminary steps. Recall that there is the E_κ-twist global Grassmannian $\mathrm{Gr}(E_\kappa)$ which classify the triples (x, E, β) where $x \in C$, E is a G-torsor and $\beta : E_\kappa|_{C-\{x\}} \simeq E|_{C-\{x\}}$ is an isomorphism. Given a dominant coweight μ (with respect to the set of simple roots we choose), it makes sense to talk about the closed substack $\mathrm{Gr}_{\leqslant \mu}(E_\kappa)$, consisting of those $\beta : E_\kappa|_{C-\{x\}} \simeq E|_{C-\{x\}}$ having relative position less than or equal to μ at x (cf. [BD91, § 5.2.2]). Let $\mathrm{Gr}_\mu(E_\kappa) = \mathrm{Gr}_{\leqslant\mu}(E_\kappa) - \bigcup_{\lambda<\mu} \mathrm{Gr}_{\leqslant\lambda}(E_\kappa)$. We have natural projection maps

$$\operatorname{Bun}_G \stackrel{pr_1}{\leftarrow} \mathrm{Gr}_{\leqslant\mu}(E_\kappa) \stackrel{pr_2}{\to} C.$$

For any $x \in C$, let

$$\mathrm{Gr}_x(E_\kappa) := \mathrm{Gr}(E_\kappa) \times_C \{x\},$$

and similarly we have $\mathrm{Gr}_{x,\leqslant\mu}(E_\kappa), \mathrm{Gr}_{x,\mu}(E_\kappa)$.

Note that for any $\tilde{x} \in \widetilde{C}$ the J-torsor $\mathrm{AJ}^{\mathscr{P},\lambda}(\tilde{x}) \in \mathscr{P}$ has a canonical trivialization s over $C - x$ (here x is the image of $\tilde{x}$ in C), thus it induces a canonical isomorphism $\beta : E_\kappa|_{C-x} \simeq E_{\tilde{x}}|_{C-x}$ (recall that $E_{\tilde{x}} := \mathrm{AJ}^{\mathscr{P},\lambda}(\tilde{x}) \times^J E_\kappa$). The assignment $\tilde{x} \to (x, E_{\tilde{x}}, \beta)$ defines a morphism $\tilde{a}_\lambda : \widetilde{C} \to \mathrm{Gr}(E_\kappa)$. We have the following key lemma.

Lemma 4.4.1. *Let μ be a dominant coweight and $\lambda \in \mathrm{W} \cdot \mu$. The morphism $\tilde{a}_\lambda$ factors through $\mathrm{Gr}_{\leqslant\mu}(E_\kappa)$, and the following diagram*

$$\begin{tikzcd} \widetilde{C} \arrow[r, "\tilde{a}_\lambda"] \arrow[dr, "a_\lambda"'] & \mathrm{Gr}_{\leqslant\mu}(E_\kappa) \arrow[d, "pr_1"] \\ & \operatorname{Bun}_G \end{tikzcd}$$

is commutative. Moreover, for any k-point $\tilde{x} \in \widetilde{U}(k)$, $\tilde{a}_\lambda(\tilde{x}) \in \mathrm{Gr}_\mu(E_\kappa)(k)$.

The proof is given at the end of this subsection. We also need the following lemma about the differential of $\tilde{a}_\lambda$.

Lemma 4.4.2. *Let $\tilde{x} \in \widetilde{U}(k)$, and let $\tilde{a}_\lambda(\tilde{x}) = (x, E_{\tilde{x}}, \beta) \in \mathrm{Gr}_\mu(E_\kappa)(k)$ (by Lemma 4.4.1). For every $v \in T_b B \subset T_{\tilde{x}} \widetilde{C} = T_x C \oplus T_b B$, we have*

$$u := (\tilde{a}_\lambda)_* v \in T_{(E_{\tilde{x}},\beta)} \mathrm{Gr}_{x,\mu}(E_\kappa) \subset T_{(x,E_{\tilde{x}},\beta)} \mathrm{Gr}_\mu(E_\kappa).$$

Proof. The subspace $T_{(E_{\tilde{x}},\beta)}\mathrm{Gr}_{x,\mu}(E_\kappa) \subset T_{(x,E_{\tilde{x}},\beta)}\mathrm{Gr}_\mu(E_\kappa)$ is equal to

$$\mathrm{Ker}((pr_2)_* : T_{(x,E_{\tilde{x}},\beta)}\mathrm{Gr}_\mu(E_\kappa) \to T_x C).$$

Therefore it is enough to show $(pr_2)_*(\tilde{a}_\lambda)_* v = 0$. Recall that we have the following commutative diagram (not Cartesian).

$$\begin{tikzcd} \widetilde{C} \arrow[r, "\tilde{a}_\lambda"] \arrow[d, "\pi"] & \mathrm{Gr}_{\leqslant\mu}(E_\kappa) \arrow[d, "pr_2"] \\ C \times B \arrow[r, "pr_C"] & C \end{tikzcd}$$

Thus we have $(pr_2)_*(\tilde{a}_\lambda)_* v = (pr_C)_*(\pi_* v) = (pr_C)_* v = 0$. This finishes the proof. □

Combining the above two lemmas we obtain that

$$\langle \theta_{\kappa,\lambda}, v\rangle = \langle \phi_{\tilde{x}}, a_{\lambda *} v\rangle = \langle \phi_{\tilde{x}}, (pr_1)_* u\rangle \tag{4.4.1}$$

where $u := (\tilde{a}_\lambda)_* v \in T_{(E_{\tilde{x}},\beta)}\mathrm{Gr}_{x,\mu}(E_\kappa)$. So we need show that the last pairing is zero. To calculate it, we need a few more notations. For any $x \in C$ we denote by $\mathcal{O}_x$ the completion of the local ring of C at x and F_x its fractional field. Let $\omega_{\mathcal{O}_x}$ (respectively ω_{F_x}) denote the completed regular (respectively rational) differentials on $\mathrm{Spec}\mathcal{O}_x$. We denote by

$$\mathrm{Res}(\,,\,) : \mathfrak{g}(\omega_{F_x}) \times \mathfrak{g}(F_x) \to k$$

the residue pairing induced by the G-invariant form $(\,,\,)$ on $\mathfrak{g}$.

Let us fix a trivialization $\gamma_\kappa : E_\kappa \simeq E^0$ on $\mathrm{Spec}\mathcal{O}_x$. Then, for every trivialization γ of E on $\mathrm{Spec}\mathcal{O}_x$, we obtain

$$g = \gamma_\kappa^{-1}\beta\gamma \in G(F_x).$$

In this way, γ_κ induces an isomorphism

$$\mathrm{Gr}_{x,\mu}(E_\kappa) \simeq \mathrm{Orb}_\mu, \quad (E,\beta) \mapsto \gamma_\kappa^{-1}\beta\gamma G(\mathcal{O}_x),$$

where Orb_μ is the $G(\mathcal{O}_x)$-orbit of $\mu \cdot G(\mathcal{O}_x) \in G(F_x)/G(\mathcal{O}_x)$. Under the isomorphism, we have the identification of the tangent spaces

$$T_{(E,\beta)}\mathrm{Gr}_{x,\mu}(E_\kappa) \simeq \mathfrak{g}(\mathcal{O}_x)/(\mathrm{Ad}_g\, \mathfrak{g}(\mathcal{O}_x) \cap \mathfrak{g}(\mathcal{O}_x)).$$

For any $u \in T_{(E,\beta)}\mathrm{Gr}_{x,\mu}(E_\kappa)$ and $\phi \in T_E\,\mathrm{Bun}_G$ the pairing $\langle \phi, (pr_1)_* u\rangle$ can be calculated as follows. Let $\tilde{u} \in \mathfrak{g}(\mathcal{O}_x)$ be a lifting of u under the above isomorphism. Let $\phi(\gamma)$ denote the $\phi : \mathrm{Spec}\mathcal{O}_x \to \mathrm{ad}E \otimes \omega_C \xrightarrow{\gamma} \mathfrak{g}(\omega_{F_x})$. Now we have

$$\langle \phi, (pr_1)_* u\rangle = \mathrm{Res}(\phi(\gamma), \mathrm{Ad}_g^{-1}\, \tilde{u}).$$

In our case $\phi = \phi_{\tilde{x}} = \mathrm{AJ}^{\mathscr{P},\lambda}(\tilde{x}) \times^J \phi_\kappa$, the following lemma will imply the vanishing of $\langle \phi_{\tilde{x}}, (pr_1)_* u\rangle$, and therefore will finish the proof of Proposition 4.3.1.

Lemma 4.4.3. *We have* $\mathrm{Ad}_g\, \phi_{\tilde{x}}(\gamma) \in \mathfrak{g}(\omega_{\mathcal{O}_x})$.

Proof. Indeed, unraveling the definitions, we have $\mathrm{Ad}_g\, \phi(\gamma) = \phi_\kappa(\gamma_\kappa)$, which is regular. □

It remains to prove Lemma 4.4.1. Let $\tilde{a}_\lambda : \widetilde{C} \to \mathrm{Gr}(E_\kappa)$ be the morphism constructed as in the lemma. Since $\widetilde{C}$ is smooth and $\widetilde{U} \subset \widetilde{C}$ is open dense, it is enough to show that $\tilde{a}_\lambda(\widetilde{U}(k)) \subset \mathrm{Gr}_\mu(E_\kappa)(k)$. Let $\tilde{x} \in \widetilde{U}(k)$ and $\tilde{a}_\lambda(\tilde{x}) = (x, E_{\tilde{x}}, \beta) \in \mathrm{Gr}(E_\kappa)(k)$ be its image, where $E_{\tilde{x}} := \mathrm{AJ}^{\mathscr{P},\lambda}(\tilde{x}) \times^J E_\kappa$ and $\beta : E_\kappa|_{C-x} \simeq E_{\tilde{x}}|_{C-x}$ is the isomorphism induced by the canonical section $s \in \mathrm{AJ}^{\mathscr{P},\lambda}(\tilde{x})(C-x)$. Let

$$\mathrm{rel} : \mathrm{Gr}_x(E_\kappa) \to \mathbb{X}_\bullet/\mathrm{W}$$

be the map sending an element (E, β) to the relative position of β (cf. [BD91, § 5.2.2]). We have $(E_{\tilde{x}}, \beta) \in \mathrm{Gr}_x(E_\kappa)$ and we need to show that $\mathrm{rel}(E_{\tilde{x}}, \beta) = \mu$. For simplicity, we will denote $\mathcal{P} := \mathrm{AJ}^{\mathscr{P},\lambda}(\tilde{x})$.

Let Gr_J (respectively Gr_T) be the global Grassmannian for the group scheme J (respectively T). By [Yun11, Lemma 3.2.5], the morphism $j^1 : \pi^*J \to T \times \widetilde{C}$ induces a W-equivariant isomorphism

$$j_{\mathrm{Gr}} : \mathrm{Gr}_J \times_{(C\times B)} \widetilde{U} \simeq \mathrm{Gr}_T \times_C \widetilde{U}$$

of group ind-schemes over $\widetilde{U}$. We denote by $j_{\tilde{x},\mathrm{Gr}} : \mathrm{Gr}_{x,J_b} \simeq \mathrm{Gr}_{x,T}$ the restriction of j_{Gr} to $\tilde{x}$. We have $(\mathcal{P}, s) \in \mathrm{Gr}_{x,J_b}(k)$ (here $s \in \mathcal{P}(C-x)$ is the canonical section) and one can check that $j_{\tilde{x},\mathrm{Gr}}(\mathcal{P}, s) = \lambda \in \mathrm{Gr}_{x,T}(k) \simeq \mathbb{X}_\bullet$. The action of Gr_{x,J_b} on (E_κ, ϕ_κ) defines a map $a_\kappa : \mathrm{Gr}_{x,J_b} \to \mathrm{Gr}_x(E_\kappa)$. We claim that the following diagram is commutative

$$\begin{tikzcd} \mathrm{Gr}_{x,J_b}(k) \arrow[r, "a_\kappa"] \arrow[d, "j_{\tilde{x},\mathrm{Gr}}"] & \mathrm{Gr}_x(E_\kappa)(k) \arrow[d, "\mathrm{rel}"] \\ \mathrm{Gr}_{x,T}(k) \simeq \mathbb{X}_\bullet \arrow[r] & \mathbb{X}_\bullet/\mathrm{W} \end{tikzcd} \tag{4.4.2}$$

Assuming the claim we see that $\mathrm{rel}(E, \beta) = \mathrm{rel}(a_\kappa(\mathcal{P}, s))$ is equal to the image of $j_{\tilde{x},\mathrm{Gr}}(\mathcal{P}, s) = \lambda \in \mathbb{X}_\bullet$ in $\mathbb{X}_\bullet/\mathrm{W}$. But by assumption $\lambda \in \mathrm{W} \cdot \mu$. This finishes the proof of Lemma 4.4.1.

To prove the claim, recall that a trivialization γ_κ of E_κ on $\mathrm{Spec}\mathcal{O}_x$ defines an isomorphism $\mathrm{Gr}_x(E_\kappa) \simeq G(F_x)/G(\mathcal{O}_x)$. Moreover, under the canonical isomorphism $\mathrm{Gr}_{x,J_b}(k) \simeq J_b(F_x)/J_b(\mathcal{O}_x)$, $\mathrm{Gr}_{x,T}(k) \simeq T(F_x)/T(\mathcal{O}_x)$ and $G(\mathcal{O}_x)\backslash G(F_x)/G(\mathcal{O}_x) = \mathbb{X}_\bullet/\mathrm{W}$, the diagram (4.4.2) can be identified with

$$\begin{tikzcd} J_b(F_x)/J_b(\mathcal{O}_x) \arrow[r] \arrow[d] & G(F_x)/G(\mathcal{O}_x) \arrow[d, "pr"] \\ T(F_x)/T(\mathcal{O}_x) \arrow[r] & G(\mathcal{O}_x)\backslash G(F_x)/G(\mathcal{O}_x) \end{tikzcd}$$

where the upper arrow is induced by the homomorphism

$$J_b \overset{a_{E_\kappa,\phi_\kappa}}{\simeq} \mathrm{Aut}(E_\kappa, \phi_\kappa) \to \mathrm{Aut}(E_\kappa) \overset{\gamma_\kappa}{\simeq} G \tag{4.4.3}$$

and the arrow in the left column is induced by the homomorphism $j^1 : \pi^*J \to T \times \widetilde{C}$. Let $b_x \in \mathfrak{c}^{rs}(\mathcal{O}_x)$ be the restriction of b to $\mathrm{Spec}\mathcal{O}_x$. Using the definition of a_{E_κ,ϕ_κ} in (2.6.1), it is not hard to see that the restriction of (4.4.3) to $\mathrm{Spec}\mathcal{O}_x$ is equal to

$$J_{b_x} \simeq I_{\mathrm{kos}(b_x)} \hookrightarrow G \times \mathrm{Spec}\mathcal{O}_x, \tag{4.4.4}$$

up to conjugation by an element in $G(\mathcal{O}_x)$. Here $\mathrm{kos}(b_x) : \mathrm{Spec}\mathcal{O}_x \overset{b_x}{\to} \mathfrak{c} \overset{\mathrm{kos}}{\to} \mathfrak{g} \in \mathfrak{g}^{\mathrm{reg}}(\mathcal{O}_x)$ and the first isomorphism is induced by the canonical isomorphism $\chi^*J|_{\mathfrak{g}^{\mathrm{reg}}} \simeq I|_{\mathfrak{g}^{\mathrm{reg}}}$ in Proposition 2.5.1.

