ON THE INJECTIVE HULLS OF CYCLIC MODULES OVER DEDEKIND DOMAINS

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As is well known, any module M over a ring possesses injective hulls, i.e., injective essential extensions [3], unique up to isomorphisms which map M identically, but the various proofs for this all require some transfinite arguments and hence provide little indication as to how these hulls may actually be constructed for a given module. There are, however, instances where they can be quite explicitly described: The injective hulls of the cyclic groups of prime order p (i.e., of the simple modules over the ring Z of integers) are the groups of type $Z(p^{\infty})$, which amounts to the fact that the subgroup

$Z[p^{-1}]/Zp$ of Q/Zp is an injective hull of Z/Zp, where Q is the rational number field, and $Z[p^{-1}]$ is generated over Z by p^{-1} as a subring of Q. The exact counterpart of this holds for any cyclic module over any principal ideal domain [1], and thus one has, in these cases, an explicit description of injective hulls in terms of familiar ring and module theoretic constructions. It is the purpose of this note to derive the corresponding analogue for arbitrary Dedekind domains from a result about more general integral domains, and to show by means of a counter-example that this description of injective hulls does not extend to Noetherian domains in general.

To begin with, let R be an arbitrary integral domain and $K \supseteq R$ a field of quotients of R. For any non-zero proper ideal J (the only case we are interested in here) let $J^{-k} = (R:J^k) = \{x \mid x \in K,\ xJ^k \subseteq R\}$ and $J^* = \Sigma J^{-k} = \cup J^{-k}$.
If J is an <u>invertible</u> ideal one has $J^{-1}J = R$ and J^{-1} is the inverse of J in the multiplicative monoid of all (non-zero) fractionary ideals of R [4]. In particular, if J is a principal ideal Rc then $J^* = R[c^{-1}]$, i.e. is obtained by

forming the subring of K generated by R and c^{-1}, since $(Rc)^{-k} = Rc^{-k}$.

PROPOSITION 1. If R is a Noetherian domain in which each proper prime ideal is maximal and J an invertible ideal of R then J^*/J is a divisible essential extension of R/J.

Proof. We first show that J^*/J is an essential extension of R/J, i.e., that for any $t+J \in J^*/J$ not in R/J there exists an $a \in R$ such that $at + J \in R/J$ and $at + J \neq J$. For any such $t+J$, let k be the first i such that $t \in J^{-i}$. Since $t \notin R$, one has $k \geq 1$ and, by the definition of k, $t \notin J^{-(k-1)}$; this means that $tJ^{k-1} \nsubseteq R$ or, equivalently, $tJ^k \nsubseteq J$ by the fact that J is invertible. Thus there exists an $a \in J^k$ for which $at \notin J$, and this has the desired property.

As to the divisibility of J^*/J, let a be any non-zero element of R and consider the factorization $Ra = BC$, obtained by suitably collecting the factors in the primary product decomposition of Ra, such that the prime ideals associated with B occur among those associated with J and those associated with C do not. Then, some power of J is contained in B since J is contained in the radical of B (note that $B = R$ is not excluded here) and hence $J^n C \subseteq Ra$ for some n; also, $C + J = R$ since C and J have no associated prime ideals in common, and hence there exists, for any given k, an element $x \in C$ such that $x - 1 \in J^{k+1}$. Now for any $t \in J^{-k}$ and this $x \in C$ one has $xt + J = t + (x-1)t + J = t + J$, and from $xt \in CJ^{-k} = CJ^n J^{-(n+k)} \subseteq aJ^{-(n+k)}$ it follows that $xt = as$ for some $s \in J^{-(n+k)}$ and hence $a(s+J) = xt + J = t + J$. This shows that J^*/J is divisible.

