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ON THE DEFICIENCIES OF COMPOSITE ENTIRE FUNCTIONS

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For any sequence (a_j) of complex numbers and for any $\rho > 1/2$, we construct an entire function F with the following properties. F has order ρ, mean type, each a_j is a deficient value of F, and F is given by $F(z) = f(g(z))$, where f and g are transcendental entire functions. This complements a result of Goldstein. We also construct, for any $\rho > 1/2$, an entire function G of order ρ, mean type, such that $\liminf_{r\to\infty} T(r,G)/T(r,G') > 1$.

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1. Introduction

We are concerned with the deficiencies of an entire function F given by $F(z) = f(g(z))$, where f and g are transcendental entire functions. It is well-known [**9**, p. 53] that if F has finite order, then f must have order zero, which implies that F has no finite Picard value. The following was proved by Goldstein [**5**] (see also [**6**]). Here the notation is that of [**9**], which we shall use throughout.

Theorem A. *Suppose that f and g are transcendental entire functions such that $F(z) = f(g(z))$ has finite order. Then*

$$\sum_a \delta(a, F) < 1, \tag{1.1}$$

where the sum is taken over all finite values a.

We remark that the hypothesis that F has finite order is necessary in Theorem A because of the obvious example $F = e^g$, which has infinite order if g is transcendental entire. The proof of Theorem A depends heavily on results of Edrei and Fuchs [**3**], and in particular on the fact that if F is entire of finite order with maximal deficiency sum, then each deficient value of F is an asymptotic value of F. As remarked in [**5**], it is possible to replace 1 on the right-hand-side of (1.1) by a constant slightly smaller than 1, but depending on the lower order of F (see [**3**]).

In the terminology of factorization theory (see [**7, 17**]), Theorem A states that an entire function F of finite order with maximal deficiency sum is pseudoprime, that is, it

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has no factorization $F(z)=f(g(z))$, where f and g are transcendental entire. The following question appears in [**17**], and is attributed to Fuchs and Song. If F is entire of finite order with $\delta(a,F)>0$ for some finite a, must F be pseudoprime? We answer this question in the negative, by constructing a non-pseudoprime entire function of finite order with infinitely many deficient values.

Now the following was proved by Arakelyan [**1**].

Theorem B. *For any sequence (a_j) of finite complex numbers, and for any $\rho>1/2$, there exists an entire function G of order ρ, mean type, such that for each j, $\delta(a_j, G)>0$.*

No such result is possible for $\rho \leqq 1/2$ (see [**9**, Ch. 4]). We remark that Eremenko [**4**] showed further that the function G may be constructed so as to have the finite deficient values a_j, but no other finite deficient values. Using in part methods similar to those of Arakelyan, we shall prove the following.

Theorem 1. *For any sequence (a_k) $(k=0,1,2,\ldots)$ of complex numbers, and for any σ with $1/2<\sigma<+\infty$, there exist transcendental entire functions f and g such that $F(z)=f(g(z))$ has order σ, mean type, and such that for each k, $\delta(a_k, F)>0$.*

As remarked by Arakelyan in [**1**], it is only necessary to prove Theorem 1 for $1/2<\sigma \leqq 1$, for otherwise we need only consider, for a suitable value of N, the function $G(z)=F(z^N)=f(g(z^N))$, which has the same deficiencies as F. To construct a composite entire function F of order σ having $\delta(0,F)>0$, we can take δ, ρ such that $(1+\delta)\rho=\sigma$, and form f as a Weierstrass product satisfying $T(r,f)=0(\log r)^{1+\delta}$. The function f has zeros of large multiplicity at the points $\exp(\lambda^n)$, where λ is large and positive. We then construct g, using a simplified version of Arakelyan's method, so that $T(r,g)=0(r^{\rho})$ and so that $|g(z)-\exp(\lambda^n)|<1$ in a subset of an annulus $\lambda^m \leqq |z| \leqq \lambda^{m+1}$. To construct a function F having infinitely many deficient values, we apply a transformation used in [**2**] to quasiconformally modify f in successive annuli so that for each k and for infinitely many n, $f(z)-a_k$ has a zero of large multiplicity close to $\exp(\lambda^n)$. It does not seem possible, however, to prove by our method that the a_k are the only finite deficient values of F.