Therefore, to prove the claim, it is enough to show that the restriction[7] of j^1 to $\mathrm{Spec}\mathcal{O}_x$ is equal to the map (4.4.4) up to left and right multiplication by elements in $G(\mathcal{O}_x)$. To see this, we first observe that the point $\tilde{x}$ defines a lifting $\tilde{b}_x \in \mathfrak{t}^{rs}(\mathcal{O}_x)$ of $b_x \in \mathfrak{c}^{rs}(\mathcal{O}_x)$. Since the map $G \times \mathfrak{g}^{rs} \to \mathfrak{g}^{rs} \times_{\mathfrak{c}} \mathfrak{g}^{rs}$, $(y, v) \to (\mathrm{Ad}\, y(v), v)$ is smooth, there exists $g \in G(\mathcal{O}_x)$ such that $\mathrm{Ad}\, g(\mathrm{kos}(b_x)) = \tilde{b}_x$. The map $\mathrm{Ad}\, g$ induced an isomorphism $\iota_g : I_{\mathrm{kos}(b_x)} \simeq I_{\tilde{b}_x} = T \times \mathrm{Spec}\mathcal{O}_x$, which is independent of the choice of g, and according to [Ngô10, Proposition 2.4.2] the restriction of j^1 to $\mathrm{Spec}\mathcal{O}_x$ is given by

$$J_{b_x} \simeq I_{\mathrm{kos}(b_x)} \overset{\iota_g}{\simeq} T \times \mathrm{Spec}\mathcal{O}_x, \tag{4.4.5}$$

where the first map is the canonical isomorphism mentioned before. The above description implies the map (4.4.4) is equal to the map (4.4.5) up to left and right multiplication by elements in $G(\mathcal{O}_x)$. This finishes the proof of the claim.

5. Main result

We assume that G is semi-simple over k whose characteristic p is positive and does not divide the order of the Weyl group of G. Let C be a smooth projective curve over k, of genus at least two. In this case, Bun_G is a 'good' stack in the sense of [BD91, § 1.1.1] (see also § B.5). Let D_{Bun_G} be the sheaf of algebras on Higgs'_G in Proposition B.5.1. Denote by $D^0_{\mathrm{Bun}_G} := D_{\mathrm{Bun}_G}|_{\mathrm{Higgs}'_G \times_{B'} B'^0}$ the restriction of D_{Bun_G} to the smooth part of the Hitchin fibration. We define $\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0$ as the category of $D^0_{\mathrm{Bun}_G}$-modules. As explained in § B.5, the category $\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0$ is a localization of the category of $\mathcal{D}$-modules on Bun_G and is canonical equivalent to the category of twisted sheaves $\mathrm{QCoh}(\mathscr{D}^0_{\mathrm{Bun}_G})_1$, where $\mathscr{D}^0_{\mathrm{Bun}_G} = \mathscr{D}_{\mathrm{Bun}_G} \times_{B'} B'^0$ and $\mathscr{D}_{\mathrm{Bun}_G}$ is the gerbe of crystalline differential operators on Higgs'_G. On the dual side, let $\mathrm{LocSys}_{\check{G}}$ be the stack of de Rham $\check{G}$-local systems on C. Recall that in [CZ15], we constructed a fibration

$$h_p : \mathrm{LocSys}_{\check{G}} \to B'$$

from $\mathrm{LocSys}_{\check{G}}$ to the Hitchin base B', which can be regraded as a deformation of the usual Hitchin fibration. We define

$$\mathrm{LocSys}^0_{\check{G}} := \mathrm{LocSys}_{\check{G}} \times_{B'} B'^0.$$

Our goal is to prove the following theorem.

Theorem 5.0.1. *Assume G is semi-simple and the genus of C is at least two. For a choice of a square root κ of ω_C, we have a canonical equivalence of bounded derived categories*

$$\mathfrak{D}_\kappa : D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0) \simeq D^b(\mathrm{QCoh}(\mathrm{LocSys}^0_{\check{G}})).$$

The proof of above theorem is divided into two steps. The first step, which involves the Langlands duality, is to establish a twisted version of the classical duality (see § 5.2). The second step, which does not involve the Langlands duality, is to establish two abelianization theorems (see § 5.3) for which we need a choice of square root κ of ω_C. Combining above two steps, our main theorem follows from a general version of the Fourier–Mukai transform (see § 5.4).

[7] Here we identify $\mathrm{Spec}\mathcal{O}_{\tilde{x}} \simeq \mathrm{Spec}\mathcal{O}_x$ and regard j^1 as a map of group schemes over $\mathrm{Spec}\mathcal{O}_x$.

5.1 The $\breve{\mathscr{P}}'$-torsor $\breve{\mathscr{H}}$

We first recall that in [CZ15], we constructed a $\breve{\mathscr{P}}'$-torsor $\breve{\mathscr{H}}$. It is defined via the following Cartesian diagram.

$$\begin{tikzcd} \breve{\mathscr{H}} \arrow[r] \arrow[d] & \mathrm{LocSys}_{\breve{J}^p} \arrow[d] \\ B' \arrow[r, "\breve{\tau}'"] & B'_{\breve{J}'} \end{tikzcd} \tag{5.1.1}$$

Here $\breve{J}^p$ is the pullback of the universal centralizer $\breve{J}'$ over $C' \times B'$ along the relative Frobenius map $F_{C' \times B'/B'} : C \times B' \to C' \times B'$. This is a group scheme with a canonical connection along C, and therefore it makes sense to talk about the stack $\mathrm{LocSys}_{\breve{J}^p}$ of $\breve{J}^p$-torsors with flat connections. In addition, it admits a map $\mathrm{LocSys}_{\breve{J}^p} \to B'_{\breve{J}'}$. We refer to [CZ15, Appendix] for generalities.

Recall that there is a description of $\mathscr{P}$ in terms of $\mathrm{Bun}_T^{\mathrm{W}}(\widetilde{C}/B)$. We give a similar description of the $\breve{\mathscr{P}}'$-torsor $\breve{\mathscr{H}}$ in terms of a $\mathrm{Bun}_{\breve{T}}^{\mathrm{W}}(\widetilde{C}/B)'$-torsor. Recall that $\breve{\tau}'$ is regarded as a section of $\Omega_{C' \times B'/B'} \otimes \mathrm{Lie}\, \breve{J}'$, which defines a $\breve{J}'$-gerbe $\mathscr{D}(\breve{\tau}')$ on $C' \times B'$ (see § B.4) and according to [CZ15, Proposition A.9], $\breve{\mathscr{H}}$ is isomorphic to $\mathscr{T}_{\mathscr{D}(\breve{\tau}')}$, the stack of splittings of $\mathscr{D}(\breve{\tau}')$ over B'. Therefore by Lemma 3.2.1 we have

$$\breve{\mathscr{H}}|_{B'^0} \simeq \mathscr{T}^{\mathrm{W},+}_{\mathscr{D}(\breve{\tau}')_{\breve{T}}}|_{B'^0}, \tag{5.1.2}$$

where $\mathscr{D}(\breve{\tau}')_{\breve{T}} := (\pi^* \mathscr{D}(\breve{\tau}'))^{\breve{j}^1}$ is the $\breve{T}$-gerbe on $\widetilde{C}'$ induced from $\mathscr{D}(\breve{\tau}')$ using maps $\pi : \widetilde{C}' \to C' \times B'$ and $\breve{j}^1 : \pi^* \breve{J}' \to \breve{T}' \times \widetilde{C}'$ (see § A.6 for the induction functor of gerbes).

On the other hand, using the definition of $\theta_{\widetilde{C}'} \in \Gamma(\widetilde{C}', \Omega_{\widetilde{C}'} \otimes \breve{\mathfrak{t}}')^{\mathrm{W}}$ in § 4.1 one can check that $\breve{j}^1_* \pi^*(\breve{\tau}') = \theta_{\widetilde{C}'}$, where $\breve{j}^1_* \pi^*(\breve{\tau}')$ is the $\breve{\mathfrak{t}}$-valued 1-form induced from $\breve{\tau}'$ using maps π and $\breve{j}^1$. Therefore, by Lemma B.4.1 we see that over B'^0 we have

$$\mathscr{D}(\breve{\tau}')_{\breve{T}} := (\pi^* \mathscr{D}(\breve{\tau}'))^{\breve{j}^1} \simeq \mathscr{D}(\breve{j}^1_* \pi^*(\breve{\tau}')) \simeq \mathscr{D}(\theta_{\widetilde{C}'}). \tag{5.1.3}$$

Hence combining (5.1.2) and (5.1.3) we get the following Galois description of $\breve{\mathscr{H}}$.

Corollary 5.1.1. *There is a canonical isomorphism of $\breve{\mathscr{P}}'$-torsors $\breve{\mathscr{H}}|_{B'^0} \simeq \mathscr{T}^{\mathrm{W},+}_{\mathscr{D}(\theta_{\widetilde{C}'})}|_{B'^0}$.*

5.2 Twisted duality

Let us construct the twisted duality. Let $\theta'_m : \mathscr{P}' \to T^*\mathscr{P}'$ denote the canonical multiplicative one form constructed in § 4.2. Let $\mathscr{D}(\theta'_m)$ denote the corresponding $\mathbb{G}_m$-gerbe on $\mathscr{P}'$ obtained by pullback of $\mathscr{D}_{\mathscr{P}}$ on $T^*\mathscr{P}'$ by θ'_m (see § B.4). According to § C.2, the gerbe $\mathscr{D}(\theta'_m)$ is canonically multiplicative. Moreover, according to § A.7, the stack of multiplicative splittings of $\mathscr{D}(\theta'_m)$ over B' is a $(\mathscr{P}')^\vee$-torsor $\mathscr{T}_{\mathscr{D}(\theta'_m)}$. Our goal is to prove the following theorem.

Theorem 5.2.1. *There is a canonical isomorphism of $\mathscr{P}'^\vee \simeq \breve{\mathscr{P}}'$-torsors*

$$\mathfrak{D} : \mathscr{T}_{\mathscr{D}(\theta'_m)}|_{B'^0} \simeq \breve{\mathscr{H}}|_{B'^0}.$$

For the rest of this subsection we will restrict everything to B'^0. Recall the Abel–Jacobi map $\mathrm{AJ}^{\mathscr{P}'} : \widetilde{C}' \times \mathbb{X}_\bullet \to \mathscr{P}'$. By Proposition 4.2.1 we have $(\mathrm{AJ}^{\mathscr{P}'})^* \theta'_m = \theta_{\widetilde{C}'}$. Therefore, Lemma B.4.1 implies that

$$(\mathrm{AJ}^{\mathscr{P}'})^* \mathscr{D}(\theta'_m) = \mathscr{D}(\theta_{\widetilde{C}'}).$$

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Since the Abel–Jacobi map $\mathrm{AJ}^{\mathscr{P}'}$ is W-equivariant, pullback via $\mathrm{AJ}^{\mathscr{P}'}$ defines a functor

$$\tilde{\mathfrak{D}}:\mathscr{T}_{\mathscr{D}(\theta'_m)}\to\mathscr{T}^{\mathrm{W}}_{\mathscr{D}(\theta_{\tilde{C}'})}.$$

We claim that $\tilde{\mathfrak{D}}$ canonically lifts to a morphism

$$\mathfrak{D}:\mathscr{T}_{\mathscr{D}(\theta'_m)}\to\mathscr{T}^{\mathrm{W},+}_{\mathscr{D}(\theta_{\tilde{C}'})}\simeq\breve{\mathscr{H}},$$

where the second isomorphism is Corollary 5.1.1. Let $E\in\mathscr{T}_{\mathscr{D}(\theta'_m)}$ be a tensor splitting of $\mathscr{D}(\theta'_m)$. We need to show that the splitting

$$(\tilde{\mathfrak{D}}(E))^{\alpha}|_{\widetilde{C}'_\alpha}=(\mathrm{AJ}^{\mathscr{P}'})^*E|_{(\widetilde{C}'_\alpha,\breve{\alpha})}$$

admits a canonical isomorphism compatible with the canonical splitting E^0_α of $\mathscr{D}(\theta_{\tilde{C}'})^{\alpha}|_{\widetilde{C}'_\alpha}=(\mathrm{AJ}^{\mathscr{P}'})^*\mathscr{D}(\theta'_m)|_{(\widetilde{C}'_\alpha,\breve{\alpha})}$. However, this follows from the fact that $\mathrm{AJ}^{\mathscr{P}'}((x,\breve{\alpha}))$ is the unit of $\mathscr{P}'$ for $x\in\widetilde{C}'_\alpha$ and a tensor splitting E of a multiplicative $\mathbb{G}_m$-gerbe $\mathscr{D}(\theta'_m)$ is canonically isomorphic to the canonical splitting E^0_α of $\mathscr{D}(\theta'_m)$ over the unit. To summarize, we have constructed the following commutative diagram.

$$\begin{tikzcd}
\mathscr{T}_{\mathscr{D}(\theta'_m)} \arrow[rr, "\mathfrak{D}"] \arrow[dr, "\tilde{\mathfrak{D}}"'] & & \breve{\mathscr{H}} \arrow[dl, "For"] \\
& \mathscr{T}^{\mathrm{W}}_{\mathscr{D}(\theta_{\tilde{C}'})} &
\end{tikzcd} \tag{5.2.1}$$

By construction, the morphism $\mathfrak{D}$ is compatible with the $\mathscr{P}'^{\vee}\simeq\breve{\mathscr{P}}'$-action, and hence is an equivalence. This finishes the proof of Theorem 5.2.1.

5.3 Abelianization theorems

We need to fix a square root κ of ω_C. Then the Kostant section for $\mathrm{Higgs}'_G\to B'$ induces a map

$$\epsilon_{\kappa'}:\mathscr{P}'\simeq{\mathrm{Higgs}'_G}^{\mathrm{reg}}\subset\mathrm{Higgs}'_G,$$

where ${\mathrm{Higgs}'_G}^{\mathrm{reg}}$ is the smooth sub-stack consisting of regular Higgs fields. The first abelianization theorem is the following.

Theorem 5.3.1. *We have a canonical isomorphism $\epsilon^*_{\kappa'}\mathscr{D}_{\mathrm{Bun}_G}\simeq\mathscr{D}(\theta'_m)$, where $\mathscr{D}_{\mathrm{Bun}_G}$ is the $\mathbb{G}_m$-gerbe on Higgs'_G of crystalline differential operators. Moreover, the pullback along the map $\epsilon_{\kappa'}$ defines an equivalence of categories of twisted sheaves*

$$\mathfrak{A}_\kappa:D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}^0_G))\simeq D^b(\mathrm{QCoh}(\mathscr{D}^0_{\mathrm{Bun}_G}))_1\overset{\epsilon^*_{\kappa'}}{\simeq}D^b(\mathrm{QCoh}(\mathscr{D}(\theta'_m)|_{B'^0}))_1.$$

Proof. By Proposition B.3.3, the restriction of $\mathscr{D}_{\mathrm{Bun}_G}$ to ${\mathrm{Higgs}'_G}^{\mathrm{reg}}$ is isomorphic to the gerbe $\mathscr{D}(\theta'_{\mathrm{can}})$ defined by the canonical 1-form θ'_{can} on ${\mathrm{Higgs}'_G}^{\mathrm{reg}}$. On the other hand, it follows from the construction of θ_m in § 4.2 that we have $\epsilon^*_{\kappa'}\theta'_{\mathrm{can}}=\theta'_m$. Hence

$$\epsilon^*_{\kappa'}\mathscr{D}_{\mathrm{Bun}_G}\simeq\epsilon^*_{\kappa'}\mathscr{D}(\theta'_{\mathrm{can}})\simeq\mathscr{D}(\epsilon^*_{\kappa'}\theta_{\mathrm{can}})\simeq\mathscr{D}(\theta'_m).$$

The last statement follows from the fact that the base change of $\epsilon_{\kappa'}:\mathscr{P}'\to\mathrm{Higgs}'_G$ to B'^0 is an isomorphism (see Proposition 2.6.1). $\square$

To state the second abelianization theorem, recall that in [CZ15] we constructed a canonical isomorphism

$$\mathfrak{C} : \breve{\mathscr{H}} \times^{\breve{\mathscr{P}}'} \mathrm{Higgs}'_{\breve{G}} \simeq \mathrm{LocSys}_{\breve{G}}\,.$$

Moreover, by [CZ15, Remark 3.14] the choice of the theta characteristic κ defines an isomorphism $\epsilon_{\kappa'} : \mathscr{P}' \simeq \mathrm{Higgs}'^{\,\mathrm{reg}}_{G}$, and hence induces an isomorphism

$$\mathfrak{C}_\kappa : \breve{\mathscr{H}} \simeq \mathrm{LocSys}^{\mathrm{reg}}_{\breve{G}} \subset \mathrm{LocSys}_{\breve{G}}\,.$$

Here $\mathrm{LocSys}^{\mathrm{reg}}_{\breve{G}}$ is the open substack consisting of $\breve{G}$-local systems with regular p-curvature, and we have $\mathrm{LocSys}^{\mathrm{reg}}_{\breve{G}}|_{B'^0} = \mathrm{LocSys}^0_{\breve{G}}$. It implies the following.