Remark 1. This proposition remains true for arbitrary Noetherian domains if one considers only such invertible ideals J for which each associated prime ideal is incomparable (in terms of inclusion) with every other proper prime ideal. In order to see this it is sufficient to find, for any non-zero $a \in R$, an ideal C such that $J^n C \subseteq Ra$ for some n and $C+J = R$, for then the above argument can be carried out. This C may be obtained by collecting, in any given primary intersection decomposition of Ra, those terms whose associated prime ideals are not associated with J. One then has $Ra = B \cap C$, B the intersection of the remaining terms, and $J^n \subseteq B$ for some n, and hence $J^n C \subseteq Ra$. $C + J = R$ holds because $C + J \subset R$ would imply $C \subseteq P$ for some prime ideal P associated with J, thus also $P' \subseteq P$ for some prime ideal P' associated with C, and therefore $P' = P$ or $P' = O$ --- a contradiction either way.

Remark 2. As the above proof shows, the invertibility of J suffices for J^*/J to be an essential extension of R/J - no matter what type of domain R is. If J is not invertible, however, this may fail to be the case, even under the same hypotheses for R as above. Take, for instance, R to be local as well and such that its maximal ideal P is not principal. Then, P is not invertible, hence $P^{-1}P \subseteq P$, and for any $x \in P^{-1}$ not in R one has $R(x+P) \cap R/P = O$ whereas $P^{-1} \supset R$.

COROLLARY. If R is a Dedekind domain then J^*/J is an injective hull of R/J for any non-zero proper ideal J of R.

Proof. Here, all proper prime ideals are maximal, all non-zero ideals invertible, and injectivity is equivalent to divisibility [4, 2].

Remark 3. An alternative proof would be via the primary decompositions of R/J and J^*/J, using the fact that injective hulls distribute over finite direct sums, and that P^*/P, for prime P, is the P-primary component of the injective torsion module K/P. However, the argument given here has the advantage that it avoids this detour.

Remark 4. We do not know whether J^*/J is still injective under the more general hypotheses of Proposition 1; similarly, we have not been able to settle whether J^*/J can contain strictly smaller divisible extensions of R/J.

We now turn to the part of the discussion which is going to provide examples of rings for which the above result no longer holds.

Let R be an arbitrary commutative ring and J any ideal of R. Associated with J one has the set $F(J) = \{a \mid (J:a) = J\}$ of all elements $a \in R$ prime to J and its subset $F_o(J) = \{a \mid Ra + J = R\}$. In general, $F_o(J)$ may well

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be properly contained in $F(J)$.

PROPOSITION 2. If J is an invertible ideal of an integral domain R such that J^*/J is divisible then $F_o(J) = F(J)$.

Proof. Suppose, by way of contradiction, there exists an $a \in F(J)$ not belonging to $F_o(J)$, i.e., such that $Ra + J \subset R$. By hypothesis it follows that $ax + J = 1 + J$ with a suitable $x \in J^*$; if k is such that $x \in J^{-k}$ but $x \notin J^{-(k-1)}$ then $k > 0$ for otherwise $1 \in Ra + J$. Now, since J is invertible this implies that $xJ^k \not\subseteq J$, i.e., there exists a $t \in J^k$ such that $xt \notin J$. Then $axt - t \in J^{k+1}$ and therefore $axt \in J$. Since $a \in F(J)$ this implies $xt \in J$, a contradiction.

For a prime ideal P one has $F(P) = CP$, the complement of P, and for any ideal J, $F_o(J) = \bigcap CM$, extended over the maximal ideals $M \supseteq J$. Thus, $F_o(P) = F(P)$ holds if and only if P is maximal. Since a polynomial ring in more than one indeterminate over a field has proper prime ideals which are principal, and hence invertible, but not maximal, one has the following:

COROLLARY. There exist Noetherian domains R which contain non-zero proper ideals J such that J^*/J is an essential extension but not an injective hull of R/J.

REFERENCES

1. B. Banaschewski, On Coverings of Modules. Math. Nachr. 31 (1966), 57-71.

2. H. Cartan and S. Eilenberg, Homological Algebra. Princeton, (1956).

3. B. Eckmann and A. Schopf, Über injektive Moduln. Archiv der Math. 4, (1953), 75-78.

4. P. Samuel and O. Zariski, Commutative Algebra. Princeton, (1958).

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