Our construction of g also has a bearing on the following problem. If f is a function meromorphic and transcendental in the plane, it is well-known [**9**] that the derivative f' satisfies

$$T(r,f') \leqq (2+o(1))\,T(r,f) \tag{1.2}$$

at least outside a set of r of finite measure, and that $2+o(1)$ may be replaced by $1+o(1)$ if f is entire. In the other direction it was recently proved by Hayman and Miles [**10**] that for any transcendental meromorphic function f and for any $K>1$, $T(r,f) \leqq 3eK\,T(r,f')$ for r lying in a set of positive lower logarithmic density. In particular

$$L(f)=\liminf_{r\to\infty} T(r,f)/T(r,f') \leqq 3e. \tag{1.3}$$

Now it was proved by Toppila in [16] that if f is transcendental and meromorphic of order zero, then $L(f)$ as defined by (1.3) is at most 1. On the other hand Toppila constructed in [16] and [15] meromorphic functions of arbitrary finite positive order and an entire function of order 1 such that $L(f)>1$. Our construction of g leads to the following.

Theorem 2. *For any ρ with $1/2<\rho<+\infty$, there exists an entire function g of order ρ, mean type, such that*

$$\liminf_{r\to\infty} T(r,g)/T(r,g')\geqq K(\rho)>1.$$

Again it is only necessary to prove Theorem 2 for $1/2<\rho\leqq 1$. In our construction, $K(\rho)\to 1$ as $\rho\to 1/2$, which suggests the possibility that if g is transcendental entire of order $\rho\leqq 1/2$, then $L(g)=1$.

2. Lemmas needed for the proofs of Theorems 1 and 2

The following Lemmas A and B are due to Keldysh and play the same role as in Arakelyan's construction [1].

Lemma A [14]. *There exist positive constants c_1, c_2 with the following property. Suppose that L is a rectifiable path of length S joining the points a and b, and that ε and d are positive. Then there is a polynomial P such that*

$$|P(1/(z-b))-1/(a-z)|<\varepsilon$$

for all z such that $\operatorname{dist}\{z,L\}\geqq d$, *and*

$$|P(1/(z-b))|<\exp(c_1(1+|\log \varepsilon d|)\exp(c_2 S/d))$$

for all z with $|z-b|\geqq d$.

Here $\operatorname{dist}\{z,L\}=\inf\{|z-w|: w \text{ on } L\}$.

Lemma B [13]. *Suppose that $1/2<\rho\leqq 1$ and $0<\alpha<\pi-\pi/2\rho$. Suppose further that H is analytic in $Re(z)\geqq 0$, and satisfies $\log^+|H(z)|=0(1+|z|^\rho)$ there. Then there exists an entire function g of at most order ρ, mean type, such that $|g(z)-H(z)|<1/2$ for all large z satisfying $|\arg z|\leqq\alpha$.*

The following lemma gives us some control over the rate at which the deficiencies $\delta(a_k,F)$ tend to zero as $k\to\infty$ in Theorem 1. The idea for the proof was suggested by the author's colleague, D. A. Burgess.

Lemma 1. *Let $\alpha_0,\alpha_1,\ldots,$ be positive real numbers such that $\sum_{k=0}^{\infty}\alpha_k<1$. Then there* *are pairwise disjoint sets T_k of non-negative integers such that each T_k is an arithmetic progression modulo p_k, where p_k satisfies $1 \leqq p_k \leqq 2/\alpha_k$.*

Proof. For each k we choose a positive integer i_k such that $1/\alpha_k \leqq 2^{i_k} \leqq 2/\alpha_k$, and we set $p_k = 2^{i_k}$. By a rearrangement if necessary, we may assume that the sequence (i_k) is non-decreasing. We set $y_0 = 0$, and $T_0 = \{np_0 : n = 0, 1, 2, \ldots\}$.

