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ON THE SIMPLE GROUP OF J. TITS

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In the series of simple groups ${}^2F_4(q)$, $q=2^{2m+1}$, discovered by Ree, Tits [4] showed that the group ${}^2F_4(2)$ was not simple but contained a simple subgroup $\mathscr{T}$ of index 2. In this note we extend the characterization of $\mathscr{T}$ obtained by the author in [3]. Namely, we prove the following result:

THEOREM. *Let G be a finite group which contains an involution z such that* $H=C_G(z)$ *has the following properties:*

(i) H *is a* $\{2, 5\}$*-group with* $O_5(H)=1$

(ii) $J=O_2(H)$ *is of order* 2^9 *and class at least* 3

(iii) H *possesses an element p of order* 5 *such that* $C_J(p) \subseteq Z(J)$.

Then $G=H\cdot O(G)$ *or* $G\simeq\mathscr{T}$.

Throughout the rest of this paper, G will denote a finite group satisfying the assumptions of the theorem, and also we assume that $G\neq H\cdot O(G)$. Using Glauberman's theorem [1], it follows that $\langle z\rangle$ is not weakly closed in H with respect to G. The notation used in this paper is standard (see [2] for example).

LEMMA 1. *We have* $\langle p\rangle=P$ *is a Sylow* 5*-group of* H, $E=\Phi(J)=Z_2(J)$ *is elementary abelian of order* 32 *and* $N_G(E)=H$. *Further, z is conjugate to some involution in* $H-E$, *and* $Z(J)=\langle z\rangle$.

Proof. Since $\langle p\rangle=P$ acts nontrivially on $J/\Phi(J)$, $|J:\Phi(J)|\geq 16$ and so $|J'|\leq |\Phi(J)|\leq 32$. As J has class at least 3, $J'\supset J'\cap Z(J)$, and as $C_J(p)\subseteq Z(J)$, $|J':Z(J)\cap J'|\geq 16$. Hence $|J:J'|=|J':Z(J)\cap J'|=16$, $Z(J)\subset J'$ so $Z(J)=\langle z\rangle$, J has class 3 and $E=J'=\Phi(J)=Z_2(J)$ has order 32. Now $E'=(J')'=1$ so E is elementary abelian as $E=C_E(p)\times[p, E]=\langle z\rangle\times[p, E]$. From $O_5(H)=1$ and $|J:\Phi(J)|=16$ it follows that $\langle p\rangle=P$ is a Sylow 5-subgroup of H.

It is clear that $C_H(E)=E$ so $C_G(E)=E$ and $N_G(E)/E$ is isomorphic to a subgroup of $GL(5, 2)$. As each involution in $E-\langle z\rangle$ has either 10 or 20 conjugates in H, z has either 1, 11, 21, or 31 conjugates in $N_G(E)$. However, $GL(5, 2)$ does not possess subgroups of order $2^i\cdot 5\cdot 11$, $2^i\cdot 3\cdot 5\cdot 7$ or $2^i\cdot 5\cdot 31$ ($i=4, 5, 6$) so $N_G(E)=H$. Suppose z is not conjugate to any involution in $H-E$. Then z is conjugate to an

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involution $t \in E-\langle z\rangle$, and note that $\langle t^x \mid x \in H\rangle = E$; i.e., the weak closure of $\langle t\rangle$ in H is E. However, under the assumption that z is not conjugate to any involution in $H-E$, E is the weak closure of $\langle z\rangle$ in $C_G(t)$, which contradicts $N_G(E)=H$, as clearly $C_G(t) \supset C_H(t)$. The lemma is proved.

From Lemma 1 it follows that H/J is isomorphic to a subgroup of F_{20}, the Frobenius group of order 20. Note also that a Sylow 2-subgroup T of H is a Sylow 2-subgroup of G since $Z(T)=\langle z\rangle$.

LEMMA 2. *We have that $H/J \simeq F_{20}$, the Frobenius group of order twenty.*

Proof. From Lemma 1 the following properties of J are derived:

(1) If $x \in J-E$ then $|C_E(x)|=16$ (as $L_3(J)=[J', J]=\langle z\rangle$ and $E=J'$).

(2) If x is an involution in $J-E$ then $\langle x, E\rangle' = \mho^1(\langle x, E\rangle)=\langle z\rangle$, so not every coset of E in J contains involutions.

