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THE STRUCTURE OF SEMISIMPLE SYMMETRIC SPACES

W. ROSSMANN

1. Introduction. A semisimple symmetric space can be defined as a homogeneous space G/H, where G is a semisimple Lie group, H an open subgroup of the fixed point group of an involutive automorphism of G. These spaces can also be characterized as the affine symmetric spaces or pseudo-Riemannian symmetric spaces or symmetric spaces in the sense of Loos [**4**] with semisimple automorphism groups [**3**, **4**]. The connected semisimple symmetric spaces are all known: they have been classified by Berger [**2**] on the basis of Cartan's classification of the Riemannian symmetric spaces. However, the list of these spaces is much too long to make a detailed case by case study feasible. In order to do analysis on semisimple symmetric spaces, for example, one needs a general structure theory, just as in the case of Riemannian symmetric spaces and semisimple Lie groups. The results of this paper should go some way to serve this purpose. For orientation and motivation I mention some special cases which have already been studied in detail:

(a) Riemannian spaces. A connected semisimple symmetric space G/H is Riemannian if and only if H is compact (presupposing that G acts effectively on G/H). The basic structural facts concerning these spaces are contained in three decomposition theorems: root space decomposition, Cartan decomposition, Iwasawa decomposition. The results of this paper show that all three of them can be generalized to arbitrary semisimple symmetric spaces.

(b) Lie groups. A Lie group G_0 is in one-to-one correspondence with a homogeneous space G/H, where $G = G_0 \times G_0$ and H is the diagonal in G (via $G/H \to G_0$, $(g_1, g_2) \to g_1g_2^{-1}$). In this way a semisimple Lie group can be regarded as a semisimple symmetric space (the involution of G being given by $(g_1, g_2) \to (g_2, g_1)$). A large part of the structure theory of semisimple Lie groups can actually be interpreted solely in terms of their symmetric space structure. A case in point is the Bruhat decomposition: it will become clear that the Bruhat decomposition plays exactly the same role for Lie groups (regarded as symmetric spaces) which the Iwasawa decomposition plays for Riemannian spaces.

(c) Real hyperbolic spaces: $0(p, q)/0(p, q - 1)$. These homogeneous spaces are semisimple symmetric spaces and can be realized as hypersurfaces $\{x \in \mathbf{R}^{p+q}: -x_1^2 - \ldots - x_p^2 + \ldots + x_{p+q}^2 = 1\}$. They are Riemannian only if $p = 0$ (spheres) or $q = 1$ (non-Euclidean geometries). For hyperbolic spaces the general structure theory has an immediate geometric meaning and

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can be used to give an intrinsic interpretation (independent of the embedding in $\mathbf{R}^{p+q}$) of the analysis on these spaces developed in [**6**].

There are three major results in this paper. The first one (Theorem 5) shows how a semisimple symmetric space (or more precisely, a semisimple Lie algebra with an involution) gives rise to a root system and identifies its Weyl group. This root system by itself does not determine the space uniquely, but does contain some important information. For example, the root system is trivial if and only if the space is compact. Both in the Riemannian case and in the Lie group case this root system is the usual system of restricted roots of a semisimple real Lie algebra.

The second result (Theorem 10) gives the decomposition of a semisimple symmetric space G/H into orbits of a maximal compact subgroup K of G. Both in the Riemannian case and in the Lie group case this decomposition reduces essentially to the Cartan decomposition $K\backslash G/K \cong W\backslash A$; in the case of the real hyperbolic spaces it reduces to the natural spherical coordinate decomposition of the hypersurface

$$\{x \in \mathbf{R}^{p+q}: -x_1^2 - \ldots - x_p^2 + \ldots + x_{p+q}^2 = 1\}$$

as the product space $S^{p-1} \times S^{q-1} \times \mathbf{R}_{\text{pos}}$.