Suppose now that disjoint sets $T_0, \ldots, T_m$ have been chosen, such that each T_k is an arithmetic progression modulo p_k, with first term $y_k \leqq p_k$. Now if $k \leqq m$, the number of elements of the set $\{1, 2, \ldots, p_{m+1}\}$ which are congruent to y_k modulo p_k is p_{m+1}/p_k. So the number of elements of the set $\{1, \ldots, p_{m+1}\}$ which are congruent to y_k modulo p_k for some $k \leqq m$ is at most

$$p_{m+1}\left(\sum_{k=0}^{m} 1/(p_k)\right) \leqq p_{m+1}\left(\sum_{k=0}^{m} \alpha_k\right) < p_{m+1}.$$

So there is some y_{m+1} in the set $\{1, \ldots, p_{m+1}\}$ which is not congruent to y_k modulo p_k for any $k \leqq m$, and we set $T_{m+1} = \{y_{m+1} + np_{m+1} : n = 0, 1, 2, \ldots\}$. In this way the sets T_k are defined inductively. Now if $k < m$ and the sets T_k and T_m are not disjoint, then y_m is congruent to y_k modulo p_k, which is impossible.

3. Construction of the function g

Our construction of the function g in the composition $F(z) = f(g(z))$ is based on the following lemma, which in turn is based on Arakelyan's method in [1].

Lemma 2. *Let $1/2 < \rho \leqq 1$ and $1 < \lambda < +\infty$. Then there exist positive sequences (θ_k), (α_k), (n_k) and (ε_k) with the following properties.*

(i) $$\sum_{k=0}^{\infty} \alpha_k < 1 \tag{3.1}$$

(ii) $$\varepsilon_k = \exp(-1/(\delta_2 \alpha_k)) \text{ for each } k, \tag{3.2}$$

where δ_2 is a positive constant.

(iii) If *$\beta_{k,n}$ are any constants satisfying*

$$\log^+ |\beta_{k,n}| \leqq \varepsilon_k \lambda^{n\rho} \tag{3.3}$$

then there exist a positive N_0 and an entire function g of at most order ρ, mean type, such that if $n \geqq \max\{n_k, N_0\}$, we have

$$|g(z) - \beta_{k,n}| < 1 \tag{3.4}$$

for z satisfying

$$\tfrac{7}{8}\lambda^n \leqq |z| \leqq \tfrac{9}{8}\lambda^{n+1}, \quad |\arg z-(-1)^n\theta_k| \leqq 2\alpha_k. \tag{3.5}$$

We remark that in Arakelyan's construction, $\beta_{k,n}=a_k$ for all k and n, and that since the conclusion of (3.4) is weaker than that required in Arakelyan's construction, we are able to dispense with the infinite product ω of [**1**, Lemma 3].

To prove Lemma 2, we take a positive α^* such that

$$\alpha^* < \pi - \pi/2\rho \tag{3.6}$$

and we choose an infinite sequence (θ_k) satisfying

$$0 < \theta_0 \leqq \theta_k < \theta_{k+1} < \alpha^* \tag{3.7}$$

for $k \geqq 0$. We choose positive constants α_k such that $\theta_0 - 8\alpha_0 > 0$ and such that the intervals $(\theta_k - 8\alpha_k,\ \theta_k + 8\alpha_k)$ are disjoint, and note that (3.1) is satisfied. We form, for $k \geqq 0$ and $n \geqq 1$, regions

$$E_{k,n} = \{z : \tfrac{3}{4}\lambda^n \leqq |z| \leqq \tfrac{5}{4}\lambda^{n+1},\ |\arg z-(-1)^n\theta_k| \leqq 4\alpha_k\} \tag{3.8}$$

and

$$F_{k,n} = \{z : \tfrac{7}{8}\lambda^n \leqq |z| \leqq \tfrac{9}{8}\lambda^{n+1},\ |\arg z-(-1)^n\theta_k| \leqq 2\alpha_k\}. \tag{3.9}$$

We note that $F_{k,n}$ is contained in $E_{k,n}$, and that the $E_{k,n}$ are disjoint and for fixed k lie alternately above and below the real axis.