Theorem 5.3.2. *For each choice of a square root κ of ω_C, we have a canonical isomorphism of $\breve{\mathscr{P}}'$-torsors $\mathfrak{C}_\kappa|_{B'^0} : \breve{\mathscr{H}}|_{B'^0} \simeq \mathrm{LocSys}^0_{\breve{G}}$ and it induces an equivalence of categories*

$$\mathfrak{C}^*_\kappa : D^b(\mathrm{QCoh}(\mathrm{LocSys}^0_{\breve{G}})) \simeq D^b(\mathrm{QCoh}(\breve{\mathscr{H}}|_{B'^0})).$$

5.4 Proof of Theorem 5.0.1

We deduce our main theorem from the twisted duality and above two abelianization theorems. By the twisted duality we have an isomorphism of $\mathscr{P}'^\vee \simeq \breve{\mathscr{P}}'$-torsors $\mathscr{T}_{\mathscr{D}(\theta'_m)}|_{B'^0} \simeq \breve{\mathscr{H}}|_{B'^0}$. Therefore the twisted Fourier–Mukai transform (Theorem A.7.2) implies an equivalence of categories

$$\mathfrak{D} : D^b(\mathrm{QCoh}(\mathscr{D}(\theta'_m)|_{B'^0}))_1 \simeq D^b(\mathrm{QCoh}(\breve{\mathscr{H}}|_{B'^0})).$$

Now combining Theorems 5.3.1 and 5.3.2 we get the desired equivalence

$$\mathfrak{D}_\kappa = (\mathfrak{C}^*_\kappa)^{-1} \circ \mathfrak{D} \circ \mathfrak{A}_\kappa : D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}^0_G)) \simeq D^b(\mathrm{QCoh}(\mathrm{LocSys}^0_{\breve{G}})).$$

5.5 A μ_2-gerbe of equivalences

In this subsection we study how those equivalences $\mathfrak{D}_\kappa : D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0) \simeq D^b(\mathrm{QCoh}(\mathrm{LocSys}^0_{\breve{G}}))$ in Theorem 5.0.1 depend on the choices of the theta characteristics κ. Our discussion is very similar to [FW08] and can be regarded as a verification of the predictions of [FW08] in our setting.

Let $\omega^{1/2}(C)$ be the groupoid of square roots of ω_C. The groupoid $\omega^{1/2}(C)$ is a torsor over the Picard category μ_2-tors(C) of μ_2-torsors on C. Let **GLC** be the groupoid of equivalences between $D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0)$ and $D^b(\mathrm{QCoh}(\mathrm{LocSys}^0_{\breve{G}}))$, i.e., objects in **GLC** are equivalences $\mathbf{E} : D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0) \simeq D^b(\mathrm{QCoh}(\mathrm{LocSys}^0_{\breve{G}}))$ and morphisms are isomorphisms between equivalences. We first construct an action of μ_2-tors(C) on **GLC**.

Let $Z = Z(G)$ be the center of G. We have a map $\alpha : \mu_2 \to Z(G)$ by restricting the cocharacter $2\rho : \mathbb{G}_m \to G$ to μ_2 (see [BD91, § 3.4.2]). Thus for each $\chi \in \mu_2$-tors(C) and $(E, \nabla) \in \mathrm{LocSys}_G$ we can twist (E, ∇) by χ using the map

$$\mu_2 \to \ Z \to \mathrm{Aut}(E, \nabla)$$

to get a new G-local system $(E \otimes \chi, \nabla_{E\otimes\chi}) \in \mathrm{LocSys}_G$. The assignment $(\chi, E, \nabla) \to (E \otimes \chi, \nabla_{E\otimes\chi})$ defines a geometric action

$$\mathrm{act}_G : \mu_2\text{-tors}(C) \times \mathrm{LocSys}_G \to \mathrm{LocSys}_G\,.$$

Likewise, there is $\mathrm{act}_G : \mu_2\text{-tors}(C) \times \mathrm{Bun}_G \to \mathrm{Bun}_G$. For $\chi \in \mu_2$-tors(C), let $a_{\chi,G} : \mathrm{Bun}_G \simeq \mathrm{Bun}_G$ (respectively, $b_{\chi,G} : \mathrm{LocSys}_G \simeq \mathrm{LocSys}_G$) be the automorphisms of Bun_G (respectively LocSys_G)

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given by $a_{\chi,G}(E) := E \otimes \chi$, (respectively $b_{\chi,G}(E,\nabla) = \mathrm{act}_G(\chi, E, \nabla)$). They induce auto-equivalences $a^*_{\chi,G}$ and $b^*_{\chi,G}$ of $D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G))$ and $D^b(\mathrm{QCoh}(\mathrm{LocSys}_G))$ respectively. Note that for the definition of $a^*_{\chi,G}$ and $b^*_{\chi,G}$, there is no restriction of the characteristic of k. However, if $\mathrm{char} k = p \nmid |\mathrm{W}|$, we have the following.

Lemma 5.5.1.

(1) *The equivalence $a^*_{\chi,G}$ preserves the full subcategory $D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0)$.*

(2) *The equivalence $b^*_{\chi,G}$ preserves the full subcategory $D^b(\mathrm{QCoh}(\mathrm{LocSys}^0_G))$.*

Proof. This lemma will be clear after we give alternative descriptions of $a^*_{\chi,G}$ and $b^*_{\chi,G}$.

First, recall that in § B.3 we introduce a $\mathbb{G}_m$-gerbe $\mathscr{D}_{\mathrm{Bun}_G}$ over $T^*\mathrm{Bun}'_G$ and the category $\mathrm{QCoh}(\mathscr{D}_{\mathrm{Bun}_G})_1$ of twisted sheaves on $\mathscr{D}_{\mathrm{Bun}_G}$ such that there is an equivalence of categories between $\mathcal{D}\text{-mod}(\mathrm{Bun}_G)$ and $\mathrm{QCoh}(\mathscr{D}_{\mathrm{Bun}_G})_1$. Let $f := da'_{\chi,G} : T^*\mathrm{Bun}'_G \simeq T^*\mathrm{Bun}'_G$ be the differential of $a'_{\chi,G}$. The map f preserves the canonical one form θ'_{can}, and thus by Lemma B.4.1, there is a canonical 1-morphism $M : f^*\mathscr{D}_{\mathrm{Bun}_G} \sim \mathscr{D}_{\mathrm{Bun}_G}$ of gerbes on $T^*\mathrm{Bun}'_G$. The 1-morphism M induces an equivalence $M : \mathrm{QCoh}(f^*\mathscr{D}_{\mathrm{Bun}_G})_1 \simeq \mathrm{QCoh}(\mathscr{D}_{\mathrm{Bun}_G})_1$ and it follows from definitions that the functor $a^*_{\chi,G}$ is isomorphic to the composition

$$D^b(\mathrm{QCoh}(\mathscr{D}_{\mathrm{Bun}_G})_1) \overset{f^*}{\simeq} D^b(\mathrm{QCoh}(f^*\mathscr{D}_{\mathrm{Bun}_G})_1) \overset{M}{\simeq} D^b(\mathrm{QCoh}(\mathscr{D}_{\mathrm{Bun}_G})_1). \tag{5.5.1}$$

Recall that the category $\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0$ is by definition the category of twisted sheaves on $\mathscr{D}^0_{\mathrm{Bun}_G} = \mathscr{D}_{\mathrm{Bun}_G}|_{B'^0}$. Therefore, part (1) follows.

To prove part (2), note that the map $\mathrm{act}_G : \mu_2\text{-tors}(C) \times \mathrm{LocSys}_G \to \mathrm{LocSys}_G$ can be also described as follows. There is a map of group schemes $(\mu_2)_{C'\times B'} \to Z(G)_{C'\times B'} \to J'$ over $C' \times B'$, which induces a morphism of Picard stacks

$$\mathfrak{l}_{\mu_2} : \mu_2\text{-tors}(C) \times B' \to \mathscr{P}', \tag{5.5.2}$$

and the action map act_G can be identified with

$$\mathrm{act}_G : \mu_2\text{-tors}(C) \times \mathrm{LocSys}_G \overset{\mathfrak{l}_{\mu_2}\times\mathrm{id}}{\to} \mathscr{P}' \times_{B'} \mathrm{LocSys}_G \to \mathrm{LocSys}_G \tag{5.5.3}$$

where the last map is the action of $\mathscr{P}'$ on LocSys_G defined in [CZ15, Proposition 3.5]. In particular, if we endow B' with the trivial : $\mu_2\text{-tors}(C)$ action, the p-Hitchin map $\mathrm{LocSys}_G \to B'$ is : $\mu_2\text{-tors}(C)$-equivariant. Therefore LocSys^0_G is invariant under the action of $b_{\chi,G}$, and part (2) follows. □

From now on we regard $a^*_{\chi,G}$ and $b^*_{\chi,G}$ as automorphisms of the category $D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0)$ and $D^b(\mathrm{Qcoh}(\mathrm{LocSys}^0_G))$.

For each $\chi \in \mu_2\,\mathrm{tors}(C)$ and $\mathbf{E} \in \mathbf{GLC}$ we define

$$\chi \cdot \mathbf{E} := b^*_{\chi,\check{G}} \circ \mathbf{E} \circ a^*_{\chi,G} \in \mathbf{GLC}.$$

The following lemma follows from the construction of $b^*_{\chi,\check{G}}$ and $a^*_{\chi,G}$.

Lemma 5.5.2. *The functor $\mu_2\text{-tors}(C) \times \mathbf{GLC} \to \mathbf{GLC}$ given by $(\chi, \mathbf{E}) \to \chi \cdot \mathbf{E}$ defines an action of the Picard category $\mu_2\text{-tors}(C)$ on $\mathbf{GLC}$.*

Now let $\mathscr{C}_1$ and $\mathscr{C}_2$ be two categories acted by a Picard category $\mathscr{G}$. A $\mathscr{G}$-module functor from $\mathscr{C}_1$ to $\mathscr{C}_2$ is a functor $N : \mathscr{C}_1 \to \mathscr{C}_2$ equipped with functorial isomorphisms $N(a \cdot c) \simeq a \cdot N(c)$ satisfying the natural compatibility condition. Here is the main result of this subsection.

Proposition 5.5.3. *The assignment $\kappa \to \mathfrak{D}_\kappa$ defines a μ_2-tors(C)-module functor*

$$\Phi : \omega^{1/2}(C) \to \mathbf{GLC}.$$

Proof. Given $\chi \in \mu_2$-tors(C) and $\kappa \in \omega^{1/2}(C)$ we need to specify a functorial isomorphism $\mathfrak{D}_{\chi\cdot\kappa} \simeq \chi \cdot \mathfrak{D}_\kappa$ satisfying the natural compatibility condition. First, observe that the maps $\epsilon_{\kappa'}$, $\epsilon_{\kappa_1'} : \mathscr{P}' \to \mathrm{Higgs}_G'$ induced by $\kappa, \kappa_1 := \chi \cdot \kappa \in \mu_2$-tors$(C)$ differ by a translation of the section $\mathfrak{l}_{\mu_2}(\{\chi\} \times B') \in \mathscr{P}'(B')$, where $\mathfrak{l}_{\mu_2}$ is the map in (5.5.2). Then it follows from the construction of $\mathfrak{A}_\kappa$ and $\mathfrak{C}_\kappa$ in § 5.3 that there are canonical functorial isomorphisms $\mathfrak{A}_{\chi\cdot\kappa} \simeq \mathfrak{A}_\kappa \circ a^*_{\chi,G}$ and $\mathfrak{C}^*_\kappa \circ b^*_{\chi,\check{G}} \simeq \mathfrak{C}^*_{\chi\cdot\kappa}$. Therefore we get a functorial isomorphism

$$\mathfrak{D}_{\chi\cdot\kappa} = (\mathfrak{C}^*_{\chi\cdot\kappa})^{-1} \circ \mathfrak{D} \circ \mathfrak{A}_{\chi\cdot\kappa} \simeq b^*_{\chi,\check{G}} \circ (\mathfrak{C}^*_\kappa)^{-1} \circ \mathfrak{D} \circ \mathfrak{A}_\kappa \circ a^*_{\chi,G} = \chi \cdot \mathfrak{D}_\kappa,$$

and one can check that it satisfies the natural compatibility condition. □

Remark 5.5.4. The above construction suggests that the geometric Langlands correspondence should be a μ_2-gerbe of equivalences between $D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G))$ and $D^b(\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}))$. This gerbe is trivial, but is not canonically trivialized. One obtains a particular trivialization of this gerbe, and hence a particular equivalence $\mathfrak{D}_\kappa$, for each choice of a square root of the canonical line bundle of C. A similar μ_2-gerbe also appears in the work of Frenkel and Witten [FW08, § 10], where the geometric Langlands correspondence is interpreted as gauge theory duality between the twisted A-model of Higgs_G and the twisted B-model of $\mathrm{Higgs}_{\check{G}}$.

5.6 The actions $a^*_{\chi,G}$ and $b^*_{\chi,G}$ as tensoring of line bundles

In this subsection we show that, under the equivalence $\mathfrak{D}_\kappa$, the geometric actions $a^*_{\chi,G}$ and $b^*_{\chi,G}$ constructed in the previous subsection become functors of tensoring with certain line bundles.

Recall that in § 3.7 we associated to every $Z(\check{G})$-torsor K on C a line bundle $\mathcal{L}_{G,K}$ on Bun_G. For any $\chi \in \mu_2$-tors(C) let $K_{G,\chi} := \chi \times^{\mu_2} Z_G \in Z(G)$-tors$(C)$ be the induced $Z(G)$-torsor via the canonical map $2\rho : \mu_2 \to Z(G)$. We denote by $\mathcal{L}_{G,\chi}$ and $\mathcal{L}_{\check{G},\chi}$ be the line bundles on Bun_G and $\mathrm{Bun}_{\check{G}}$ corresponding to $K_{\check{G},\chi}$ and $K_{G,\chi}$. Since the line bundle $\mathcal{L}_{G,\chi}$ carries a canonical connection with zero p-curvature, tensoring with $\mathcal{L}_{G,\chi}$ defines an auto-equivalence $\mathcal{L}_{G,\chi}\otimes ?$ of $D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0)$.

For any $\kappa \in \omega^{1/2}(C)$ let $\mathfrak{D}_\kappa : D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0) \simeq D^b(\mathrm{QCoh}(\mathrm{LocSys}^0_{\check{G}}))$ be the equivalence in Theorem 5.0.1.

Theorem 5.6.1. *We have the following.*

(1) *The equivalence $\mathfrak{D}_\kappa$ intertwines the auto-equivalence $\mathcal{L}_{G,\chi}\otimes ?$ of $D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0)$ and the auto-equivalence $b^*_{\chi,\check{G}}$ on $D^b(\mathrm{QCoh}(\mathrm{LocSys}^0_{\check{G}}))$ constructed in § 5.5.*

(2) *The equivalence $\mathfrak{D}_\kappa$ intertwines the auto-equivalence $a^*_{\chi,G}$ of $D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)^0)$ as in § 5.5 and the auto-equivalence $\mathcal{L}_{\check{G},\chi}\otimes ?$ on $D^b(\mathrm{QCoh}(\mathrm{LocSys}^0_{\check{G}}))$ (here we regard $\mathcal{L}_{\check{G},\chi}$ as a line bundle on $\mathrm{LocSys}^0_{\check{G}}$ via the projection $\mathrm{LocSys}_{\check{G}} \to \mathrm{Bun}_{\check{G}}$).*

Remark 5.6.2. Similar actions by tensoring line bundles on LocSys_G and on Higgs_G also appear in the work of Frenkel and Witten [FW08, § 10.4]. Moreover, the authors also predict that the geometric Langlands correspondence should interchange these actions.