(3) If $J \supset J_1 \supset J_2 \supset J_3 \supset E$ is any (maximal) chain of subgroups from J to E then J_i', $Z(J_i) \subseteq E$ $(i=1, 2, 3)$, $|Z(J_1)|=4$, $|Z(J_2)|=8$ and $|Z(J_3)|=16$, and $|J_1'| \geq 8$. (This last fact may be proved by noting that we may choose $a_i \in J-E$, $i=1, \ldots, 4$, so that $J_1=\langle E, a_1, a_2, a_3\rangle$, $a_4=a_1^p$ and $J=\langle a_i \mid i=1, \ldots, 4\rangle$. Also $\{z, [a_i, a_j] \mid$ for suitable $i, j \leq 4\}$ is a basis for $E=J'$. If $|J_1'| \leq 4$ then $|C_{J_1}(a_i)| \geq 2^7$, $i=1, 2, 3$ whence $|C_J(a_4)| \geq 2^7$. However if $J_2'=\langle z, t\rangle$ (of order four), it follows that z, t, $[a_1, a_4]$, $[a_2, a_4]$, $[a_3, a_4]$ are not linearly independent and so $J' \subset E$, a contradiction.)

(4) For $x \in J-E$, $2^5 \leq |C_J(x)| \leq 2^6$. (The last inequality follows from the fact that if $|C_J(x)|=2^7$, $C_J(x) \cdot E$ is maximal in J and $(C_J(x) \cdot E)'$ has order at most four; while if $|C_J(x)|=2^8$, $C_J(x) \supseteq E=\Phi(J)$ which is impossible.)

(5) If T is a Sylow 2-subgroup of H and $M/E=Z(T/E)$ then any coset xE of E in H is conjugate to a coset of E in M, when $|T/J|<4$.

The proof of the lemma is by way of contradiction, so we suppose H/J is isomorphic to a proper subgroup of F_{20}. The proof is divided into 7 steps.

(I) *These are involutions in $T-J$, hence $H/J \simeq D_{10}$, the dihedral group of order ten.*

Recall that by Lemma 1, z is conjugate to some involution in $H-E$. We suppose that for a Sylow-2-subgroup T of H, $\Omega_1(T)=J$ so that $z \underset{G}{\sim} a$ for some involution $a \in J-E$. By (5), we may suppose $aE \subseteq Z(T/E)$ so that $\langle a, E\rangle < T$. Put $F= \langle a\rangle \times C_E(a)$ so $F \lhd T$ as $\langle a, E\rangle$ contains precisely two elementary subgroups of order 32, namely E and F.

By assumption, $z \underset{G}{\sim} a$ so $C_T(a)=C_H(a)$ is a proper subgroup of some Sylow 2-subgroup of $C_G(a)$ whence $N_G(C_T(a)) \supset N_T(C_T(a))$. It follows that $\langle z\rangle$ is not a characteristic subgroup of $C_T(a)$. If $\Omega_1(C_T(a)) \supset F$ then $\Omega_1(C_T(a))=C_J(a)$ by (4) and our assumption; however in this case $\langle z\rangle=C_J(a)'$ so $\langle z\rangle$ char $C_T(a)$. Thus $\Omega_1(C_T(a))=F$ so $N_G(F) \supset N_H(F)=T$, as $F \lhd N_G(C_T(a))$. As $C_G(F)=F$, $N_G(F)/F$ is isomorphic to a subgroup of $GL(5, 2)$. The structure of $GL(5, 2)$ yields that

$|T:O_2(N_G(F))|\leq 2$, so if $K=O_2(N_G(F))$, $N_G(F)/K\simeq S_3$ (the symmetric group on three letters) as $z\in Z(K)$ is of order at most four if $|T:K|\leq 2$. Thus in this case $Z(K)$ is of order four so $|J:K\cap J|=2$.

Now $E\subseteq K\cap J$ and as $E\not\triangleleft N_G(F)$, $W=\Omega_1(K)\supseteq\langle F, E^Q\rangle$, where Q is a Sylow 3-subgroup of $N_G(F)$, is of order $\geq 2^2|F|$. As $\Omega_1(K)\subseteq J\cap K$, either $W=J\cap K$ or $|W:F|=4$. In the first case $|(J\cap K)'|\geq 8$ by (3) so that $L_3(W)=\langle z\rangle$. In the last case, each coset of E in W contains involutions so $\mho^1(W)=\langle z\rangle$ by (2). Thus in both cases $\langle z\rangle\triangleleft N_G(F)$ so $N_G(F)\subseteq H$, a contradiction.

(II) *These are involution in* $J-E$.