Now let $Y_{k,n}$ be the boundary of $E_{k,n}$. We join $Y_{k,n}$ to the point $-\lambda^{n+1/2}$ by an arc $L_{k,n}$ of the circle $|z| = \lambda^{n+1/2}$, taking always the shortest possible such arc. Now

$$s_{k,n} = \text{length}(Y_{k,n} \cup L_{k,n}) < c_1\lambda^n \tag{3.10}$$

where $c_1, c_2, \ldots,$ henceforth denote positive constants which are independent of k and n. Also

$$\text{dist}(F_{k,n},\ Y_{k,n} \cup L_{k,n}) > c_2\alpha_k\lambda^n \tag{3.11}$$

where $\text{dist}(A, B) = \inf\{|z-w| : z \text{ in } A,\ w \text{ in } B\}$. We set

$$d_{k,n} = \delta_1\alpha_k\lambda^n \tag{3.12}$$

where δ_1 is positive, but small compared to c_2, and set

$$D_{k,n} = \{z : \text{dist}\{z,\ Y_{k,n} \cup L_{k,n}\} < d_{k,n}\}. \tag{3.13}$$

Provided δ_1 is small enough, the $D_{k,n}$ are disjoint for fixed k, and

$$F_{k,n} \cap D_{k,n} = \phi. \tag{3.14}$$

We set

$$\varepsilon_k = \exp(-1/(\delta_2 \alpha_k)) \tag{3.15}$$

where δ_2 is small and positive. We apply Lemma A with $d = d_{k,n}$, $a = u$, where u lies on $Y_{k,n}$, and $b = -\lambda^{n+\frac{1}{2}}$. We obtain functions $Q_{k,n}(u,z)$, each of which is for fixed u a polynomial in $1/(z+\lambda^{n+\frac{1}{2}})$, such that for u on $Y_{k,n}$ and z not in $D_{k,n}$,

$$|Q_{k,n}(u,z) - 1/(u-z)| < \tfrac{16}{25}\, \lambda^{-2n-2} \exp(-\varepsilon_k \lambda^{n\rho}). \tag{3.16}$$

Here we have chosen ε in Lemma A to equal the right-hand-side of (3.16). Provided $n \geqq n^*(k)$, say, and u lies on $Y_{k,n}$, $Q_{k,n}(u,z)$ is analytic in $Re(z) \geqq -1$ and satisfies, for such z,

$$|Q_{k,n}(u,z)| < \exp(c_3 \varepsilon_k \lambda^{n\rho} \exp(c_4/\alpha_k)) < \exp(\lambda^{n\rho}) \tag{3.17}$$

provided δ_2 is small enough in (3.15). By an obvious compactness argument, we can assume that $Q_{k,n}(u,z)$ is piecewise constant in u.

Now suppose that the constants $\beta_{k,n}$ satisfy

$$\log^+ |\beta_{k,n}| \leqq \varepsilon_k \lambda^{n\rho} \tag{3.18}$$

and set

$$h(z) = \beta_{k,n} \quad \text{for } z \text{ in } E_{k,n}. \tag{3.19}$$

We choose integers $n_k \geqq n^*(k)$ so large that

$$\sum_{k=0}^{\infty} \sum_{n=n_k}^{\infty} \int_{Y_{k,n}} 1/|u|^2\, |du| < 1/2 \tag{3.20}$$

and we set

$$V_n = \bigcup_{\substack{k,m \\ n_k \leqq m \leqq n}} Y_{k,m}. \tag{3.21}$$

In addition we set

$$H_n(z)=(1/2\pi i)\int_{V_n} h(u)Q(u,z)\,du, \tag{3.22}$$

where $Q(u,z)=Q_{k,m}(u,z)$ for $Re(z)\geqq -1$, u on $Y_{k,m}$. Here all integrals are taken in the positive sense, and H_n is analytic in $Re(z)\geqq -1$.

Suppose now that $|z|\leqq \lambda^N$, and that $m>n>N+c_5$. Then

$$\int_{V_m\setminus V_n} 1/(u-z)\,du=0$$

and so

$$\begin{aligned}|H_m(z)-H_n(z)|&\leqq(1/2\pi)\left|\int_{V_m\setminus V_n} h(u)\left(Q(u,z)-\frac{1}{u-z}\right)du\right|\\ &\leqq(1/2\pi)\int_{V_m\setminus V_n} 1/|u|^2|du|\to 0 \quad \text{as } n\to\infty.\end{aligned} \tag{3.23}$$

using (3.16), (3.18), (3.19) and (3.20). Thus

$$H(z)=\lim_{n\to\infty} H_n(z)$$

is analytic in $Re(z)\geqq -1$.