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Combining Theorems 5.5.3 and 5.6.1 we have the following.

COROLLARY 5.6.3. *Let $\kappa_1, \kappa_2 \in \omega^{1/2}(C)$. Then there is a natural isomorphism of equivalences*

$$\mathfrak{D}_{\kappa_1} \simeq (\mathcal{L}_{\breve{G},\chi} \otimes\, ?) \circ \mathfrak{D}_{\kappa_2} \circ (\mathcal{L}_{G,\chi} \otimes\, ?).$$

Here $\chi \in \mu_2$-tors(C) such that $\kappa_1 = \chi \cdot \kappa_2$ and $\mathcal{L}_{G,\chi} \otimes\, ?$ (respectively $\mathcal{L}_{\breve{G},\chi} \otimes\, ?$) is the functor of tensoring with the line bundle $\mathcal{L}_{G,\chi}$ (respectively $\mathcal{L}_{\breve{G},\chi}$).

The rest of this subsection is devoted to the proof of this theorem.

We first introduce a morphism of Picard stack

$$\tilde{\mathfrak{l}} : Z(G)\text{-tors}(C) \times B' \to \mathrm{Pic}(\breve{\mathscr{H}})$$

and prove a twisted version of Proposition 3.7.4. We begin with the construction of $\tilde{\mathfrak{l}}$. Let Bun_{J^p} be the Picard stack of J^p-torsors over C. We have the generalized Chern class map $\tilde{c}_{\breve{J}^p} : \mathrm{Bun}_{\breve{J}^p} \to \Pi_{\breve{G}}(1)\text{-gerbes}(X) \times B'$ and a Picard functor $\mathfrak{l}_{\breve{J}^p} : Z(G)\text{-tors}(C) \times B' \to \mathrm{Pic}(\mathrm{Bun}_{\breve{J}^p})$. We define

$$\tilde{\mathfrak{l}} : Z(G)\text{-tors}(C) \times B' \stackrel{\mathfrak{l}_{\breve{J}^p}}{\to} \mathrm{Pic}(\mathrm{Bun}_{\breve{J}^p}) \to \mathrm{Pic}(\breve{\mathscr{H}})$$

where the last map is induced by the restriction map $\breve{\mathscr{H}} = \mathrm{LocSys}_{\breve{J}^p}(\tau') \to \mathrm{Bun}_{\breve{J}^p}$.

Recall the morphism $\breve{\mathfrak{l}}_{\breve{J}} : Z_G\text{-tors}(C) \times B' \to \mathscr{P}'$ constructed in § 3.7. For any $Z(G)$-torsor K over C, we define

$$\mathcal{L}_{\breve{J}^p,K} := \tilde{\mathfrak{l}}(\{K\} \times B') \in \mathrm{Pic}(\breve{\mathscr{H}}).$$

Let K' denote the Frobenius descendent of K (as $C_{\text{ét}} \simeq C'_{\text{ét}}$), and let

$$K'_{J'} = \breve{\mathfrak{l}}_{\breve{J}}(\{K'\} \times B') \in \mathscr{P}'(B').$$

We will relate $\mathcal{L}_{\breve{J}^p,K}$ with $K'_{J'}$ via the twisted duality. From the definition of θ'_m in § 4.2, one can easily check that the restriction of θ'_m to $K'_{J'}$ is zero. Thus the restriction of the $\mathbb{G}_m$-gerbe $\mathscr{D}(\theta'_m)$ to $K'_{J'}$ is canonical trivial and we can regard the structure sheaf $\delta_{K'_{J'}} \in \mathrm{QCoh}(\mathscr{P}')$ as an object in $\mathrm{QCoh}(\mathscr{D}(\theta'_m))_1$. Let $\tilde{\mathcal{L}}_K = \mathfrak{D}(\delta_{K'_{J'}}) \in \mathrm{Pic}(\breve{\mathscr{H}})$ be the image of $\delta_{K'_{J'}}$ under the twisted duality $\mathfrak{D} : D^b(\mathrm{QCoh}(\mathscr{D}(\theta')))_1 \simeq D^b(\mathrm{QCoh}(\breve{\mathscr{H}}))$.

LEMMA 5.6.4. *We have $\tilde{\mathcal{L}}_K \simeq \mathcal{L}_{\breve{J}^p,K}$.*

Proof. Let $\breve{\mathfrak{G}} := \mathscr{D}(\theta'_m)^\vee$. We have a short exact sequence of Beilinson 1-motives $0 \to \breve{\mathscr{P}}' \to \breve{\mathfrak{G}} \stackrel{p}{\to} \mathbb{Z} \to 0$ and $\breve{\mathscr{H}} = p^{-1}(1)$. The construction of duality for torsors in § A.7 implies that there is a multiplicative line bundle $\tilde{\mathcal{L}}_{\breve{\mathfrak{G}},K}$ on $\breve{\mathfrak{G}}$ such that $\tilde{\mathcal{L}}_{\breve{\mathfrak{G}},K}|_{\breve{\mathscr{H}}} \simeq \tilde{\mathcal{L}}_K$. Moreover, this line bundle is characterized by the property that $\tilde{\mathcal{L}}_{\breve{\mathfrak{G}},K}|_{\breve{\mathscr{P}}'} \simeq \breve{\mathfrak{D}}_{\mathrm{cl}}^{-1}(K'_{J'})$. Observe that we have a natural map $\breve{\mathfrak{G}} \to \mathrm{Bun}_{\breve{J}^p}$ of Picard stacks[8] such that the composition $\breve{\mathscr{H}} \to \breve{\mathfrak{G}} \to \mathrm{Bun}_{\breve{J}^p}$ is the natural inclusion. Thus the morphism $\tilde{\mathfrak{l}} : Z_G\text{-tors}(C) \times B' \to \mathrm{Pic}(\breve{\mathscr{H}})$ factors through a morphism $\tilde{\mathfrak{l}}_{\breve{\mathfrak{G}}} : Z_G\text{-tors}(C) \times B' \to \breve{\mathfrak{G}}^\vee$, and the corresponding multiplicative line bundle $\mathcal{L}_{\breve{\mathfrak{G}},K} := \tilde{\mathfrak{l}}_{\breve{\mathfrak{G}}}(\{K\} \times B') \in \breve{\mathcal{P}}^\vee(B')$ satisfies $\mathcal{L}_{\breve{\mathfrak{G}},K}|_{\breve{\mathscr{H}}} \simeq \mathcal{L}_{\breve{J}^p,K}$. It is enough to show that $\tilde{\mathcal{L}}_{\breve{\mathfrak{G}},K} \simeq \mathcal{L}_{\breve{\mathfrak{G}},K}$. From the characterization of $\tilde{\mathcal{L}}_{\breve{\mathfrak{G}},K}$, it is enough to show that $\mathcal{L}_{\breve{\mathfrak{G}},K}|_{\breve{\mathscr{P}}'} \simeq \breve{\mathfrak{D}}_{\mathrm{cl}}^{-1}(K'_{J'})$. But this follows from Proposition 3.7.4 and the fact that $\mathcal{L}_{\breve{\mathfrak{G}},K}|_{\breve{\mathscr{P}}'}$ is isomorphic to $\mathcal{L}_{\breve{J}',K'}$. □

[8] We have $\breve{\mathfrak{G}} = \{(n,t) | n \in \mathbb{Z},\ t \in \breve{\mathscr{H}}^{\otimes n}\}$ and $\breve{\mathscr{H}}^{\otimes n}$ is isomorphic to $\mathrm{LocSys}_{\breve{J}^p}(n \cdot \tau')$, the base change of $\mathrm{LocSys}_{\breve{J}^p} \to B_{\breve{J}'}$ along the section $n \cdot \tau' : B' \to B_{\breve{J}'}$. Thus there is a natural map $\breve{\mathscr{H}}^{\otimes n} \to \mathrm{Bun}_{\breve{J}^p}$ and the map $\breve{\mathfrak{G}} \to \mathrm{Bun}_{\breve{J}^p}$ is given by $\breve{\mathfrak{G}} \to \breve{\mathscr{H}}^{\otimes n} \to \mathrm{Bun}_{\breve{J}^p}$.

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Recall that a choice of $\kappa \in \omega^{1/2}(C)$ defines an isomorphism $\mathfrak{C}_\kappa : \breve{\mathscr{H}} \simeq \mathrm{LocSys}_{\breve{G}}^{\mathrm{reg}}$. More precisely, we have $\mathfrak{C}_\kappa(P,\nabla) = (P \otimes F_C^* E_{\kappa'}, \nabla_{P\otimes F_C^* E_{\kappa'}})$ where $P \otimes F_C^* E_{\kappa'} := P \times^{J^p} F_C^* E_{\kappa'}$ and $\nabla_{P\otimes F_C^* E_{\kappa'}}$ is the product connection.

Lemma 5.6.5. *The pullback of the line bundle $\mathcal{L}_{\breve{G},K}$ along the map $\breve{\mathscr{H}} \xrightarrow{\mathfrak{C}_\kappa} \mathrm{LocSys}_{\breve{G}} \xrightarrow{\mathrm{pr}} \mathrm{Bun}_{\breve{G}}$ is isomorphic to $\tilde{\mathcal{L}}_K$. That is, we have $\tilde{\mathcal{L}}_K \simeq \mathfrak{C}_\kappa^* \circ \mathrm{pr}^* \mathcal{L}_{\breve{G},K}$.*

Proof. The proof is similar to the proof of Lemma 3.7.2. Recall that the line bundles $\mathcal{L}_{\breve{G},K}$ and $\tilde{\mathcal{L}}_K \simeq \mathcal{L}_{\breve{J}^p,K}$ are induced by the generalized Chern class map $\tilde{c}_{\breve{G}}$, $\tilde{c}_{\breve{J}^p}$. Therefore it is enough to show that for any $(P,\nabla) \in \breve{\mathscr{H}}$ there is a canonical isomorphism $\tilde{c}_{\breve{J}^p}(P) \simeq \tilde{c}_{\breve{G}}(\mathfrak{C}_\kappa(P))$ of $\Pi_{\breve{G}}$-gerbes, where $\mathfrak{C}_\kappa(P) = P \times^{J^p} F_C^* E_{\kappa'}$. Let $\tilde{P} \in \tilde{c}_{\breve{J}^p}(P)$ and $\tilde{E}_{\kappa'}$ be the canonical lifting of the Kostant section appearing in Lemma 3.7.2. The G_{sc}-torsor $\tilde{P} \times^{(J^p_{\mathrm{sc}})} F_C^* \tilde{E}_{\kappa'}$ is a lifting of $\mathfrak{C}_\kappa(P)$ and the assignment $\tilde{P} \to \tilde{P} \times^{(J^p_{\mathrm{sc}})} F_C^* \tilde{E}_{\kappa'}$ defines an isomorphism between $\tilde{c}_{J^p}(P)$ and $\tilde{c}_{\breve{G}}(\mathfrak{C}_\kappa(P))$. This finishes the proof. $\square$

Now we prove Theorem 5.6.1. Recall that we have $\mathfrak{D}_\kappa = (\mathfrak{C}_\kappa^*)^{-1} \circ \mathfrak{D} \circ \mathfrak{A}_\kappa$ where $\mathfrak{A}_\kappa$ and $\mathfrak{C}_\kappa^*$ are equivalences constructed in § 5.3. It follows from the definition that under the equivalence $\mathfrak{C}_\kappa^*$ the functor $b^*_{\breve{G},\chi}$ becomes the functor induced by the geometric action of $K'_{\breve{G},\chi} \in Z(\breve{G})\text{-tors}(C')$ on $\breve{\mathscr{H}}$.[9] Now Theorem A.7.2 implies, under the equivalence

$$\mathfrak{D} : D^b(\mathrm{QCoh}(\mathscr{D}(\theta'_m)|_{B'^0}))_1 \simeq D^b(\mathrm{QCoh}(\breve{\mathscr{H}}|_{B'^0})),$$

the above geometric action becomes the functor of tensoring with the line bundle $\mathcal{L}'_{J,\chi} := \mathfrak{D}_{\mathrm{cl}}^{-1}(K'_{\breve{G},\chi}) \in (\mathrm{Bun}_{J'})^\vee$.[10] By Lemmas 3.7.2 and 3.7.4, the line bundle $\mathcal{L}'_{J,\chi}$ is equal to the pullback of $\mathcal{L}'_{G,\chi}$ under the map $\mathscr{P}' \xrightarrow{\epsilon'_\kappa} \mathrm{Higgs}'_G \to \mathrm{Bun}'_G$. On the other hand, since the equivalence $\mathfrak{A}_\kappa : D^b(\mathcal{D}\text{-mod}(\mathrm{Bun}_G^0)) \simeq D^b(\mathrm{QCoh}(\mathscr{D}(\theta'_m)|_{B'^0}))_1$ is induced by pullback along the morphism $\epsilon_\kappa : \mathscr{P} \to \mathrm{Higgs}_G$, an easy exercise shows that under the equivalence $\mathfrak{A}_\kappa$ the functor of tensoring with $\mathcal{L}'_{J,\chi}$ becomes the functor of tensoring with $\mathcal{L}_{G,\chi}$. This implies part (1) of Theorem 5.6.1.

The proof of part (2) of Theorem 5.6.1 is similar to part (1). Unraveling the definition of $a^*_{G,\chi}$ and the construction of $\mathfrak{A}_\kappa$, one sees that $\mathfrak{A}_\kappa$ interchanges the functor $a^*_{G,\chi}$ with the functor of convolution product with $\delta_{K'_{G,\chi}} \in \mathrm{QCoh}(\mathscr{P}')$. Now Theorem A.7.2 implies that, under the equivalence $\mathfrak{D}$, the above convolution action becomes the functor of tensoring with the line bundle $\tilde{\mathcal{L}}_{K_{G,\chi}} := \mathfrak{D}(K'_{G,\chi}) \in \mathrm{Pic}\,\breve{\mathscr{H}}$. By Lemmas 5.6.4 and 5.6.5, the line bundle $\tilde{\mathcal{L}}_{K_{G,\chi}}$ is isomorphic to the pullback of $\mathcal{L}_{\breve{G},\chi}$ under the map $\breve{\mathscr{H}} \xrightarrow{\mathfrak{C}_\kappa} \mathrm{LocSys}_{\breve{G}} \xrightarrow{\mathrm{pr}} \mathrm{Bun}_{\breve{G}}$. It implies that $\mathfrak{C}_\kappa^* \circ (\mathrm{pr}^* \mathcal{L}_{\breve{G},\chi} \otimes ?) \simeq (\tilde{\mathcal{L}}_{K_{G,\chi}} \otimes ?) \circ \mathfrak{C}_\kappa^*$.

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[9] Recall that $K_{\breve{G},\chi}$ carries a canonical connection with zero p-curvature and $K'_{\breve{G},\chi}$ is its Frobenius descent.

[10] Here we use the fact that $\omega_{\mathscr{P}^\vee/B} \cong pr_B^*(e^*\omega_{\mathscr{P}^\vee/B})$ is trivial. Indeed, since B is isomorphic to an affine space we have $e^*\omega_{\mathscr{P}^\vee/B} \in \mathrm{Pic}\,(B) = 0$.

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Appendix A. Beilinson 1-motive

In this section, we review the duality theory of Beilinson 1-motives. The main references are [Ari08, DP12, DP08, Lau96].