By (I) there are involutions in $T-J$. Suzuki's lemma ([2, p. 328, Example 2]) yields that any involution in $H-J$ inverts an element of order 5 in H. Thus by Sylow's theorem there is an involution j in $N_H(P)$ with $jpj=p^{-1}$. Therefore, a Sylow 2-subgroup $\langle j, z\rangle$ of $N_H(P)$ is elementary abelian of order four, and any involution in $H-J$ is conjugate in H to either j or jz (or perhaps to both).

Since j inverts P, $|C_E(j)|=8$ and $Z(T/E)=(T/E)'=\Phi(T/E)=M/E$ is elementary of order four. Further, precisely four cosets of E in $T-J$ contain involutions so we have two possibilities: either $j\underset{H}{\sim}jz$ and $|C_H(j)|=2^5$ or $j\underset{H}{\nsim}jz$ and $|C_H(j)|=2^6$. Finally, if $C_E(j)=\langle z, t, v\rangle$ then $C_T(t)$ and $C_T(v)$ are maximal subgroups of T so that $C_T(\langle v, t\rangle)/E\supseteq\Phi(T/E)$ whence $C_J(\langle t, v\rangle)=M$; i.e. $Z(M)=C_E(j)$.

Suppose there are no involutions in $J-E$. Then E is the only elementary subgroup of order 32 in T whence z is not conjugate to any involution in $E-\langle z\rangle$ (using the same argument as in Lemma 1). We may suppose therefore, that $z\underset{G}{\sim}j$ so that $C_T(j)$ is a proper subgroup of $C_G(j)$. It follows that $W=\Omega_1(C_T(j))=\langle j, Z(M)\rangle$ char $C_T(j)$ so $N_G(W)\supset N_T(W)=\langle M, j\rangle$. If j has 8 conjugates in $N_T(W)$ then z has 9 conjugates in $N_G(W)$ (as z is not conjugate to any element in $E-\langle z\rangle$); but now $\{Z(M)-\langle z\rangle\}\triangleleft N_G(W)$ so $Z(M)\triangleleft N_G(W)$, a contradiction. Therefore, we may assume j has only four conjugates in $N_T(W)$ and z has 5 conjugates in $N_G(W)$. It follows that $C_T(W)$ covers M/E and $N_G(W)/C_G(W)$ has order 20 (and of course is isomorphic to a subgroup of $A_8\simeq GL(4, 2)$). However $E\cdot C_G(W)/C_G(W)$ is a Sylow 2-subgroup of $N_G(W)/C_G(W)$ which contradicts the structure of A_8.

(III) *If m is an involution in $M-E$ and $F=\langle m\rangle\times C_E(m)$, then $N_G(F)=T$.*

We argue by way of contradiction, noting that T is a Sylow 2-subgroup of $N_G(F)$ and $C_G(F)=F$. Under the assumption $N_G(F)\supset T$, the structure of $GL(5, 2)$ implies that $|O_2(N_G(F))|=2^9$ (see [3, Lemma 6] for a similar argument using the structure of $GL(5, 2)$). Put $K=O_2(N_G(F))$, and as $z\in Z(K)$ and $Z(K)$ is of order at most four, $N_G(F)/K\simeq S_3$ and $Z(K)=\langle z, v\rangle$ for some involution $v\in Z(M)-\langle z\rangle$. If $Z(M)\triangleleft N_G(F)$, then $N_G(F)/C_G(Z(M))$ is a subgroup of order 12 since $|K:C_K(Z(M))|=2$. However this contradicts the structure of $GL(3, 2)\simeq PSL(2, 7)$ (as the group of order 12 is not 2-closed), whence $Z(M)\not\triangleleft N_G(F)$.

Let d be an involution in $J-K$ (note that $J\subseteq N_G(F)$) and let R be a Sylow 3-subgroup of $N_G(F)$ inverted by d. As $|C_F(d)\cap E|=8$, we have $C_F(R)\cap E\neq 1$.

Since $C(R) \cap Z(M)=1$ (otherwise $Z(M) \lhd N_G(F)$), there is an $e \in C_F(R) \cap E$ with $e \notin Z(M)$. Now $C_K(e)=\langle F, E^R\rangle$ has order 2^7 (since $EF \not\lhd N_G(F)$) and $C_K(e)$ is R-invariant. However, each coset of E in $C_K(e)$ contains involutions, which means $\langle z\rangle=\mho^1(C_K(e))$ is R-invariant (see [2]). ($\langle F, E^R\rangle$ has $31+3\cdot 16=79$ involutions; hence as $\langle F, E^R\rangle \subseteq T$ and $|\langle F, E^R\rangle : E|=4$, each coset of E in $\langle F, E^R\rangle$ must contain involutions.) As $R \nsubseteq H$, we have a contradiction.