The third result (Theorem 12) describes the decomposition of G/H into a finite number of orbits of a parabolic subgroup P of G and identifies the open orbits of P. It turns out that the orbits of P in G/H are in one-to-one correspondence with a certain finite symmetric space $W_{G/H}$ ("symmetric" in the sense of Loos [**4**]), which plays the role of the Weyl group in a Bruhat decomposition of a semisimple Lie group. In fact, if $G_BH = G_0$ "is" a semisimple Lie group (example (b) above), then the decomposition of G/H under P is essentially a Bruhat decomposition of G_0, and $W_{G/H}$ is essentially the Weyl group of G_0. In the Riemannian case the decomposition of G/H under P reduces to an Iwasawa decomposition of G; for the real hyperbolic spaces it reduces to the decomposition of the hypersurface

$$\{x \in \mathbf{R}^{p+q}: -x_1^2 - \ldots - x_p^2 + \ldots + x_{p+q}^2 = 1\}$$

obtained by intersecting it with a hyperplane tangent to its asymptotic cone $\{x \in \mathbf{R}^{p+q}: -x_1^2 - \ldots - x_p^2 + \ldots + x_{p+q}^2 = 0\}$. Another special case ($G$ complex, H a real form of G) has been treated by Wolf in [**10**]. The fact that $P\backslash G/H$ is always finite is also proved in [**11**].

The way the structure theory developed here can be applied to the analysis on semisimple symmetric spaces can be seen from the special cases (a), (b), (c) mentioned above: the fundamental role of the decomposition theorems of Cartan, Iwasawa, and Bruhat in the analysis on Riemannian symmetric spaces and on semisimple Lie groups is well known; the use of the analogous decompositions of the real hyperbolic spaces is illustrated in [**6**]. I hope to give further applications elsewhere.

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2. Lie Algebras with involutions. Let $\mathfrak{G}$ be a semisimple real Lie algebra, σ an involutive automorphism of $\mathfrak{G}$. σ is called a *Cartan involution* of $\mathfrak{G}$ if the bilinear form B_σ defined by $B_\sigma(x, y) = B(x, \sigma(y))$ (B the Killing form of $\mathfrak{G}$) is negative definite; or equivalently if the fixed point set of σ is a maximal compactly embedded subalgebra of $\mathfrak{G}$. In any case, there always is a Cartan involution which commutes with σ, and such a Cartan involution is in fact unique up to conjugation by an inner automorphism of the form exp (ad (x)), where x is an element of $\mathfrak{G}$ fixed by σ [**4**, vol. I, p. 153].

An abelian subalgebra $\mathfrak{A}$ of $\mathfrak{G}$ is said to (be) *split* (in $\mathfrak{G}$) if $\mathfrak{G}$ decomposes as the sum of the eigenspaces ("root spaces") of ad $(\mathfrak{A})$. One knows that an abelian subalgebra $\mathfrak{A}$ of $\mathfrak{G}$ splits if and only if there is a Cartan involution of $\mathfrak{G}$ which acts on $\mathfrak{A}$ by multiplication by -1. The maximal split abelian subalgebras of $\mathfrak{G}$ are therefore precisely the maximal abelian algebras in the (-1)-eigenspaces of Cartan involutions. It is well known that any two maximal split abelian subalgebras of $\mathfrak{G}$ are conjugate under $G = \operatorname{Int}(\mathfrak{G})$. Also, if σ is Cartan, then the σ-stable maximal abelian subalgebras of $\mathfrak{G}$ coincide with the maximal abelian algebras in the (-1)-eigenspace of σ and are all conjugate under the connected subgroup H of G with Lie algebra $\mathfrak{H} = \{x \in \mathfrak{G}: \sigma(x) = x\}$. This need not be the case for an arbitrary involution. For an overview of the general situation fix a Cartan involution τ which commutes with σ and denote by K the (maximal compact, connected) subgroup of G with Lie algebra $\mathfrak{K} = \{x \in \mathfrak{G}: \tau(x) = x\}$. For any subspace $\mathfrak{S}$ of $\mathfrak{G}$ set

$$\mathfrak{S}^{\pm\sigma} = \{x \in \mathfrak{G}: \sigma(x) = \pm x\}$$

and for $u, v = \pm 1$ set $\mathfrak{S}^{u\sigma, v\tau} = \{x \in \mathfrak{S}: \sigma(x) = ux, \tau(x) = vx\}$. Thus $\mathfrak{H} = \mathfrak{G}^\sigma$, $\mathfrak{K} = \mathfrak{G}^\tau$, and $\mathfrak{G} = \sum_{u,v=\pm 1} \mathfrak{G}^{u\sigma, v\tau}$.