Now denote by F the union of the sets $F_{k,n}$, for $k\geqq 0$ and $n\geqq n_k$. Then if z is in F, and n is large,

$$(1/2\pi i)\int_{V_n} h(u)\,1/(u-z)\,du=h(z),$$

because z lies inside precisely one $Y_{k,m}$, and further, using (3.16),

$$\begin{aligned}|H_n(z)-h(z)|&=(1/2\pi)\left|\int_{V_n} h(u)\left(Q(u,z)-\frac{1}{u-z}\right)du\right|\\ &\leqq(1/2\pi)\int_{V_n} 1/|u|^2\,|du|<1/4\pi.\end{aligned}$$

Here we have used the fact that by (3.14) F does not meet any $D_{k,n}$. Thus for all z in F,

$$|H(z)-h(z)|<1/10. \tag{3.24}$$

Now we estimate the growth of H. Assume that $\lambda^{N-1}\leqq|z|\leqq\lambda^N$, and set $M=N+c_6$. If c_6 is chosen large enough then as in the proof of (3.23), $m>M$ gives $|H_m(z)-H_M(z)|<1/4\pi$, using (3.20). Also

$$|H_M(z)| \leqq (1/2\pi) \int_{V_M} 1/|u|^2 \, |du| \max_{u \text{ in } V_M} |u|^2 |h(u)Q(u,z)|$$

$$= 0(|z|^2 \exp(c_7 \lambda^{N\rho}))$$

using (3.17) and (3.18). This gives

$$\log^+ |H(z)| = 0(1 + |z|^\rho). \tag{3.25}$$

Now using Lemma B and (3.6) and (3.7) we can choose an entire function g of at most order ρ, mean type, such that $|g(z) - H(z)| < 1/2$ for large z with $|\arg z| \leqq \alpha^*$. Now using (3.19) and (3.24) we obtain (3.4). This completes the proof of Lemma 2.

4. Proof of Theorem 2

We choose

$$\beta_{0,n} = \exp(\varepsilon_0 \lambda^{n\rho}) \tag{4.1}$$

in (3.18). Constructing g as in Section 3, Cauchy's integral formula gives $|g'(z)| < 1$ for large z satisfying

$$\lambda^n \leqq |z| \leqq \lambda^{n+1}, \; |\arg z - (-1)^n \theta_0| \leqq \alpha_0. \tag{4.2}$$

On the other hand, if n is large and z satisfies (4.2),

$$|g(z) - \beta_{0,n}| < 1. \tag{4.3}$$

For large n, and r in $[\lambda^n, \lambda^{n+1}]$, we denote by L_r the set of θ in $[0, 2\pi]$ such that $re^{i\theta}$ satisfies (4.2). Then clearly

$$(1/2\pi) \int_{L_r} \log^+ |g(re^{i\theta})| \, d\theta > c_1 r^\rho \tag{4.4}$$

for some positive constant c_1, while L_r makes no contribution to $m(r, g')$. On the other hand, if $M_r = [0, 2\pi] \backslash L_r$, then

$$(1/2\pi) \int_{M_r} \log^+ |g'(re^{i\theta})| \, d\theta \leqq (1/2\pi) \int_{M_r} \log^+ |g(re^{i\theta})| \, d\theta + m(r, g'/g) = 0(r^\rho). \tag{4.5}$$

It follows at once that g has order ρ, mean type and that

$$L(g) = \liminf_{r \to \infty} T(r,g)/T(r,g') > 1.$$

We remark that for the proof of Theorem 2 we do not in fact need the sets $F_{k,n}$ for $k \geqq 1$. However we still need to apply Lemma B in order to obtain the entire function g from the analytic function H. Since Lemma B restricts the angle in which we may approximate H by g, we see that for our examples $L(g) \to 1$ as $\rho \to 1/2$.

5. The construction of f

The entire function f required for the proof of Theorem 2 will be constructed using the following lemma.