A.1 Picard stack

Let us first review the theory of Picard stacks. The standard reference is [Del73, § 1.4]. Let $\mathcal{T}$ be a given site. Recall that a Picard Stack is a stack $\mathscr{P}$ over $\mathcal{T}$ together with a bi-functor

$$\otimes : \mathscr{P} \times \mathscr{P} \to \mathscr{P},$$

and the associativity and commutative constraints

$$a : \otimes \circ (\otimes \times 1) \simeq \otimes \circ (1 \times \otimes), \quad c : \otimes \simeq \otimes \circ \text{flip},$$

such that for every $U \in \mathcal{T}$, $\mathscr{P}(U)$ form a Picard groupoid (i.e., symmetrical monoidal groupoid such that every object has a monoidal inverse). The Picard stack is called strictly commutative if $c_{x,x} = \mathrm{id}_x$ for every $x \in \mathscr{P}$. In the paper, Picard stacks will always mean strictly commutative ones.

Let us denote by $\mathcal{PS}/\mathcal{T}$ the 2-category of Picard stacks over $\mathcal{T}$. This means that if $\mathscr{P}_1, \mathscr{P}_2$ are two Picard stacks over $\mathcal{T}$, $\mathrm{Hom}_{\mathcal{PS}/\mathcal{T}}(\mathscr{P}_1, \mathscr{P}_2)$ form a category. Indeed, $\mathcal{PS}/\mathcal{T}$ is canonically enriched over itself. For $\mathscr{P}_1, \mathscr{P}_2 \in \mathcal{PS}/\mathcal{T}$, we use $\underline{\mathrm{Hom}}(\mathscr{P}_1, \mathscr{P}_2)$ to denote the Picard stack of 1-homomorphisms from $\mathscr{P}_1$ to $\mathscr{P}_2$ over $\mathcal{T}$ (cf. [Del73, § 1.4.7]). On the other hand, let $C^{[-1,0]}$ be the 2-category of 2-term complexes of sheaves of abelian groups $d : \mathcal{K}^{-1} \to \mathcal{K}^0$ with $\mathcal{K}^{-1}$ injective and 1-morphisms are morphisms of chain complexes (and 2-morphisms are homotopy of chain complexes).[11] Let $\mathcal{K} \in C^{[-1,0]}$. We associate to it a Picard prestack $\mathrm{pch}(\mathcal{K})$ whose U point is the following Picard category.

(1) Objects of $\mathrm{pch}(\mathcal{K})(U)$ are equal to $\mathcal{K}^0(U)$.

(2) If $x, y \in \mathcal{K}^0(U)$, a morphism from x to y is an element $f \in \mathcal{K}^{-1}(U)$ such that $df = y - x$.

Let $\mathrm{ch}(\mathcal{K})$ be the stackification of $\mathrm{pch}(\mathcal{K})$. Then a theorem of Deligne (cf. [Del73, Corollaire 1.4.17]) says that the functor

$$\mathrm{ch} : \mathrm{C}^{[-1,0]} \to \mathcal{PS}/\mathcal{T}$$

is an equivalence of 2-categories.

Let us fix an inverse functor $(\,)^\flat$ of the above equivalence. So for $\mathscr{P}$ a Picard stack, we have a 2-term complex of sheaves of abelian groups $\mathscr{P}^\flat := \mathcal{K}^{-1} \to \mathcal{K}^0$. For example, if A is an abelian group in $\mathcal{T}$, then its classifying stack BA is a natural Picard stack and $(BA)^\flat$ can be represented by a 2-term complex quasi-isomorphic to $A[1]$. The following result of Deligne (cf. [Del73, Construction 1.4.18]) is convenient for computations:

$$(\underline{\mathrm{Hom}}(\mathscr{P}_1, \mathscr{P}_2))^\flat \simeq \tau_{\leqslant 0} \,\mathrm{R}\, \underline{\mathrm{Hom}}(\mathscr{P}_1^\flat, \mathscr{P}_2^\flat). \tag{A.1.1}$$

[11] The 2-category $C^{[-1,0]}$ is an enhancement of the subcategory $D^{[-1,0]} \subset D$ of the derived category consisting of complexes concentrated in cohomological degrees $[-1, 0]$. That is, the homotopy category of $C^{[-1,0]}$ is equivalent to $D^{[-1,0]}$.

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A.2 Short exact sequences of Picard stacks

Let $a : \mathscr{P}_1 \to \mathscr{P}_2$ be a homomorphism of Picard stacks. We define $\ker(a)$ as the fiber $\mathscr{P}_1 \times_{\mathscr{P}_2} \{e\}$, where $e \in \mathscr{P}_2$ is the unit. Then $\ker(a)$ acquires a natural Picard stack structure. The next lemma follows from the construction of ch.

Lemma A.2.1. *There is a natural isomorphism* $\ker(a)^\flat \simeq \tau_{\leqslant 0} C(a^\flat)[-1]$, *where* $C(a^\flat)$ *is the cone of the morphism of complexes*

$$a^\flat : \mathscr{P}_1^\flat \to \mathscr{P}_2^\flat.$$

A left exact sequence of Picard stacks, usually denoted by

$$1 \to \mathscr{P}_1 \xrightarrow{a} \mathscr{P}_2 \xrightarrow{b} \mathscr{P}_3,$$

is a sequence of homomorphisms of Picard stacks that exhibits $\mathscr{P}_1$ as $\ker(b)$. If, in addition locally on $\mathcal{T}$, b is essentially surjective, we call such a sequence exact and denote it by

$$1 \to \mathscr{P}_1 \xrightarrow{a} \mathscr{P}_2 \xrightarrow{b} \mathscr{P}_3 \to 1.$$

Sometimes, we also call $\mathscr{P}_2$ an *extension* of $\mathscr{P}_3$ by $\mathscr{P}_1$. The following lemma is used in several places in the paper.

Lemma A.2.2. *The sequence of homomorphisms* $\mathscr{P}_1 \to \mathscr{P}_2 \to \mathscr{P}_3$ *is exact if and only if*

$$\mathscr{P}_1^\flat \to \mathscr{P}_2^\flat \to \mathscr{P}_3^\flat \to$$

is a distinguished triangle.

A.3 Duality of Picard stacks

Let S be a Noetherian scheme. We consider the category Sch/S of schemes over S. We will endow Sch/S with *fpqc* topology in the following discussion.

Definition A.3.1. For a Picard stack $\mathscr{P}$, we define the dual Picard stack as

$$\mathscr{P}^\vee := \underline{\mathrm{Hom}}(\mathscr{P}, B\mathbb{G}_m).$$

Example A.3.2. Let $A \to S$ be an abelian scheme over S. Then by definition $A^\vee := \underline{\mathrm{Hom}}(A, B\mathbb{G}_m) = \underline{\mathrm{Ext}}^1(A, \mathbb{G}_m)$ classifies the multiplicative line bundles on A, is represented by an abelian scheme over S, called the dual abelian scheme of A.

Example A.3.3. Let Γ be a finitely generated abelian group over S. By definition, this means locally on S, Γ is isomorphic to the constant sheaf M_S, where M is a finitely generate abelian group (in the naive sense). Recall that the Cartier dual of Γ, denoted by $\mathrm{D}(\Gamma)$ is the sheaf which assigns every scheme U over S the group $\mathrm{Hom}(\Gamma \times_S U, \mathbb{G}_m)$, which is represented by an affine group scheme over S. We claim that $\Gamma^\vee \simeq B\,\mathrm{D}(\Gamma)$. By (A.1.1), it is enough to show that $\mathrm{R}^i\,\underline{\mathrm{Hom}}(\Gamma, \mathbb{G}_m) = 0$ if $i > 0$. This is clear since locally on S, Γ is represented by a 2-term complex $\mathbb{Z}_S^m \to \mathbb{Z}_S^n$.

Example A.3.4. Let G be a group of multiplicative type over S, i.e., $G = D(\Gamma)$ for some finitely generated abelian group Γ over S. Let $\mathscr{P} = BG$, the classifying stack of G. We have

$$\mathscr{P}^\vee \simeq \tau_{\leqslant 0}\,\mathrm{R}\,\underline{\mathrm{Hom}}(BG, B\mathbb{G}_m) \simeq \underline{\mathrm{Hom}}(G, \mathbb{G}_m) \simeq \Gamma.$$

Definition A.3.5. Let $\mathscr{P}$ be a Picard stack. We say that $\mathscr{P}$ is dualizable if the canonical 1-morphism $\mathscr{P} \to \mathscr{P}^{\vee\vee}$ is an isomorphism.

By the above examples, abelian schemes, finitely generated abelian groups, and the classifying stacks of groups of multiplicative type are dualizable.

Let $\mathscr{P}$ be a dualizable Picard stack. There is the Poincare line bundle $\mathcal{L}_{\mathscr{P}}$ over $\mathscr{P} \times_S \mathscr{P}^\vee$. Let $D^b(\mathrm{QCoh}(\mathscr{P}))$ denote the bounded derived category of quasi coherent sheaves on $\mathscr{P}$. We define the Fourier–Mukai functor

$$\Phi_{\mathscr{P}} : D^b(\mathrm{QCoh}(\mathscr{P})) \to D^b(\mathrm{QCoh}(\mathscr{P}^\vee)), \quad \Phi_{\mathscr{P}}(F) = (\mathrm{R}\, p_2)_*(\mathrm{L}\, p_1^* F \otimes \mathcal{L}_{\mathscr{P}}).$$

Here $p_1 : \mathscr{P} \times_S \mathscr{P}^\vee \to \mathscr{P}$ and $p_2 : \mathscr{P} \times_S \mathscr{P}^\vee \to \mathscr{P}^\vee$ denote the natural projections. It is easy to see in the case when $\mathscr{P}$ is of the form given in the above examples, $\Phi_{\mathscr{P}}$ is an equivalence of categories. Indeed, the case when $\mathscr{P} = A$ follows from the results of Mukai; the case when $\mathscr{P} = \Gamma$ or BG is clear.

It is not clear to us whether $\Phi_{\mathscr{P}}$ is an equivalence for all dualizable Picard stacks. In the following subsection, we select out a particular class of Picard stacks, called the Beilinson 1-motive (following [DP12] and Arinkin's appendix to [DP08]), for which the Fourier–Mukai transforms are equivalences.

A.4 Beilinson 1-motives

Let $\mathscr{P}_1, \mathscr{P}_2$ be two Picard stacks. We say that $\mathscr{P}_1 \subset \mathscr{P}_2$ if there is a 1-morphism $\phi : \mathscr{P}_1 \to \mathscr{P}_2$, which is a faithful embedding.

Definition A.4.1. We called a Picard stack $\mathscr{P}$ a Beilinson 1-motive if it admits a two step filtration $W_\bullet \mathscr{P}$:

$$W_{-1} = 0 \subset W_0 \subset W_1 \subset W_2 = \mathscr{P}$$

such that: (i) $\mathrm{Gr}_0^W \simeq BG$ is the classifying stack of a group G of multiplicative type; (ii) $\mathrm{Gr}_1^W \simeq A$ is an abelian scheme; and (iii) $\mathrm{Gr}_2^W \simeq \Gamma$ is a finitely generated abelian group.

Lemma A.4.2. *The dual of a Beilinson* 1*-motive is a Beilinson* 1*-motive and Beilinson* 1*-motives are dualizable.*

Proof. This is proved via the induction on the length of the filtration. We use the following fact. Let

$$0 \to \mathscr{P}' \to \mathscr{P} \to \mathscr{P}'' \to 0$$

be a short exact sequence of Picard stacks. Then

$$0 \to (\mathscr{P}'')^\vee \to \mathscr{P}^\vee \to (\mathscr{P}')^\vee$$

with the right arrow surjective if $\mathrm{R}^2\, \underline{\mathrm{Hom}}((\mathscr{P}'')^\flat, \mathbb{G}_m) = 0$.

If $\mathscr{P} = W_0\mathscr{P}$, this is given by Example A.3.4. If $\mathscr{P} = W_1\mathscr{P}$, we have the following exact sequence

$$0 \to BG \to \mathscr{P} \to A \to 0.$$

Using the fact that $\underline{\mathrm{Ext}}^2(A, \mathbb{G}_m) = 0$ (see [Bre75, Remark 6]), we know that $\mathscr{P}$ is also a Beilinson 1-motive. In general, we have

$$0 \to W_1\mathscr{P} \to \mathscr{P} \to \Gamma \to 0,$$

and the lemma follows from the fact $\underline{\mathrm{Ext}}^2(\Gamma, \mathbb{G}_m) = 0$ (see Example A.3.3). □

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Corollary A.4.3. *Let $\mathscr{P}$ be a Beilinson 1-motive, and $\mathscr{P}^\vee$ be its dual. Then* $\mathrm{D}(\mathrm{Aut}_{\mathscr{P}}(e)) = \pi_0(\mathscr{P}^\vee)$, *where e denotes the unit of $\mathscr{P}$ and π_0 denotes the group of connected components of $\mathscr{P}^\vee$.*

Lemma A.4.4. *Let $\mathscr{P}$ be a Beilinson 1-motive. Then, locally on S,*

$$\mathscr{P} \simeq A \times BG \times \Gamma.$$

Proof. It is enough to prove that

$$\underline{\mathrm{Ext}}^1(\Gamma, BG) = \underline{\mathrm{Ext}}^1(\Gamma, A) = \underline{\mathrm{Ext}}^1(A, BG) = 0.$$

Clearly, $\underline{\mathrm{Ext}}^1(\Gamma, BG) = \underline{\mathrm{Ext}}^2(\Gamma, G) = 0$. To see that $\underline{\mathrm{Ext}}^1(\Gamma, A) = 0$, we can assume that $\Gamma = \mathbb{Z}/n\mathbb{Z}$. Then it follows that $A \xrightarrow{n} A$ is surjective in the flat topology that $\underline{\mathrm{Ext}}^1(\Gamma, A) = 0$.

To see that $\underline{\mathrm{Ext}}^1(A, BG) = 0$, let $\mathscr{P}$ to the Beilinson 1-motive corresponding to a class in $\underline{\mathrm{Ext}}^1(A, BG)$. Taking the dual, we have $0 \to A^\vee \to \mathscr{P}^\vee \to \mathrm{D}(G) \to 0$. Therefore, locally on S, $\mathscr{P}^\vee \simeq A^\vee \times \mathrm{D}(G)$, and therefore, locally on S, $\mathscr{P}^{\vee\vee} \simeq A \times BG$. □

Definition A.4.5 (Cf. [Ari08]). We say that a Picard stack $\mathscr{P}$ is *good* if it satisfies the following two conditions.

(1) The Picard stack $\mathscr{P}$ is dualizable, i.e., the map $r : \mathscr{P} \to \mathscr{P}^{\vee\vee}$ is an isomorphism of Picard stacks.
(2) The functor $\Phi_{\mathscr{P}} : D^b(\mathrm{QCoh}(\mathscr{P})) \to D^b(\mathrm{QCoh}(\mathscr{P}^\vee))$ is an equivalence of categories.

As explained in § A.3 (see also [BB07]), examples of good Picard stacks include BG, Γ and abelian schemes over S, as well as fiber products over S of such. More generally, we have the following theorem.

Theorem A.4.6 [Ari08, Proposition A.6]. *Let $\mathscr{P}$ be a Beilinson 1-motive. Then $\mathscr{P}$ is 'good' in the sense of Definition A.4.5. In particular, the functor $\Phi_{\mathscr{P}}$ is an equivalence of categories.*

Proof. Indeed, the property of being good is fpqc-local on S. This can be seen by lifting $\Phi_{\mathscr{P}}$ to a functor between stable ∞-categories of quasi-coherent sheaves and then applying a descent argument. Therefore, the theorem follows from Lemma A.4.4 and the above examples.