THEOREM 1. (1) *Every σ-stable maximal split abelian subalgebra of $\mathfrak{G}$ is H-conjugate to a σ, τ-stable one.*

(2) *For two σ, τ-stable maximal split abelian subalgebras $\mathfrak{A}_1$, $\mathfrak{A}_2$ of $\mathfrak{G}$ the following conditions are equivalent:*

(a) *$\mathfrak{A}_1$ and $\mathfrak{A}_2$ are H-conjugate*

(b) *$\mathfrak{A}_1$ and $\mathfrak{A}_2$ are $H \cap K$-conjugate.*

(c) *$\mathfrak{A}_1^\sigma$ and $\mathfrak{A}_2^\sigma$ are $H \cap K$-conjugate.*

(d) *$\mathfrak{A}_1^{-\sigma}$ and $\mathfrak{A}_2^{-\sigma}$ are $H \cap K$-conjugate.*

Proof. (1) Let $\mathfrak{A}$ be a σ, τ-stable maximal split abelian subalgebra of $\mathfrak{G}$ for which $\mathfrak{A}^{-\sigma}$ is maximal split abelian in $\mathfrak{G}^{-\sigma}$. (Such an $\mathfrak{A}$ always exists: choose any maximal abelian algebra $\mathfrak{A}^{-\sigma}$ in $\mathfrak{G}^{-\sigma,-\tau}$ and extend it to a maximal abelian algebra in $\mathfrak{G}^{-\tau}$). Set $A = \exp(\mathfrak{A})$, $A^- = \exp(\mathfrak{A}^{-\sigma})$ ($\exp = \exp_G$). We shall see later that $G = K \cdot A^- \cdot H$ (Theorem 10). Because of the G-conjugacy of the maximal split abelian sub-algebras of $\mathfrak{G}$, any σ-stable one is of the form $g \cdot \mathfrak{A}$, where $g \in G$ satisfies $\sigma(g) = \sigma(g) \cdot \mathfrak{A} = \mathfrak{A}$, i.e. $g^{-1}\sigma(g) \in N_G(A)$. (Here $\sigma(g) = \sigma \cdot g \cdot \sigma$ and $N_G(A)$ is the normalizer of A in G.) So if we write $g = hak$ with $h \in H$, $a \in A^-$, $k \in K$, then $g^{-1}\sigma(g) = k^{-1}a^{-2}\sigma(k)$ lies in $N_G(A)$. Using the

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well known fact that $K \times \mathfrak{G}^{-\tau} \to G$, $(k, x) \to k \exp(x)$ is a bijection and that $N_G(A) = N_K(A) \cdot A$ we see that $k^{-1}a^{-2}\sigma(k) \in N_G(A)$ implies $k^{-1}\sigma(k) \in N_k(A)$ and $k^{-1}ak \in A$ (because $k^{-1}a^{-2}\sigma(k) = k^{-1}\sigma(k) \cdot (k^{-1}a^{-2}k)$, $k^{-1}\sigma(k) \in K$, $\sigma(k^{-1}a^{-2}k) \in \exp(\mathfrak{G}^{-\tau})$). So $g \cdot \mathfrak{A} = hk(k^{-1}ak) \cdot \mathfrak{A} = hk \cdot \mathfrak{A}$ is H-conjugate to $k \cdot \mathfrak{A}$, which is τ-stable.

(2) The proof of the equivalence of (a)-(d) depends on the following:

LEMMA 2. *Let $\mathfrak{S}$ be a subset of $\mathfrak{G}^{-\tau}$. If $g \cdot \mathfrak{S}$ lies also in $\mathfrak{G}^{-\tau}$ for some $g \in G$, then there is a $k \in K$ satisfying $g \cdot x = k \cdot x$ for all $x \in \mathfrak{S}$.*

In particular, any two subspaces of $\mathfrak{G}^{-\tau}$ which are conjugate under G are also conjugate under K.

Proof. By assumption, $\mathfrak{S}$ and $g \cdot \mathfrak{S}$ lie both in $\mathfrak{G}^{-\tau}$. So if $x \in \mathfrak{S}$, then $\tau(g \cdot x) = -g \cdot x$ and also $\tau(g \cdot x) = \tau(g) \cdot \tau(x) = -\tau(g) \cdot x$, hence $g^{-1}\tau(g) \cdot x = x$, i.e. $g^{-1}\tau(g) \in C_G(\mathfrak{S})$ (the centralizer of $\mathfrak{S}$ in G). Since $C_G(\mathfrak{S})$ is a τ-stable subgroup of G, the decomposition $G = K \cdot \exp(\mathfrak{G}^{-\tau})$ of G induces the decomposition $C_G(\mathfrak{S}) = C_K(\mathfrak{S}) \cdot \exp(\mathfrak{C}_{\mathfrak{G}}(\mathfrak{S})^{-\tau})$ of $C_G(\mathfrak{S})$. So we can write $g = kp$ with $k \in K$ and $p \in \exp(\mathfrak{G}^{-\tau})$; then $g^{-1}\sigma(g) = p^{-2} \in C_G(\mathfrak{S}) \cap \exp(\mathfrak{G}^{-\tau}) = \exp(\mathfrak{C}_{\mathfrak{G}}(\mathfrak{S})^{-\tau})$, which gives $p \in \exp(\mathfrak{C}_{\mathfrak{G}}(\mathfrak{S})^{-\tau})$ (because exp is one-to-one on $\mathfrak{G}^{-\tau}$). Consequently, $g = kp \in kC_G(\mathfrak{S})$.