Lemma 3. *Let $a_0, a_1, \ldots$ be any complex numbers. Let $0 < \delta \leqq 1$, and let $\lambda > 1$ be such that λ^δ is an integer and*

$$\lambda^\delta < (\lambda^\delta - 1)(1 + \delta). \tag{5.1}$$

In addition, let $T_0, T_1, \ldots$ be pairwise disjoint infinite sets of non-negative integers. Then there exists an entire function f with the following properties.

(i)
$$\log M(r, f) = 0(\log r)^{1+\delta}. \tag{5.2}$$

(ii) *There exist positive constants c and n^* and a sequence (B_n) satisfying*

$$\left|\log|B_n| - \lambda^n\right| < 1 \tag{5.3}$$

for $n \geqq n^*$, and such that if $n \geqq n^*$ and n is in T_k and $\lambda^n \geqq \log^+|a_k|$, we have

$$\log|f(z) - a_k| < -c\lambda^{n(1+\delta)} \tag{5.4}$$

for all z with $|z - B_n| < 1$.

Proof. We set

$$h(z) = \prod_{n=0}^{\infty} (1 - z/e^{\lambda^n})^{\lambda^{n\delta}}. \tag{5.5}$$

Now if $\exp(\lambda^n) \leqq r < \exp(\lambda^{n+1})$, the number of zeros of h in $|z| \leqq r$ is $1 + \lambda^\delta + \cdots + \lambda^{n\delta}$ which is less than $\lambda^\delta(\lambda^\delta - 1)^{-1} (\log r)^\delta$. Hence the counting function $n(r, 1/h)$ of the zeros of h satisfies

$$n(r, 1/h) < \lambda^\delta(\lambda^\delta - 1)^{-1} (\log r)^\delta \tag{5.6}$$

for all positive r. Thus by [9, p. 28] we see that $\log M(r, h) = 0(\log r)^2$. For a more precise estimate, it follows from a result of Hayman [8] that for r outside a set of finite logarithmic measure, $\log M(r,h) \sim T(r,h) = N(r,1/h) + 0(1)$ so that for some d with $0 < d < 1$ and for all large r we have, using (5.1) and (5.6),

$$\log M(r,h) < d(\log r)^{1+\delta}. \tag{5.7}$$

Now suppose that $|z - \exp(\lambda^n)| = (1/4)\exp(\lambda^n)$, with n large. Then (5.7) gives

$$\log |h(z)(z - \exp(\lambda^n))^{-\lambda^{n\delta}}| < d(\log(\tfrac{5}{4}\exp(\lambda^n)))^{1+\delta}$$
$$- \lambda^{n\delta}\log(\tfrac{1}{4}\exp(\lambda^n)) < -8c\lambda^{n(1+\delta)}$$

for some positive c. Consequently if

$$\log |z - \exp(\lambda^n)| < c\lambda^n \tag{5.8}$$

we have

$$\log |h(z)| < -7c\lambda^{n(1+\delta)}. \tag{5.9}$$

Now consider, for $R > 0$ and $|a| \leqq R/2$, the transformation

$$w = w_R(z) = \frac{R^2(z+a)}{(R^2 + \bar{z}a)} \tag{5.10}$$

which maps $|z| \leqq R$ one-to-one onto $|w| \leqq R$, such that $w(0) = a$ and $w_R(z) = z$ when $|z| = R$. (This transformation is used, for example, in [**2**].) Also

$$|w_{\bar{z}}/w_z| = |aw/R^2| \leqq |a|/R \tag{5.11}$$

and

$$|w(z) - a| \leqq 4|z| \tag{5.12}$$

for $|z| < R$. We modify f using the transformations (5.10). If $n \geqq n_0$ for some large n_0, we certainly have, by [**8**],

$$\tfrac{1}{2}|h(z)| > \exp(2\lambda^n) = R_n \tag{5.13}$$

on

$$|z - \exp(\lambda^n)| = \tfrac{1}{4}\exp(\lambda^n). \tag{5.14}$$

Suppose that $n \geqq n_0$, that n is in the set T_k, and that $R_n \geqq |a_k|^2$. Then in the open disc

$$D_n = \{z: |z - \exp(\lambda^n)| < \tfrac{1}{4}\exp(\lambda^n)\} \tag{5.15}$$

we set

$$H(z) = \frac{R_n^2(h(z) + a_k)}{R_n^2 + \overline{h(z)}a_k} \tag{5.16}$$

if $|h(z)| \leqq R_n$, and $H(z) = h(z)$ otherwise. If $n < n_0$ or if $R_n < |a_k|^2$ or if n lies in none of the sets T_k, or if z lies outside the union of the discs D_n, we just set $H(z) = h(z)$.