(IV) *We have z is not conjugate to any involution in $E-\langle z\rangle$ in G.*

Suppose z is conjugate to some involution in $E-\langle z\rangle$. Since $Z(M)$ contains a representative of each conjugate class of involutions of E in H, we have $z \underset{G}{\sim} t$ for some $t \in Z(M)-\langle z\rangle$. Let $C=C_T(t)$, so that $Z(C)=\langle t, z\rangle \lhd T$ and $N_G(C)/C \simeq S_3$. We see that $(C/E)'$ has order two (see the remarks at the beginning of (II)) and as $C'' \subseteq \langle z\rangle$, C' is abelian for $\langle z\rangle \not\lhd N_G(C)$. If C' is not elementary, $\mho^1(C')=\langle v\rangle$ for some involution $v \in \langle t, z\rangle$ which is impossible. Thus $C' \subseteq \langle c\rangle \times C_E(c)$ for some involution $c \in C'-E$. Therefore $C' \subset \langle c\rangle \times C_E(c)$ by (III) and so $C'=\langle c\rangle \times Z(M)$. (We know $Z(M) \subseteq C'$ for $Z(M)=\langle z\rangle \times [j, E]$, $j \in C-J$ (see II) and obviously $z \in C'$). If C/C' is not elementary abelian then $C' \subset \mho^1(C) \subseteq \langle c, E\rangle$. This leads to a contradiction as above for $\mho^1(C) \lhd N_G(C)$.

Finally if C/C' is elementary (of order 32) then $|C_{C/C'}(i)| \geq 8$ and $|(T/C')'| \geq 2^4$ for any involution $i \in T-C$. This is also a contradiction as it implies $|T:T'| \geq 2^4$ while $T'=M$. (We know $(T/E)'=M$ and $J'=E$ so $T'=M$.)

(V) *For any involution $a \in J-E$, we have $a \underset{G}{\nsim} z$.*

If we assume $z \underset{G}{\sim} a \in J-E$ we may suppose $a \in M-E$ by (5). Since $z \underset{G}{\sim} a$, if $\Omega_1(C_T(a))=Y$ then $N_G(Y) \supset N_T(Y)$ so $Y \supset F_1=\langle a\rangle \times C_E(a)$ by (III). If $C_J(a)=F_1$ then $C_T(a)$ covers T/J and $Z(C_T(a))=\langle Z(M), a\rangle=Z$. However a has 8 conjugates in $N_T(Z)$ under this assumption, whence z has 9 conjugates in $N_G(Z)$ (using (IV)). This is a contradiction as before as we now have $\{Z(M)-\langle z\rangle\} \lhd N_G(Z)$ and so $Z(M) \lhd N_G(Z)$. Further if $Y \subset J$ then $Y'=\langle z\rangle$ so $N_G(Y) \subseteq H$, which is also impossible. Thus $Y=C_T(a)$ has order 2^7, so $Z(Y)=\langle z, t, a\rangle$ for some $t \in Z(M)-\langle z\rangle$. Also $\langle a_1, z\rangle \subseteq Z(Y) \cap Y'$ for some $a_1 \in Z(Y)-\langle t, z\rangle$ for otherwise $z \in Z(Y) \cap Y' \subseteq E$ which implies z is conjugate to an involution in $E-\langle z\rangle$ against (IV), or $\langle z\rangle \lhd N_G(Y)$. Thus $|Z(Y):Z(Y) \cap Y'| \leq 2$.

Put $V=Z(Y)$ and note that $N_G(V) \supset N_T(V)$. Thus z has only 3 conjugates in $N_G(V)$ which implies $N_G(V)/C_G(V) \simeq S_3$. (As $z \nsim t$, tz by (IV), and $N_G(V) \supset N_T(V)$, z has 3 or 5 conjugates in $N_G(V)$. The latter case yields $\{zt, t\} \lhd N_G(V)$ and so $\langle zt, t\rangle=\langle z, t\rangle \lhd N_G(V)$, a contradiction.) Note that $C_G(V)=C_T(a)=Y$.