Remark. The lemma obviously remains true if $\mathfrak{G}$ is only reductive, provided one understands by a Cartan involution of a reductive Lie algebra $\mathfrak{G}$ an involution whose restriction to $[\mathfrak{G}, \mathfrak{G}]$ is Cartan. This observation will be used below.

Returning now to the proof of Theorem 2 we first prove:

(a) $\Rightarrow$ (c). Suppose $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are H-conjugate, say $\mathfrak{A}_1 = h \cdot \mathfrak{A}_2$ with $h \in H$. Since H commutes with σ, $\mathfrak{A}_1^\sigma = h \cdot \mathfrak{A}_2^\sigma$ (and also $\mathfrak{A}_1^{-\sigma} = \mathfrak{A}_2^{-\sigma}$). Thus $\mathfrak{A}_1^\sigma$ and $\mathfrak{A}_2^\sigma$ are two H-conjugate subspaces of $\mathfrak{H}^{-\tau}$. Being τ-stable, $\mathfrak{H}$ is reductive (in $\mathfrak{G}$), and $\tau|\mathfrak{H}$ is a Cartan involution of $\mathfrak{H}$. By the lemma, $\mathfrak{A}_1^\sigma$ and $\mathfrak{A}_2^\sigma$ are also $H \cap K$-conjugate.

(c) $\Rightarrow$ (d), (b). Suppose $\mathfrak{A}_1^\sigma$ and $\mathfrak{A}_2^\sigma$ are $H \cap K$-conjugate. We may as well assume $\mathfrak{A}_1^\sigma = \mathfrak{A}_2^\sigma$. $\mathfrak{A}_1^{-\sigma}$ and $\mathfrak{A}_2^{-\sigma}$ are then both maximal abelian in $\mathfrak{C}_{\mathfrak{G}}(\mathfrak{A}_1^\sigma)^{\sigma\tau,-\tau}$. Since $\mathfrak{C}_{\mathfrak{G}}(\mathfrak{A}_1^\sigma)^{\sigma\tau}$ (the fixed point set of the composite involution $\sigma\tau$ in the centralizer $\mathfrak{C}_{\mathfrak{G}}(\mathfrak{A}_1^\sigma)$ of $\mathfrak{A}_1^\sigma$ in $\mathfrak{G}$) is a τ-stable (hence reductive) subalgebra of $\mathfrak{G}$, $\mathfrak{A}_1^{-\sigma}$ and $\mathfrak{A}_2^{-\sigma}$ are conjugate by an element in $H \cap K$ which leaves $\mathfrak{A}_1^\sigma$ pointwise fixed. This element therefore also maps $\mathfrak{A}_1 = \mathfrak{A}_1^\sigma + \mathfrak{A}_1^{-\sigma}$ onto $\mathfrak{A}_2 = \mathfrak{A}_1^\sigma + \mathfrak{A}_2^{-\sigma}$.

(d) $\Rightarrow$ (c)(b). This is shown by an obvious modification of the proof of (c) $\Rightarrow$ (d)(b). (Interchange the roles of $\mathfrak{A}_i^\sigma$ and $\mathfrak{A}_i^{-\sigma}$ and replace $\mathfrak{C}_{\mathfrak{G}}(\mathfrak{A}_1^\sigma)^{\sigma\tau}$ by $\mathfrak{C}_{\mathfrak{G}}(\mathfrak{A}_2^{-\sigma})^\sigma$.)

(b) $\Rightarrow$ (a) trivially.