It follows that H is continuous in the plane and is analytic outside the union of the discs D_n, while in D_n H is quasiregular with complex dilatation bounded by $R_n^{-1/2} = \exp(-\lambda^n)$. Also for n large with n in T_k and $R_n \geqq |a_k|^2$,

$$\log|z - \exp(\lambda^n)| < c\lambda^n \tag{5.17}$$

implies that

$$\log|H(z) - a_k| < -6c\lambda^{n(1+\delta)}, \tag{5.18}$$

using (5.9) and (5.12).

Let $\sigma(z) = H_{\bar{z}}/H_z$ be the complex dilatation of H, which exists almost everywhere (see [**2**] or [**11**]), and is zero outside the union of the discs D_n. Now $\sigma(z) = 0(1/|z|)$ *as* $z \to \infty$, by the construction of H, so that $\int_{|z|>1} \sigma(z)/|z|^2\, dx\, dy < \infty$. Proceeding as in [**2**], the Teichmüller–Belinskii theorem [**11**, p. 227] implies that there is a solution Λ of the Beltrami equation

$$\Lambda_{\bar{z}} = \sigma(z)\Lambda_z$$

which is a quasiconformal homeomorphism of the extended plane onto itself such that $\Lambda(0) = 0$, $\Lambda(\infty) = \infty$, and

$$\Lambda(z) = z(1 + o(1)) \quad \text{as } z \to \infty. \tag{5.19}$$

Denoting by $J = \Lambda^{-1}$ the inverse function of Λ, we note that $f(z) = H(J(z))$ is almost everywhere conformal and so entire.

Now define B_n by

$$B_n = \Lambda(\exp(\lambda^n)) \tag{5.21}$$

so that (5.3) follows at once for n sufficiently large, using (5.19). We consider now the distortion properties of $J = \Lambda^{-1}$. Now $J(B_n) = \exp(\lambda^n)$, and $J(z) = z(1 + o(1))$, so for n large set

$$J_n(z) = (J(B_n + \tfrac{1}{4}|B_n|z) - e^{\lambda^n})\exp(-\lambda^n). \tag{5.22}$$

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Now each J_n maps $|z|<1$ into itself, with $J_n(0)=0$. Further, in terms of the real dilatation, J_n is K_n quasiconformal, where $K_n=(1+\kappa_n)/(1-\kappa_n)$, and $\kappa_n=0(1/|B_n|)$. Distortion theorems (see, for example, [12, p. 6]) give the following. If $|z|<4/|B_n|$ then we have $|J_n(z)|\leqq c_1(1/|B_n|)^{1/K_n}$ where c_1, c_2 henceforth denote positive absolute constants, so that

$$|e^{\lambda^n}J_n(z)|\leqq c_2|B_n|^{1-1/K_n}\leqq c_3|B_n|^{c_4|B_n|^{-1}},$$

for such z. Thus $|z-B_n|<1$ gives

$$|J(z)-e^{\lambda^n}|<\exp(c\lambda^n) \tag{5.23}$$

if n is large. Now (5.17), (5.18) and (5.23) imply that if $n\geqq n^*$, say, if n is in the set T_k and if $\lambda^n\geqq\log^+|a_k|$, then $|z-B_n|<1$ gives

$$\log|f(z)-a_k|=\log|H(J(z))-a_k|<-6c\lambda^{n(1+\delta)}$$

which proves (5.4).

To estimate the growth of f we just note that if n is large then in $2\exp(\lambda^n)\leqq|z|\leqq\frac{1}{2}\exp(\lambda^{n+1})$, we have $H(z)=h(z)$. So for $3\exp(\lambda^n)\leqq|z|\leqq\frac{1}{3}\exp(\lambda^{n+1})$, we have

$$\log|f(z)|\leqq\log M(|z|(1+o(1)),h)=0(\log|z|)^{1+\delta}$$

which proves (5.2), and completes the proof of Lemma 3.