Clearly $E \subseteq N_G(V)$. Take $e \in E-C_G(V)$ and, as above, e inverts a Sylow 3-subgroup Q of $N_G(V)$ by Suzuki's lemma. Now e fixes 8 cosets of V in $C_G(V)$ (all of which lie in $J \cap C_G(V)$) whence $C(Q) \cap C_G(V)$ has order 8. Put $S=V\cdot (C(Q) \cap C_G(V))$ and note that $|S \cap F_1| \geq 16$, whence $|S \cap F_1|=16$ (as $|S|=32$ and S could not be equal to F_1 (S is Q-invariant) by (III)). It follows that there exists $f\in(F_1\cap C(Q))-V$, whence $C(f)\cap C_G(V)=S\cdot F_1$ is Q-invariant. (Note that S is abelian as $S'=\langle z\rangle$.) However $|C_G(V):S\cdot F_1|=2\geq|V:V\cap C_G(V)'|$ so Q stabilizes the chain $C_G(V)\supset S\cdot F_1\supset S\supset V\supset V\cap C_G(V)'$ which implies Q centralizes $C_G(V)/V\cap C_G(V)'$. Thus Q centralizes $C_G(V)=Y$ and hence $Q\subseteq C_G(Y)\subseteq H$, clearly a contradiction.

(VI) *We have that z is not conjugate to any involution in $T-J$.*

As usual the proof is by way of contradiction, so we may suppose $z\underset{G}{\sim}j$. Since $C_T(j)\subseteq\langle M,j\rangle$ and $\langle j,Z(M)\rangle\subseteq C_T(j)$ we have $\langle j,Z(M)\rangle\subseteq W=Z(\Omega_1(C_T(j)))$. Note that (IV) and (V) imply that z is not conjugate (in G) to any involution in $J-\langle z\rangle$.

If $W=\langle j,Z(M)\rangle$ then z has 5 conjugates in $N_G(W)$ (as $z\underset{G}{\sim}j$ and $N_G(W)\supset N_T(W)$), otherwise z has 9 conjugates in $N_G(W)$ which implies $\{Z(M)-\langle z\rangle\}\lhd N_G(W)$ and therefore $\langle z\rangle\lhd N_G(W)$. Since $E\cdot C_G(W)/C_G(W)$ is elementary of order 4 and $|N_G(W):C_G(W)|\mid 2^4\cdot 5$, it follows from the structure of $GL(4,2)\simeq A_8$ that $N_G(W)$ is 2-closed. However $N_T(W)$ is a Sylow 2-subgroup of $N_G(W)$ and $z\in Z(N_T(W))\subseteq E$, clearly a contradiction.

If $|W:\langle j,Z(M)\rangle|=2$ then $|T:N_T(W)|=2$ so that j has 8 or 16 conjugates in $N_G(W)$. Thus z has 9 or 17 conjugates in $N_G(W)$. Here $|W|=32$ so $N_G(W)/C_G(W)$ is isomorphic to a subgroup of $GL(5,2)$. We see that $|N_G(W)|=2^9\cdot 9$ and $|N_G(W):C_G(W)|=2^i\cdot 9$, $i=3$ or 4. If x is an involution in $N_T(W)-C_T(j)\cdot E$ then $[x,E]=\langle z\rangle$ so $N_G(W)/C_G(W)$ contains an elementary abelian subgroup of order 8. The structure of $GL(5,2)$ (see [3, §1] for example) implies $N_G(W)$ is 2-closed. This gives a contradiction as above.

Finally if $W=C_T(j)$ (i.e. $|W:\langle j,Z(M)\rangle|=4$) then each coset of E in M would contain involutions, against (2) and (5). We have completed the proof of (VI).

(VII) *The subgroup $\langle z\rangle$ is weakly closed in H (with respect to G).*

This follows immediately since (IV), (V), (VI) yield that z is not conjugate to any involution in $T-\langle z\rangle$ in G, whence $\langle z\rangle$ is weakly closed in H by Sylow's theorem.

The proof of Lemma 2 is complete as (VII) and Glauberman's theorem [1] implies that $G=H\cdot O(G)$, contradicting our assumption.

LEMMA 3. *We have that $G\simeq\mathscr{T}$, the simple group of J. Tits.*

Proof. From Lemmas 1 and 2, H satisfies the following properties:

(i) $O_2(H)=J$ is of order 2^9 and class 3

(ii) $H/J\simeq F_{20}$, the Frobenius group of order 20

(iii) If P is a Sylow 5-subgroup of H, then $C_H(P)\subseteq Z(J)$.

Hence the assumptions of the theorem of [3] are satisfied, and this yields immediately that $G\simeq\mathscr{T}$.

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