Alternatively, similarly to the usual Fourier–Mukai transform, one can show directly that in our generality there is still an isomorphism of functors $\Phi_{\mathscr{P}^\vee} \circ \Phi_{\mathscr{P}} \simeq \omega^{-1}_{\mathscr{P}/S} \otimes (-1)^*[-g]$, where $\omega_{\mathscr{P}/S}$ is the canonical sheaf and g is the relative dimension of $\mathscr{P}/S$.[12] By the argument as in [Muk85] (see also [Lau96]), one reduces to show that the kernel complex

$$\mathrm{R}\, p_{12*}(\mathrm{L}\, p_{13}^*\mathcal{L}_{\mathscr{P}} \otimes \mathrm{L}\, p_{23}^*\mathcal{L}_{\mathscr{P}}) \simeq m^* \,\mathrm{R}\, p_{1*}\mathcal{L}_{\mathscr{P}}$$

for the functor $\Phi_{\mathscr{P}^\vee} \circ \Phi_{\mathscr{P}}$ is isomorphic to the kernel complex

$$\sigma_*(\omega^{-1}_{\mathscr{P}/S})[-g] \simeq m^* e_*(e^*\omega^{-1}_{\mathscr{P}/S})[-g]$$

for $\omega^{-1}_{\mathscr{P}/S} \otimes (-1)^*[-g]$. Here p_{ij} are the projections of $\mathscr{P} \times_S \mathscr{P} \times_S \mathscr{P}^\vee$ on the (i,j)-factors, $\sigma : \mathscr{P} \to \mathscr{P} \times_S \mathscr{P}, x \to (x, x^{-1})$ and $e : S \to \mathscr{P}$ is the unit morphism.

[12] This argument was suggested to us by the referees.

To prove this, we first observe that there is a natural map

$$\mathrm{R}\,p_{1*}\mathcal{L}_{\mathscr{P}} \simeq \mathrm{R}^g\,p_{1*}\mathcal{L}_{\mathscr{P}}[-g] \to e_*\,\mathrm{R}^g\,pr_{S*}\mathcal{O}_{\mathscr{P}^\vee}[-g] \to e_*(e^*\omega^{-1}_{\mathscr{P}/S})[-g].$$

We claim that the map above is an isomorphism, and hence induces $m^*\,\mathrm{R}\,p_{1*}\mathcal{L}_{\mathscr{P}} \simeq m^*e_*(e^*\omega^{-1}_{\mathscr{P}/S})[-g]$. To prove the claim, we observe that, using Lemma A.4.4 and fpqc base change, we can assume $\mathscr{P} \simeq A \times BG \times \Gamma$ and the claim follows from the results in [Muk85, p. 519] or [Lau96, Lemma 1.2.5]. □

Entirely similar arguments as in [Muk81, p. 160] and [Lau96, Corollary 1.3.3] give us the following.

THEOREM A.4.7. *Let $\mathscr{P}$ be a Beilinson 1-motive. Let*

$$*: D^b(\mathrm{QCoh}(\mathscr{P})) \times D^b(\mathrm{QCoh}(\mathscr{P})) \to D^b(\mathrm{QCoh}(\mathscr{P}))$$

be the functor defined by $\mathcal{F}_1 * \mathcal{F}_2 := \mathrm{R}\,m_*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$. We called $*$ the convolution product. Then there are canonical isomorphisms

$$\Phi_{\mathscr{P}}(\mathcal{F}_1 * \mathcal{F}_2) \simeq \Phi_{\mathscr{P}}(\mathcal{F}_1) \otimes \Phi_{\mathscr{P}}(\mathcal{F}_2)$$

and

$$\Phi_{\mathscr{P}}(\mathcal{F}_1 \otimes \mathcal{F}_2) \simeq (\Phi_{\mathscr{P}}(\mathcal{F}_1) * \Phi_{\mathscr{P}}(\mathcal{F}_2)) \otimes \omega_{\mathscr{P}^\vee/S}[g].$$

A.5 Multiplicative torsors and extensions of Beilinson 1-motives

Let us return to the general set-up. Let $\mathcal{T}$ be a fixed site and let $\mathscr{P}$ be a Picard stack over $\mathcal{T}$. A torsor of $\mathscr{P}$ is a stack $\mathscr{Q}$ over $\mathcal{T}$, together with a bi-functor

$$\mathrm{Action}: \mathscr{P} \times \mathscr{Q} \to \mathscr{Q},$$

satisfying the following properties.

(i) The bi-functor Action defines a monoidal action of $\mathscr{P}$ on $\mathscr{Q}$.

(ii) For every $V \in \mathcal{T}$, there exists a covering $U \to V$, such that $\mathscr{Q}(U)$ is non-empty.

(iii) For every $U \in \mathcal{T}$ such that $\mathscr{Q}(U)$ is non-empty and let $D \in \mathscr{Q}(U)$, the functor

$$\mathscr{P}(U) \to \mathscr{Q}(U), \quad C \mapsto \mathrm{Action}(C, D)$$

is an equivalence.

In the case when $\mathscr{P}$ is the Picard stack of G-torsors for some sheaf of abelian groups G, people usually call a $\mathscr{P}$-torsor $\mathscr{Q}$ a G-gerbe.

All $\mathscr{P}$-torsors form a 2-category, denoted by $B\mathscr{P}$, which is canonically enriched over itself [OZ11, § 2.3]. That is, given two $\mathscr{P}$-torsors $\mathscr{Q}_1, \mathscr{Q}_2$, $\underline{\mathrm{Hom}}_{\mathscr{P}}(\mathscr{Q}_1, \mathscr{Q}_2)$ is a natural $\mathscr{P}$-torsor. An object in $\underline{\mathrm{Hom}}_{\mathscr{P}}(\mathscr{Q}_1, \mathscr{Q}_2)$ induces an equivalence between $\mathscr{Q}_1$ and $\mathscr{Q}_2$. In addition, there is a monoidal structure on $B\mathscr{P}$ making $B\mathscr{P}$ a Picard 2-stack.

Remark A.5.1. Let $1 \to \mathscr{P}_1 \to \mathscr{P} \to \mathbb{Z}_S \to 1$ be an exact sequence of Picard stacks. Then $\mathscr{T} := \mathscr{P} \times_{\mathbb{Z}_S} \{1\}$ is naturally a $\mathscr{P}_1$-torsor. As explained in [Ari08] and [Tra11, § 3.1], the correspondence $\mathscr{P} \to \mathscr{T}$ induces an equivalence of 2-categories between extensions of $\mathbb{Z}_S$ by $\mathscr{P}_1$ and $\mathscr{P}_1$-torsors.

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Now, let $\mathscr{P}$ and $\mathscr{P}_1$ be two Picard stacks and let $\mathscr{G}$ be a $\mathscr{P}_1$-torsor over $\mathscr{P}$. Let $m : \mathscr{P}\times\mathscr{P}\to\mathscr{P}$, $e : \mathfrak{T}\to\mathscr{P}$ be the multiplication morphism and the unit morphism respectively, and let $\sigma : \mathscr{P}\times\mathscr{P}\to\mathscr{P}\times\mathscr{P}$ be the flip map $\sigma(x, y) = (y, x)$.

Definition A.5.2. A commutative group structure on $\mathscr{G}$ consists of the following data:

(i) an equivalence $M : \mathscr{G}\boxtimes\mathscr{G}\simeq m^*\mathscr{G}$ of $\mathscr{P}_1$-torsors over $\mathscr{P}\times\mathscr{P}$;

(ii) a 2-morphism γ between the resulting two 1-morphisms between $\mathscr{G}\boxtimes\mathscr{G}\boxtimes\mathscr{G}$ and $m^*\mathscr{G}$ over $\mathscr{P}\times\mathscr{P}\times\mathscr{P}$, which satisfies the cocycle condition;

(iii) a 2-morphism $i : \sigma^*M\simeq M$ such that $i^2 = \mathrm{id}$.

(Note that $\sigma^*(M)$ is another 1-morphism between $m^*\mathscr{G}$ and $\mathscr{G}\boxtimes\mathscr{G}$.)

Clearly, all $\mathscr{P}_1$-torsors over $\mathscr{P}$ with a commutative group structure also form a 2-category. We have the following lemma.

Lemma A.5.3. *A commutative group structure on $\mathscr{G}$ makes $\mathscr{G}$ into a Picard stack which fits into the following short exact sequence:*

$$0\to\mathscr{P}_1\to\mathscr{G}\to\mathscr{P}\to 0.$$

In particular, if $\mathscr{P}$ is a Beilinson 1-motive, and $\mathscr{P}_1 = B\mathbb{G}_m$, then $\mathscr{G}$ is a Beilinson 1-motive. In this case, we also call $\mathscr{G}$ a multiplicative $\mathbb{G}_m$-gerbe over $\mathscr{P}$.

Definition A.5.4. A multiplicative splitting of a $\mathscr{P}_1$-torsor $\mathscr{G}$ over $\mathscr{P}$ with a commutative group structure is a 1-morphism (in the category of all $\mathscr{P}_1$-torsors over $\mathscr{P}$ with a commutative group structure): $\mathscr{P}\to\mathscr{G}$.

A.6 Induction functor

Let $\phi : \mathscr{P}\to\mathscr{P}_1$ be a morphism of Picard stacks. Then to each $\mathscr{P}$-torsor $\mathscr{Q}$ we may associate a $\mathscr{P}_1$-torsor $\mathscr{Q}^\phi := \underline{\mathrm{Hom}}_{\mathscr{P}}(\mathscr{Q}^{-1}, \mathscr{P}_1)$ whose sections are $\mathscr{P}$-equivariant functors from $\mathscr{Q}^{-1} := \underline{\mathrm{Hom}}_{\mathscr{P}}(\mathscr{Q}, \mathscr{P})$ to $\mathscr{P}_1$ (here $\mathscr{P}$ acts on $\mathscr{P}_1$ via ϕ) and whose morphisms are natural transformations of such functors.

We have a canonical functor $\mathscr{Q}\to\mathscr{Q}^\phi$, compatible with their $\mathscr{P}$ and $\mathscr{P}_1$-structure via ϕ. For any section E of $\mathscr{Q}$ we denote by E^ϕ the section of $\mathscr{Q}^\phi$ induced by the canonical map $\mathscr{Q}\to\mathscr{Q}^\phi$.

A.7 Duality for torsors

Let $\mathscr{Y}$ be an algebraic stack. Let $\widetilde{\mathscr{Y}}$ be a $\mathbb{G}_m$-gerbe over $\mathscr{Y}$, i.e., $\widetilde{\mathscr{Y}}$ is a $B\mathbb{G}_m$-torsor over $\mathscr{Y}$. We say $\widetilde{\mathscr{Y}}$ is split if it is isomorphic to $\mathscr{Y}\times B\mathbb{G}_m$. Let $D^b(\mathrm{QCoh}(\widetilde{\mathscr{Y}}))$ be the bounded derived category of quasi coherent sheaves on $\widetilde{\mathscr{Y}}$. If $\widetilde{\mathscr{Y}}$ is split, there is a decomposition

$$D^b(\mathrm{QCoh}(\widetilde{\mathscr{Y}})) = \bigoplus_{n\in\mathbb{Z}} D^b(\mathrm{QCoh}(\widetilde{\mathscr{Y}}))_n \tag{A.7.1}$$

according to the character of $\mathbb{G}_m$.[13] In general we still have such a decomposition given as follows: $\mathcal{M}\in D^b(\mathrm{QCoh}(\widetilde{\mathscr{Y}}))_n$ if only if $a^*(\mathcal{M})\in D^b(\mathrm{QCoh}(\widetilde{\mathscr{Y}}))_n$, where $a : B\mathbb{G}_m\times\widetilde{\mathscr{Y}}\to\widetilde{\mathscr{Y}}$ is the action map.

[13] The direct sum in (A.7.1) means that every object in $D^b(\mathrm{QCoh}(\widetilde{\mathscr{Y}}))$ decomposes as a direct sum of objects in the subcategories $D^b(\mathrm{QCoh}(\widetilde{\mathscr{Y}}))_n$.

Definition A.7.1. The direct summand $D^b(\mathrm{QCoh}(\widetilde{\mathscr{Y}}))_1$ is called the category of twisted sheaves on $\widetilde{\mathscr{Y}}$.

Now we further assume $\mathscr{Y} = \mathscr{P}$ is a Beilinson 1-motive over S and $\widetilde{\mathscr{Y}} = \mathscr{D}$ is a multiplicative $\mathbb{G}_m$-gerbe over $\mathscr{P}$. Let $\mathscr{P}$ and $\mathscr{D}$ as above. Then by Lemma A.5.3 we have the following short exact sequence

$$0 \to B\mathbb{G}_m \xrightarrow{i} \mathscr{D} \xrightarrow{p} \mathscr{P} \to 0 \tag{A.7.2}$$

as Picard stacks. Note that in this case $\mathscr{D}$ is also a Beilinson 1-motive. Let $\mathscr{D}^\vee$ be the dual Beilinson 1-motive. It fits into the short exact sequence

$$0 \to \mathscr{P}^\vee \to \mathscr{D}^\vee \xrightarrow{\pi} \mathbb{Z}_S \to 0.$$

Let

$$\mathscr{T}_{\mathscr{D}} = \pi^{-1}(1) \tag{A.7.3}$$

be the $\mathscr{P}^\vee$-torsor associated to above extension. We call $\mathscr{T}_{\mathscr{D}}$ the stack of multiplicative splitting of $\mathscr{D}$. To justify the name, let us give an alternative description of $\mathscr{T}_{\mathscr{D}}$. By definition the dual of $\mathscr{D}$ is

$$\mathscr{D}^\vee = \underline{\mathrm{Hom}}(\mathscr{D}, B\mathbb{G}_m).$$

An element $s \in \mathscr{D}^\vee$ belongs to $\mathscr{T}_{\mathscr{D}}$ if and only if the composition

$$B\mathbb{G}_m \xrightarrow{i} \mathscr{D} \xrightarrow{s} B\mathbb{G}_m$$

is equal to the identity. Equivalently, $s \in \mathscr{T}_{\mathscr{D}}$ gives a splitting of the exact sequence (A.7.2) and according to Definition A.5.4 it is a multiplicative splitting of $\mathscr{D}$.

The following theorem follows immediately from Theorem A.4.7.

Theorem A.7.2 ([Ari08], [Tra11, § 3.2]). (1) *The Fourier–Mukai functor $\Phi_{\mathscr{D}}$ restricts to an equivalence*

$$\Phi_{\mathscr{D}} : D^b(\mathrm{QCoh}(\mathscr{D}))_1 \simeq D^b(\mathrm{QCoh}(\mathscr{T}_{\mathscr{D}})).$$

(2) *There is an action of $D^b(\mathrm{Qcoh}(\mathscr{P}))$ on $D^b(\mathrm{QCoh}(\mathscr{D}))_1$ by tensoring and an action of $D^b(\mathrm{QCoh}(\mathscr{P}^\vee))$ on $D^b(\mathrm{QCoh}(\mathscr{T}_{\mathscr{D}}))$ by convolution. Those two actions are compatible with the above equivalence in the following sense: there is a canonical isomorphism*

$$\Phi_{\mathscr{D}}(\mathcal{F}_1 \otimes \mathcal{F}_2) \simeq (\Phi_{\mathscr{D}}(\mathcal{F}_1) * \Phi_{\mathscr{D}}(\mathcal{F}_2)) \otimes \omega_{\mathscr{P}^\vee/S}[g]$$

for $\mathcal{F}_1 \in D^b(\mathrm{Qcoh}(\mathscr{P}))$ and $\mathcal{F}_2 \in D^b(\mathrm{QCoh}(\mathscr{D}))_1$. Here $\omega_{\mathscr{P}^\vee/S}$ is the canonical sheaf and g is the relative dimension of $\mathscr{P}/S$.

(3) *The convolution product $*$ on $D^b(\mathrm{QCoh}(\mathscr{D}))$ induces a convolution product on $D^b(\mathrm{QCoh}(\mathscr{D}))_1$ (by abuse of notation we still denote it by $*$). On the other hand, the category $D^b(\mathrm{QCoh}(\mathscr{T}_{\mathscr{D}}))$ has the usual monoidal structure by tensoring. The equivalence $\Phi_{\mathscr{D}}$ is compatible with those monoidal structures: there is a canonical isomorphism*

$$\Phi_{\mathscr{D}}(\mathcal{F}_1 * \mathcal{F}_2) \simeq \Phi_{\mathscr{D}}(\mathcal{F}_1) \otimes \Phi_{\mathscr{D}}(\mathcal{F}_2)$$

for $\mathcal{F}_1, \mathcal{F}_2 \in D^b(\mathrm{QCoh}(\mathscr{D}))_1$.