6. Proof of Theorem 1

Given σ with $1/2<\sigma\leqq 1$, we first choose ρ and δ with $1/2<\rho<1$ and $0<\delta<1$ and $\rho(1+\delta)=\sigma$. We choose $\lambda>1$ so that λ^δ is an integer, and such that (5.1) is satisfied. Now let the sequences (θ_k), (α_k), (n_k) and (ε_k) be as in the statement of Lemma 2. Since (3.1) is satisfied, we can, by Lemma 1, find pairwise disjoint infinite sets T_k of non-negative integers such that each T_k is an arithmetic progression modulo p_k, where $p_k\leqq 2/\alpha_k$. We now apply Lemma 3, to construct an entire function f such that (5.2) holds, and such that for some sequence (B_n) satisfying (5.3) the following holds. If $n\geqq n^*$, if n is in T_k and $\lambda^n\geqq\log^+|a_k|$, then (5.4) holds for all z with $|z-B_n|<1$.

Now we define the constants $\beta_{k,n}$ as follows. Given $k\geqq 0$ and $n\geqq n^*$, let m be the largest member of T_k such that

$$\lambda^m+1\leqq\varepsilon_k\lambda^{n\rho}, \tag{6.1}$$

and set $\beta_{k,n}=B_m$. If no such m exists, or if $k\geqq 0$ and $0\leqq n<n^*$, we set $\beta_{k,n}=0$. Using Lemma 2 we construct an entire function g of at most order ρ, mean type, such that if $n\geqq\max\{n_k, N_0\}$ we have

$$|g(z)-\beta_{k,n}|<1 \tag{6.2}$$

for

$$\lambda^n \leqq |z| \leqq \lambda^{n+1}, |\arg z-(-1)^n\theta_k| \leqq 2\alpha_k. \tag{6.3}$$

We set $F(z)=f(g(z))$. It follows at once, using (5.2), that

$$T(r,F)=0(r^\sigma). \tag{6.4}$$

Now suppose that $k \geqq 0$, and that n is large compared to k. Then for z satisfying (6.3), we have $|g(z)-B_m|<1$, where m is as defined by (6.1). Now (5.4) gives, for such z, provided n is large enough,

$$\log|F(z)-a_k|<-c\lambda^{m(1+\delta)}. \tag{6.5}$$

But T_k is an arithmetic progression modulo p_k, so that if n is large enough,

$$\lambda^m \geqq (\varepsilon_k \lambda^{n\rho}-1)\lambda^{-2/\alpha_k} \geqq \tfrac{1}{2}\, \varepsilon_k \lambda^{n\rho} \lambda^{-2/\alpha_k}.$$

Denoting by $d_1, d_2, \ldots$ positive constants which do not depend on n or k, (6.5) now gives, for z satisfying (6.3),

$$-\log|F(z)-a_k|>d_1 \lambda^{n\rho(1+\delta)}(\varepsilon_k)^{d_2} \lambda^{-d_3/\alpha_k}>d_4|z|^\sigma(\varepsilon_k)^{d_2}\lambda^{-d_3/\alpha_k}.$$

Thus F has order σ, mean type, each a_k is a deficient value for F, and, using (3.2),

$$(\log 1/\delta(a_k,F))^{-1} \geqq d_5 \alpha_k. \tag{6.6}$$

Concluding Remarks. The estimate (6.6) shows that the deficiencies tend to zero at a rate comparable to that in Arakelyan's construction [**1**], this rate being essentially only determined by (3.1).

It does not seem possible to prove by the present method that the a_k are the only finite deficient values. In Eremenko's extension of Theorem B (which also uses the Teichmüller–Belinskii theorem) it is shown that the function G may be constructed with the following property. There exist arbitrarily large circles on which, with the exception of a set of arbitrarily small angular measure, G is either large or is close to one of the a_k. For our problem, if A is not one of the a_k, we would need to estimate the proximity of F to A in terms of the proximity of g to the roots of $f(w)=A$.

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