COROLLARY 3. *Any two σ-stable maximal split abelian subalgebras $\mathfrak{A}_1$, $\mathfrak{A}_2$ of $\mathfrak{G}$ with the property that $\mathfrak{A}_1^\sigma$ and $\mathfrak{A}_2^\sigma$ (resp. $\mathfrak{A}_1^{-\sigma}$ and $\mathfrak{A}_2^{-\sigma}$) are both maximal split abelian in $\mathfrak{G}^\sigma$ (resp. in $\mathfrak{G}^{-\sigma}$) are conjugate under H.*

Proof. In view of part (1) of the theorem, we may assume that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are also τ-stable. The hypothesis then implies that $\mathfrak{A}_1^\sigma$ and $\mathfrak{A}_2^\sigma$ (resp. $\mathfrak{A}_1^{-\sigma}$ and $\mathfrak{A}_2^{-\sigma}$) are both maximal abelian in $\mathfrak{G}^{\sigma,-\tau}$ (resp. in $\mathfrak{G}^{\sigma\tau,-\tau} = \mathfrak{G}^{-\sigma,-\tau}$). Since $\mathfrak{G}^\sigma$ (resp. $\mathfrak{G}^{\sigma\tau}$) is τ-stable (hence reductive), $\mathfrak{A}_1^\sigma$ and $\mathfrak{A}_2^\sigma$ (resp. $\mathfrak{A}_1^{-\sigma}$ and $\mathfrak{A}_2^{-\sigma}$) are $H \cap K$-conjugate. Part (2) of the theorem shows that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are then also $H \cap K$-conjugate.

COROLLARY 4. *There is only a finite number of H-conjugacy classes of σ-stable maximal split abelian subalgebras of $\mathfrak{G}$.*

Proof. Fix a σ-stable maximal abelian algebra $\mathfrak{A}_0$ in $\mathfrak{G}^{-\tau}$ with the property that $\mathfrak{A}_0^{-\sigma}$ is maximal abelian in $\mathfrak{G}^{-\sigma,-\tau}$. Any abelian algebra in $\mathfrak{G}^{-\sigma,-\tau}$ is then $H \cap K$-conjugate to a subalgebra of $\mathfrak{A}_0^{-\sigma}$. In view of the theorem it therefore suffices to show that there are only a finite number of subspaces of $\mathfrak{A}_0^{-\sigma}$ of the form $\mathfrak{A}^{-\sigma}$ for some σ-stable maximal abelian $\mathfrak{A}$ in $\mathfrak{G}^{-\tau}$.

To see this, let $\mathfrak{G} = \sum \{\mathfrak{G}_\alpha : \alpha \in \mathfrak{A}_0^*\}$ be the root space decomposition of $\mathfrak{G}$ under $\mathfrak{A}_0$. If $\mathfrak{A}$ is σ-stable, maximal abelian in $\mathfrak{G}^{-\tau}$ with $\mathfrak{A}^{-\sigma} \subset \mathfrak{A}_0^{-\sigma}$, then

$$\mathfrak{A}^{-\sigma} = \bigcap \{\ker (\alpha|\mathfrak{A}_0^{-\sigma}) : \alpha|\mathfrak{A} \cap \mathfrak{A}_0 = 0, \mathfrak{G}_\alpha \neq 0\}.$$

(In fact, if $x \in \mathfrak{A}_0^{-\sigma}$ and $\alpha(x) = 0$ whenever $\alpha|\mathfrak{A} \cap \mathfrak{A}_0 = 0$ and $\mathfrak{G}_\alpha \neq 0$, then $[x, y] = 0$ whenever $y \in \sum \{\mathfrak{G}_\alpha : \alpha|\mathfrak{A} \cap \mathfrak{A}_0 = 0\} = \mathfrak{C}_\mathfrak{G}(\mathfrak{A} \cap \mathfrak{A}_0)$. In particular $[x, \mathfrak{A}] = 0$, so $x \in \mathfrak{A}$ by the maximality property of $\mathfrak{A}$. Thus $x \in \mathfrak{A} \cap \mathfrak{A}_0^{-\sigma} = \mathfrak{A}^{-\sigma}$). It follows that $\mathfrak{A}^{-\sigma}$ is uniquely determined by a certain set of roots of $\mathfrak{A}_0$ in $\mathfrak{G}$, which shows that there are only finitely many such subspaces $\mathfrak{A}^{-\sigma}$ of $\mathfrak{A}_0^{-\sigma}$.