Appendix B. $\mathcal{D}$-modules on stacks and Azumaya property

In this section we review some basic facts about $\mathcal{D}$-modules on algebraic stacks and the Azumaya property of the sheaf of differential operators. Standard references are [BD91] and [BB07].

B.1 Azumaya algebras and twisted sheaves

Let us begin with a review of the basic definition of Azumaya algebras and the category of twisted sheaves. Let S be a Noetherian scheme. Let $\mathscr{X}$ be an algebraic stack over S. Recall that an Azumaya algebra $\mathcal{A}$ over $\mathscr{X}$ is a quasi-coherent sheaf of $\mathcal{O}_{\mathscr{X}}$-algebras, which is locally in smooth topology isomorphic to $\mathcal{E}nd(\mathcal{V})$ for some vector bundle $\mathcal{V}$ on $\mathscr{X}$. Such an isomorphism between $\mathcal{A}$ and the matrix algebra is called a splitting of $\mathcal{A}$. Given an Azumaya algebra $\mathcal{A}$ on $\mathscr{X}$, one can associate to it the $\mathbb{G}_m$-gerbe $\mathscr{D}_{\mathcal{A}}$ of splittings over $\mathscr{X}$, i.e., for any $U \to S$ we have

$$\mathscr{D}_{\mathcal{A}}(U) = \{(x, \mathcal{V}, i) | x \in \mathscr{X}(U), i : \mathcal{E}nd(\mathcal{V}) \simeq x^*(\mathcal{A})\}. \tag{B.1.1}$$

We will use the following proposition in what follows.

Proposition B.1.1 [DP08, § 2.1.2]. *Let $\mathcal{A}$ be a sheaf of Azumaya algebras on $\mathscr{X}$. There is the following equivalence of categories*

$$\mathrm{QCoh}(\mathscr{D}_{\mathcal{A}})_1 \simeq \mathcal{A}\text{-mod}(\mathrm{QCoh}(\mathscr{X})),$$

where $\mathcal{A}$-mod(Qcoh($\mathscr{X}$)) is the category of $\mathcal{A}$-modules which is quasi-coherent as $\mathcal{O}_{\mathscr{X}}$-modules.

B.2 $\mathcal{D}$-module on scheme

Let X be a scheme smooth over S. Let $\mathcal{D}_{X/S}$ be the sheaf of crystalline differential operators on X, i.e., $\mathcal{D}_{X/S}$ is the universal enveloping $\mathcal{D}$-algebra associated to the relative tangent Lie algebroid $T_{X/S}$. By definition, the category of $\mathcal{D}$-modules on X is the category of modules over $\mathcal{D}_{X/S}$ that are quasi-coherent as $\mathcal{O}_X$-modules. We denote by $\mathcal{D}$-mod(X) the category of $\mathcal{D}$-modules on X. In the case $p\mathcal{O}_S = 0$, we have the following fundamental observation.

Theorem B.2.1 [BMR08, §§ 1.3.2, 2.2.3]. *The center of $(F_{X/S})_*\mathcal{D}_{X/S}$ is isomorphic to $\mathcal{O}_{T^*(X'/S)}$ and there is an Azumaya algebra $D_{X/S}$ on $T^*(X'/S)$ such that*

$$(F_{X/S})_*\mathcal{D}_{X/S} \simeq (\tau_{X'})_*D_{X/S}.$$

where $\tau_{X'} : T^*(X'/S) \to X'$ is the natural projection.

In particular, we have the following.

Corollary B.2.2. *There is a canonical equivalence of categories*

$$\mathcal{D}\text{-mod}(X) \simeq \mathrm{QCoh}(\mathscr{D}_{D_{X/S}})_1$$

where $\mathscr{D}_{D_{X/S}}$ is the gerbe of splittings of $D_{X/S}$.

In what follows, the gerbe $\mathscr{D}_{D_{X/S}}$ will be denoted by $\mathscr{D}_{X/S}$ for simplicity.

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B.3 $\mathcal{D}$-module on stack

Let S be a Noetherian scheme and $p\mathcal{O}_S = 0$. Let $\mathscr{X}$ be a smooth algebraic stack over S. A $\mathcal{D}$-module M on $\mathscr{X}$ is an assignment for each $U \to \mathscr{X}$ in $\mathscr{X}_{\mathrm{sm}}$, a $\mathcal{D}_{U/S}$-module M_U and for each morphism $f : V \to U$ in $\mathscr{X}_{\mathrm{sm}}$ an isomorphism $\phi_f : f^*M_U \simeq M_V$ which satisfies the cocycle conditions. We denote the category of $\mathcal{D}$-modules on $\mathscr{X}$ by $\mathcal{D}\text{-mod}(\mathscr{X})$.

Unlike the case of schemes, in general there does not exist a sheaf of algebras $\mathcal{D}_{\mathscr{X}/S}$ on $\mathscr{X}$ such that the category of $\mathcal{D}$-modules on $\mathscr{X}$ is equivalent to the category of modules over $\mathcal{D}_{\mathscr{X}/S}$, and therefore the naive stacky generalization of Theorem B.2.1 is wrong. On the other hand, it is shown in [Tra11] that the obvious stacky version of Corollary B.2.2 is correct.

PROPOSITION B.3.1. *There exists a $\mathbb{G}_m$-gerbe $\mathscr{D}_{\mathscr{X}/S}$ on $T^*(\mathscr{X}'/S)$ such that the category of twisted sheaves on $\mathscr{D}_{\mathscr{X}/S}$ is equivalent to the category of $\mathcal{D}$-modules on $\mathscr{X}$, i.e., we have*

$$\mathcal{D}\text{-mod}(\mathscr{X}) \simeq \mathrm{QCoh}(\mathscr{D}_{\mathscr{X}/S})_1.$$

Remark B.3.2. It is a theorem of Gabber that on a quasi-projective scheme X every torsion element in $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$ can be constructed from an Azumaya algebra via (B.1.1). However, this fails for non-separated schemes. A theorem of Töen [Toe12] shows that in a very general situation, every $\mathbb{G}_m$-gerbe arises from a derived Azumaya algebra. Although Töen's theorem does not directly apply to $T^*(\mathscr{X}'/S)$, it suggests that the derived category of $\mathcal{D}$-modules on $\mathscr{X}$ (which is not the derived category of $\mathcal{D}\text{-mod}(\mathscr{X})$ in general) probably should be equivalent to the category of modules over some derived Azumaya algebra $D^{\mathrm{dr}}_{\mathscr{X}/S}$ on $T^*(\mathscr{X}'/S)$.

Let us sketch the construction of the $\mathbb{G}_m$-gerbe $\mathscr{D}_{\mathscr{X}/S}$ on $T^*(\mathscr{X}'/S)$. As gerbes satisfy smooth descent, it is enough to supply a $\mathbb{G}_m$-gerbe $(\mathscr{D}_{\mathscr{X}/S})_U$ on $T^*(\mathscr{X}/S) \times_{\mathscr{X}'} U'$ for every $U \to \mathscr{X}$ in $\mathscr{X}_{\mathrm{sm}}$ and compatible isomorphisms for any $\beta : U \to V$ in $\mathscr{X}_{\mathrm{sm}}$. But for any $f : U \to \mathscr{X}$ in $\mathscr{X}_{\mathrm{sm}}$ we have

$$(f'_U)_d : T^*(\mathscr{X}/S) \times_{\mathscr{X}'} U' \to T^*(U'/S).$$

We have a $\mathbb{G}_m$-gerbe $\mathscr{D}_{U/S}$ on $T^*(U'/S)$ corresponding to the sheaf of relative differential operators $\mathcal{D}_{U/S}$. We define a $\mathbb{G}_m$-gerbe $(\mathscr{D}_{\mathscr{X}/S})_U$ on $T^*(\mathscr{X}/S) \times_{\mathscr{X}'} U'$ as the pullback of $\mathscr{D}_{U/S}$ along $(f'_U)_d$. One can check that these gerbes $(\mathscr{D}_{\mathscr{X}/S})_U$ are compatible under pullbacks, and therefore define a $\mathbb{G}_m$-gerbe $\mathscr{D}_{\mathscr{X}/S}$ on $\mathscr{X}$.

Let $f : \mathscr{X} \to \mathscr{Y}$ be a schematic morphism between two smooth algebraic stacks. From the above construction, the following lemma clearly follows from its scheme theoretic version.

LEMMA B.3.3 [Tra11]. (1) *There is a canonical* 1*-morphism of $\mathbb{G}_m$-gerbe on $T^*(\mathscr{Y}'/S) \times_{\mathscr{Y}'} \mathscr{X}'$*

$$M_f : (f'_p)^*\mathscr{D}_{\mathscr{Y}/S} \simeq (f'_d)^*\mathscr{D}_{\mathscr{X}/S}.$$

(2) *For a pair of morphisms $\mathscr{X} \xrightarrow{g} \mathscr{Z} \xrightarrow{h} \mathscr{Y}$ and their composition $f = h \circ g : \mathscr{X} \to \mathscr{Y}$, there is a canonical* 1*-morphisms of $\mathbb{G}_m$-gerbe on $T^*(\mathscr{Y}'/S) \times_{\mathscr{Y}'} \mathscr{X}'$*

$$M_{g,h} : (f'_p)^*\mathscr{D}_{\mathscr{Y}/S} \simeq (f'_d)^*\mathscr{D}_{\mathscr{X}/S},$$

together with a canonical 2*-morphism between $M_{h\circ g}$ and $M_{g,h}$.*

(3) *We have a canonical* 1*-morphism of $\mathbb{G}_m$-gerbe on $T^*(\mathscr{X}'/S)^{\mathrm{sm}}$:*

$$\mathscr{D}_{\mathscr{X}/S}|_{T^*(\mathscr{X}'/S)^{\mathrm{sm}}} \simeq \mathscr{D}_{T^*(\mathscr{X}'/S)^{\mathrm{sm}}/S}(\theta_{\mathrm{can}}) := \theta^*_{\mathrm{can}}(\mathscr{D}_{T^*(\mathscr{X}'/S)^{\mathrm{sm}}/S}),$$

where $T^*(\mathscr{X}'/S)^{\mathrm{sm}}$ is the maximal smooth open substack of $T^*(\mathscr{X}'/S)$ and $\theta_{\mathrm{can}} : T^*(\mathscr{X}'/S)^{\mathrm{sm}} \to T^*(T^*(\mathscr{X}'/S)^{\mathrm{sm}})$ is the canonical one form.

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Let us discuss a stacky version of [OV07, § 4.3]. Let $\mathscr{X}/S$ be a smooth algebraic stack as above and let $\mathscr{P}ic^{\natural}(\mathscr{X}/S)$ be the Picard stack of invertible sheaves on $\mathscr{X}$ equipped with a connection (i.e., objects in $\mathscr{P}ic^{\natural}(\mathscr{X}/S)$ are $\mathcal{D}$-modules on $\mathscr{X}$ whose underlying quasi-coherent sheaves are invertible). Let $B'_S = \mathrm{Sect}_S(\mathscr{X}', T^*(\mathscr{X}'/S))$. Note that the following proposition does not need the representability of $\mathscr{P}ic^{\natural}(\mathscr{X}/S)$.

Proposition B.3.4.

(1) *There is a natural morphism* $\psi : \mathscr{P}ic^{\natural}(\mathscr{X}/S) \to B'_S$.

(2) *The pullback of the gerbe* $\mathscr{D}_{\mathscr{X}/S}$ *along*

$$\mathscr{X}' \times_S \mathscr{P}ic^{\natural}(\mathscr{X}/S) \stackrel{\mathrm{id}\times\psi}{\to} \mathscr{X}' \times_S B'_S \to T^*(\mathscr{X}'/S)$$

is canonically trivialized.

Proof. For part (1), recall that if $\mathscr{X}$ is a scheme, the morphism ψ is given by the p-curvature map (see [OV07, § 4.3]). We explain how to generalize this map to stacks. Let $U \to \mathscr{X}$ be a smooth morphism. Then via pullback, we obtain a morphism $\mathscr{P}ic^{\natural}(\mathscr{X}/S) \to \mathscr{P}ic^{\natural}(U/S) \to \mathrm{Sect}_S(U', T^*(U'/S))$. By considering further pullbacks to $V = U \times_{\mathscr{X}} U$, we find that the above maps fit into the following commutative diagram.

$$\begin{tikzcd}
\mathscr{P}ic^{\natural}(\mathscr{X}/S) \arrow[r] \arrow[d, "\psi_U"'] & \mathscr{P}ic^{\natural}(U/S) \arrow[d] \\
\mathrm{Sect}_S(U', T^*(\mathscr{X}'/S) \times_{\mathscr{X}'} U') \arrow[r] & \mathrm{Sect}_S(U', T^*(U'/S))
\end{tikzcd}$$

These ψ_U are compatible under pullbacks and define the $\pi : \mathscr{P}ic^{\natural}(\mathscr{X}/S) \to B'_S$.

For part (2), again let $U \to \mathscr{X}$ be a smooth morphism. Note that the pullback of the gerbe $\mathscr{D}_{U/S}$ along $U' \times_S \mathscr{P}ic^{\natural}(U/S) \to T^*(U'/S)$ is canonically trivialized by the object $F_*(\mathcal{L}, \nabla)$, where $(\mathcal{L}, \nabla)$ is the universal object on $U \times_S \mathscr{P}ic^{\natural}(U/S)$. Combining this with Lemma B.3.3 and the proof of part (1), this shows that the pullback of $\mathscr{D}_{\mathscr{X}/S}$ along $U' \times_S \mathscr{P}ic^{\natural}(\mathscr{X}/S) \to \mathscr{X}' \times_S \mathscr{P}ic^{\natural}(\mathscr{X}/S)$ is canonically trivialized. These trivializations glue together and give a canonical trivialization of $\mathscr{D}_{\mathscr{X}/S}$ on $\mathscr{X}' \times_S \mathscr{P}ic^{\natural}(\mathscr{X}/S)$. □

B.4 1-forms

In this subsection we make a digression into the construction of gerbes using 1-forms. We refer to [CZ15, Appendix A.8] for more details. Recall that for any smooth algebraic stack $\mathscr{X}/S$ we can associate to it a $\mathbb{G}_m$-gerbe $\mathscr{D}_{\mathscr{X}/S}$ on $T^*(\mathscr{X}'/S)$. Thus giving a 1-form $\theta : \mathscr{X}' \to T^*(\mathscr{X}'/S)$ we can construct a $\mathbb{G}_m$-gerbe $\mathscr{D}(\theta) := \theta^*\mathscr{D}_{\mathscr{X}/S}$ on $\mathscr{X}'$ by pulling back $\mathscr{D}_{\mathscr{X}/S}$ along θ.

When $\mathscr{X} = X$ is a smooth Noetherian scheme, above construction can be generalized as follows. Let $\mathcal{G}$ be a smooth affine commutative group scheme over X. For any section θ of $\mathrm{Lie}\,\mathcal{G}' \otimes \Omega_{X'/S}$ we can associate to it a $\mathcal{G}$-gerbe $\mathscr{D}(\theta)$ on X' using the four term exact sequence constructed in *loc. cit.* In the case $\mathcal{G} = \mathbb{G}_m$, the $\mathbb{G}_m$-gerbe $\mathscr{D}(\theta)$ is isomorphic to $\theta^*\mathscr{D}_{X,S}$ the pullback of $\mathscr{D}_{X/S}$ along $\theta : \mathscr{X}' \to T^*(X'/S)$. We have the following functorial properties.

Lemma B.4.1. (1) *Let $\mathscr{Y}$ be another smooth algebraic stack over S and let $f : \mathscr{Y} \to \mathscr{X}$ be a morphism. Let θ be a 1-form on $\mathscr{X}$. There is a canonical equivalence of $\mathbb{G}_m$-gerbes on $\mathscr{Y}'$*

$$f'^*\mathscr{D}(\theta) \simeq \mathscr{D}(f'^*\theta).$$

(2) *Let X be a smooth Noetherian scheme and let $\phi : \mathcal{G} \to \mathcal{H}$ be a morphism of smooth commutative affine group schemes over X. For any section θ of $\operatorname{Lie} \mathcal{G}' \otimes \Omega_{X'}$ let $\phi'_*\theta$ denote its image $\operatorname{Lie} \mathcal{H}' \otimes \Omega_{X'/S}$ under the map induced by ϕ. There is a canonical equivalence of $\mathcal{H}'$-gerbes on X',*

$$\mathscr{D}(\theta)^{\phi'} \simeq \mathscr{D}(\phi'_*\theta),$$

where $\mathscr{D}(\theta)^{\phi'}$ is the $\mathcal{H}'$-gerbe induced form $\mathscr{D}(\theta)$ using the map ϕ' (see § A.6).

B.5 Azumaya property of differential operators on good stacks

Recall that a smooth algebraic stack $\mathscr{X}$ over S of relative dimension d is called relatively good if it satisfies the following equivalent properties:

(1) $\dim(T^*(\mathscr{X}/S)) = 2d$;
(2) $\operatorname{codim}\{x \in \mathscr{X} \mid \dim \operatorname{Aut}(x) = n\} \geqslant n$ for all $n > 0$;
(3) for any $U \to \mathscr{X}$ in $\mathscr{X}_{\mathrm{sm}}$, the complex

$$\operatorname{Sym}(T_{U/\mathscr{X}} \to T_{U/S})$$

has cohomology concentrated in degree 0 and

$$H^0(\operatorname{Sym}(T_{U/\mathscr{X}} \to T_{U/S})) \simeq \operatorname{Sym}(T_{U/S})/T_{U/\mathscr{X}} \operatorname{Sym}(T_{U/S}).$$

The following proposition is proved in [BB07] (see also [Tra11]).

Proposition B.5.1. *Let $\mathscr{X}$ be a relatively good stack. Let $\pi_{\mathscr{X}} : T^*(\mathscr{X}/S) \to \mathscr{X}$ be the natural projection and $\pi_{\mathscr{X}'}$ be its Frobenius twist. Let $T^*(\mathscr{X}'/S)^0$ be the maximal smooth open substack of $T^*(\mathscr{X}'/S)$. Then we have the following.*

(i) *There is a natural coherent sheaf of algebras $D_{\mathscr{X}/S}$ on $T^*(\mathscr{X}'/S)$ such that the restriction of $D_{\mathscr{X}/S}$ to $T^*(\mathscr{X}'/S)^0$ is an Azumaya algebra on $T^*(\mathscr{X}'/S)^0$ of rank $p^{2\dim(\mathscr{X}/S)}$.*
(ii) *The $\mathbb{G}_m$-gerbe $\mathscr{D}^0_{\mathscr{X}/S} := \mathscr{D}_{\mathscr{X}/S}|_{T^*(\mathscr{X}'/S)^0}$ is isomorphic to $\mathscr{D}_{D^0_{\mathscr{X}/S}}$, the gerbe of splittings of $D^0_{\mathscr{X}/S}$. In particular, we have*

$$D^0_{\mathscr{X}/S}\text{-mod} \simeq \operatorname{QCoh}(\mathscr{D}^0_{\mathscr{X}/S})_1.$$

Remark B.5.2. By Proposition B.3.1, the category $D^0_{\mathscr{X}/S}$-mod can be thought as a localization of the category of $\mathcal{D}$-modules on $\mathscr{X}$.

Appendix C. Abelian duality

C.1 Abelian duality for Beilinson 1-motives

Assume that S is a scheme and $p\mathcal{O}_S = 0$. Let $\mathscr{A}$ be a Picard stack over S. In this subsection, we denote the base change of $\mathscr{A}$ along $Fr_S : S \to S$ by $\mathscr{A}'$ instead of $\mathscr{A}^{(S)}$. Let $\mathbb{T}^*_e\mathscr{A}'$ be the vector bundle on S, which is the restriction of the relative (to S) cotangent bundle of $\mathscr{A}'$ along $e : S \to \mathscr{A}'$. Then there is a canonical isomorphism

$$\mathscr{A}' \times_S \mathbb{T}^*_e\mathscr{A}' \simeq T^*(\mathscr{A}'/S).$$

Therefore, via the map $\pi_S : T^*(\mathscr{A}'/S) \simeq \mathscr{A}' \times_S \mathbb{T}^*_e\mathscr{A}' \to \mathbb{T}^*_e\mathscr{A}'$, $T^*(\mathscr{A}'/S)$ becomes a Picard stack over $\mathbb{T}^*_e\mathscr{A}'$ and we denote by m_S the multiplication map:

$$m_S : T^*(\mathscr{A}'/S) \times_{\mathbb{T}^*_e\mathscr{A}'} T^*(\mathscr{A}'/S) \to T^*(\mathscr{A}'/S).$$

Recall that it makes sense to talk about a gerbe on a Picard stack with a commutative group structure (cf. Definition A.5.2).

LEMMA C.1.1. *The gerbe $\mathscr{D}_{\mathscr{A}/S}$ on $T^*(\mathscr{A}'/S)$ admits a canonical commutative group structure.*

Proof. Let us sketch the construction of the multiplicative structure M and the 2-morphisms γ and i in Definition A.5.2. The multiplication $m : \mathscr{A} \times_S \mathscr{A} \to \mathscr{A}$ induces the following diagram.

$$\begin{tikzcd} T^*(\mathscr{A}'/S) \times_{\mathscr{A}'} (\mathscr{A}' \times_S \mathscr{A}') \arrow[r, "m_d"] \arrow[d, "m_p"] & T^*(\mathscr{A}' \times_S \mathscr{A}'/S) \\ T^*(\mathscr{A}'/S) & \end{tikzcd}$$

Observe that the map $m_d : T^*(\mathscr{A}'/S) \times_{\mathscr{A}'} (\mathscr{A}' \times_S \mathscr{A}') \to T^*(\mathscr{A}' \times_S \mathscr{A}'/S) \simeq T^*(\mathscr{A}'/S) \times_S T^*(\mathscr{A}'/S)$ induces an isomorphism

$$T^*(\mathscr{A}'/S) \times_{\mathscr{A}'} (\mathscr{A}' \times_S \mathscr{A}') \simeq T^*(\mathscr{A}'/S) \times_{\mathbb{T}_e^*\mathscr{A}'} T^*(\mathscr{A}'/S) \to T^*(\mathscr{A}'/S) \times_S T^*(\mathscr{A}'/S).$$

Under this isomorphism m_p becomes the multiplication map m_S. Now the canonical 1-morphism between $m_S^*\mathscr{D}_{\mathscr{A}/S}$ and $\mathscr{D}_{\mathscr{A}/S} \boxtimes \mathscr{D}_{\mathscr{A}/S}$ comes from Lemma B.3.3. We have two different factorizations of the multiplicative morphism $\mathscr{A} \times_S \mathscr{A} \times_S \mathscr{A} \to \mathscr{A}$ and the 2-morphisms γ comes from the 2-morphisms for corresponding equivalences of Lemma B.3.3. Finally, the 2-morphism $i : \sigma^*M \simeq M$ can be constructed by applying Lemma B.3.3 to the morphism $\mathscr{A} \times_S \mathscr{A} \xrightarrow{\sigma} \mathscr{A} \times_S \mathscr{A} \xrightarrow{m} \mathscr{A}$. $\square$

Now we assume that $\mathscr{A}$ is a Beilinson 1-motive and is good when regarded as an algebraic stack. Let $\mathscr{A}^\natural := \mathscr{P}ic^\natural(\mathscr{A})$ be the Picard stack of multiplicative invertible sheaves on $\mathscr{A}$ with a connection (cf. [Lau96]), and let $\psi_S : \mathscr{A}^\natural \to \mathbb{T}_e^*\mathscr{A}'$ be the p-curvature morphism as given in Proposition B.3.4(1). By [OV07, § 4.3], there is a natural action of $T^*(\mathscr{A}'/S)^\vee \simeq (\mathscr{A}')^\vee \times_S \mathbb{T}_e^*\mathscr{A}'$ on $\mathscr{A}^\natural$. Concretely, for any $b : U \to \mathbb{T}_e^*\mathscr{A}'$ objects in $\mathscr{A}^\natural \times_{\mathbb{T}_e^*\mathscr{A}'} U$ consist of multiplicative line bundles on $\mathscr{A} \times_S U$ with a connection whose p-curvature is equal to b. Then for any $\mathcal{L}' \in (\mathscr{A}')^\vee \times_S U \simeq T^*(\mathscr{A}'/S)^\vee \times_S U$ and $(\mathcal{L}, \nabla) \in \mathscr{A}^\natural \times_{B'_S} U$ we define

$$\mathcal{L}' \cdot (\mathcal{L}, \nabla) := (F_{\mathscr{A}}^*\mathcal{L}' \otimes \mathcal{L}, \nabla_{F_{\mathscr{A}}^*\mathcal{L}'} \otimes \nabla),$$

where $\nabla_{F_{\mathscr{A}}^*\mathcal{L}'}$ is the canonical connection on $F_{\mathscr{A}}^*\mathcal{L}'$ giving by the Cartier descent. It also follows from the Cartier descent that $\mathscr{A}^\natural$ is a $T(\mathscr{A}'/S)^\vee$-torsor under this action.

On the other hand, recall that for a $\mathbb{G}_m$-gerbe $\mathscr{D}$ with commutative group structure on a Beilinson 1-motive $\mathscr{P}$, we defined the $\mathscr{P}^\vee$-torsor $\mathscr{T}_{\mathscr{D}}$ of multiplicative splittings of $\mathscr{D}$ (cf. § A.7).

PROPOSITION C.1.2. *There is a canonical $(T^*(\mathscr{A}'/S))^\vee$-equivariant isomorphism $\mathscr{A}^\natural \to \mathscr{T}_{\mathscr{D}_{\mathscr{A}/S}}$.*

Proof. We sketch the proof. Write $\mathscr{T}_{\mathscr{D}_{\mathscr{A}/S}}$ by $\mathscr{T}_{\mathscr{D}}$ for simplicity. Recall that for $U \to \mathbb{T}_e^*\mathscr{A}'$, $\mathscr{T}_{D_A}(U)$ is the groupoid of splittings of $\mathscr{D}_{\mathscr{A}/S}$ over $U \times_{\mathbb{T}_e^*\mathscr{A}'} T^*(\mathscr{A}'/S)$ which are compatible with the commutative group structure of $\mathscr{D}_{\mathscr{A}/S}$. Note that

$$U \times_{\mathbb{T}_e^*\mathscr{A}'} T^*(\mathscr{A}'/S) \simeq U \times_{\mathbb{T}_e^*\mathscr{A}'} (\mathbb{T}_e^*\mathscr{A}' \times_S \mathscr{A}') \simeq \mathscr{A}' \times_S U,$$

and under this isomorphism, the projection of left-hand side to the second factor is identified with

$$\mathscr{A}' \times_S U \to \mathscr{A}' \times_S \mathscr{A}^\natural \to T^*(\mathscr{A}'/S).$$

Now by Lemma B.3.4, the pullback of $\mathscr{D}_{\mathscr{A}/S}$ to $\mathscr{A}' \times_S U$ has a canonical splitting $\mathcal{L}_{U,\alpha}$. Moreover, one can check that this canonical splitting is compatible with the commutative group structure of $\mathscr{D}_{\mathscr{A}/S}$. Thus the assignment $(U, \alpha) \to \mathcal{L}_{U,\alpha}$ defines a map from $\mathscr{A}^\natural$ to $\mathscr{T}_{\mathscr{D}}$ which is compatible with their $T(\mathscr{A}'/S)^\vee$-torsor structures hence an equivalence. $\square$

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As a corollary, we obtain the following theorem.

Theorem C.1.3. *Let $\mathscr{A}$ be a good Beilinson* 1*-motive. Then there is a canonical equivalence of categories*

$$D^b(\mathcal{D}\text{-mod}(\mathscr{A})) \simeq D^b(\mathrm{QCoh}(\mathscr{A}^\natural)).$$

Proof. This is the combination of Theorem A.7.2 and Proposition B.5.1. □

Remark C.1.4. Note that in [Lau96], this theorem is proved for abelian schemes over S of characteristic zero. In fact, Laumon's construction applies to any 'good' Beilinson 1-motive over a locally Noetherian base. When $p\mathcal{O}_S = 0$, it is easy to see that Laumon's equivalence and the equivalence constructed above are the same.

In particular, let $\theta : \mathscr{A}' \to \mathbb{T}^*\mathscr{A}'$ be a section obtained by base change $\tau : S \to \mathbb{T}^*_e\mathscr{A}'$. Let $\mathscr{D}_{\mathscr{A}/S}(\theta) := \theta^*\mathscr{D}_{\mathscr{A}/S}$. Then $\mathscr{D}_{\mathscr{A}/S,\theta}$ is a $\mathbb{G}_m$-gerbe on $\mathscr{A}'$ equipped with a canonical commutative group structure, and the ${\mathscr{A}'}^\vee$-torsor $\mathscr{T}_{\mathscr{D}_{\mathscr{A}/S,\theta}}$ of multiplicative splittings can be identified with $\mathscr{A}^\natural \times_{\mathbb{T}^*_e\mathscr{A}',\tau} S$.

C.2 A variant

In the main body of the paper, however, we need a variant of the above construction. Let k be an algebraically closed field of characteristic p. For a k-scheme X, we denote by X' its Frobenius base change along $Fr : k \to k$. Let S be a smooth k-scheme. For an S-scheme $X \to S$, we denote by $X^{(S)}$ its base change along $Fr_S : S \to S$. Let $\mathscr{A} \to S$ be a Picard stack with multiplication $m : \mathscr{A} \times_S \mathscr{A} \to \mathscr{A}$. The goal of this subsection is to construct certain multiplicative gerbe $\mathscr{D}_{\mathscr{A}}(\theta)$ on $\mathscr{A}'$ (rather than on $\mathscr{A}^{(S)}$ as done at the end of the previous subsection).

Let $\theta : \mathscr{A}' \to T^*\mathscr{A}'$ be a section, where $T^*\mathscr{A}'$ is the cotangent bundle of $\mathscr{A}'$ relative to k. We say θ is multiplicative if the upper right corner of the following diagram is commutative.

$$\begin{tikzcd}
T^*\mathscr{A}' \times T^*\mathscr{A}' & T^*\mathscr{A}' \times T^*\mathscr{A}'|_{\mathscr{A}' \times_{S'} \mathscr{A}'} \arrow[l] \arrow[r] & T^*(\mathscr{A}' \times_{S'} \mathscr{A}') \\
\mathscr{A}' \times \mathscr{A}' \arrow[u, "\theta\times\theta"] & \mathscr{A}' \times_{S'} \mathscr{A}' \arrow[l] \arrow[u, "\theta\times\theta"] \arrow[r, "m^*\theta"] \arrow[d, "m"] & T^*\mathscr{A}' \times_{\mathscr{A}'} (\mathscr{A}' \times_{S'} \mathscr{A}') \arrow[u, "m_d"] \arrow[d, "m_p"] \\
 & \mathscr{A}' \arrow[r, "\theta"] & T^*\mathscr{A}'
\end{tikzcd}$$

Let $\mathscr{D}_{\mathscr{A}}(\theta) = \theta^*\mathscr{D}_{\mathscr{A}}$ be the pullback of $\mathscr{D}_{\mathscr{A}}$ to $\mathscr{A}'$. Then by the same argument as in Lemma C.1.1, we have the following.

Lemma C.2.1 (See also [BB07, Lemma 3.16]). *Let $\theta : \mathscr{A}' \to T^*\mathscr{A}'$ be a multiplicative section. Then $\mathscr{D}_{\mathscr{A}}(\theta)$ is a $\mathbb{G}_m$-gerbe on $\mathscr{A}'$ with a commutative group structure.*

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