Remark. Wolf shows in [**11**] that the corollary remains true if "maximal split abelian" is replaced by "Cartan". His proof uses algebraic geometry and a reduction to special cases. The proof given here is more related to M. Sugiura's method [**8**]. One will note that the $\mathfrak{A}_0$ in this proof could equally well have been chosen so that $\mathfrak{A}_0^\sigma$ is maximal abelian in $\mathfrak{G}^{\sigma,-\tau}$.

Define a *maximal split abelian algebra in* $\mathfrak{G}^{-\sigma}$ to be a split abelian subalgebra of $\mathfrak{G}$ which lies in $\mathfrak{G}^{-\sigma}$ and is maximal with this property. Of course, such an algebra need not be maximal split abelian in $\mathfrak{G}$, but can always be extended to a σ-stable maximal split abelian subalgebra of $\mathfrak{G}$. It follows from Corollary 3 that any two maximal split abelian algebras in $\mathfrak{G}^{-\sigma}$ are conjugate under H. Moreover, as a consequence of Theorem 1, any maximal split abelian algebra in $\mathfrak{G}^{-\sigma}$ is H-conjugate to a τ-stable one (i.e. to a maximal abelian algebra in $\mathfrak{G}^{-\sigma,-\tau}$) and any two τ-stable ones are conjugate under $H \cap K$. For the remainder of this section $\mathfrak{A}$ denotes a fixed maximal abelian algebra in $\mathfrak{G}^{-\sigma,-\tau}$.

Since $\mathfrak{A}$ splits in $\mathfrak{G}$, $\mathfrak{G}$ decomposes into root spaces of $\mathfrak{A}$: $\mathfrak{G} = \sum \{\mathfrak{G}_\alpha : \alpha \in \mathfrak{A}^*\}$, where

$$\mathfrak{G}_\alpha = \{x \in \mathfrak{G} : \text{ad } (y)x = \alpha(y)x \quad \text{for all } y \in \mathfrak{A}\},$$

and in particular $\mathfrak{G}_0 = \mathfrak{C}_{\mathfrak{G}}(\mathfrak{A})$, the centralizer of $\mathfrak{A}$ in $\mathfrak{G}$. Set $R = \{\alpha \in \mathfrak{A}^*, \alpha \neq 0 \text{ and } \mathfrak{G}_\alpha \neq 0\}$. We shall show:

THEOREM 5. *R is a root system in $\mathfrak{A}^*$ with Weyl group $W \cong N_G(\mathfrak{A})/C_G(\mathfrak{A}) \cong N_K(\mathfrak{A})/C_K(\mathfrak{A})$.*

Before going into the proof of this theorem we need some preliminary observations. First of all, following a familiar pattern one proves:

LEMMA 6. (a) *R spans $\mathfrak{A}^*$.*
(b) $[\mathfrak{G}_\alpha, \mathfrak{G}_\beta] \subset \mathfrak{G}_{\alpha+\beta}$.
(c) $\sigma(\mathfrak{G}_\alpha) = \mathfrak{G}_{-\alpha} = \tau(\mathfrak{G}_\alpha)$.
(d) $B(\mathfrak{G}_\alpha, \mathfrak{G}_\beta) = 0$ *unless* $\alpha = -\beta$.
(e) *If $x \in \mathfrak{A}$ satisfies $\alpha(x) \neq 0$ for all $\alpha \in R$, then $\mathfrak{C}_{\mathfrak{G}}(\mathfrak{A}) = \mathfrak{C}_{\mathfrak{G}}(x)$.*

One sees from (c) that the subspaces $\mathfrak{G}_\alpha + \mathfrak{G}_{-\alpha}$ of $\mathfrak{G}$ are stable under σ and τ, while the root space $\mathfrak{G}_\alpha$ themselves are stable under the composite involution $\sigma\tau$. These subspaces of $\mathfrak{G}$ are related by

LEMMA 7. *For any $\alpha \in R$ and any $u, v = \pm 1$ there are linear isomorphisms:*
(a) *$\mathfrak{G}_\alpha \to (\mathfrak{G}_\alpha + \mathfrak{G}_{-\alpha})^{u\sigma}$ and $\mathfrak{G}_\alpha \to (\mathfrak{G}_\alpha + \mathfrak{G}_{-\alpha})^{u\tau}$ given by $x \to (x + u\sigma(x))$ and $x \to (x + u\tau(x))$, respectively.*
(b) *$\mathfrak{G}_\alpha^{uv\sigma\tau} \to (\mathfrak{G}_\alpha + \mathfrak{G}_{-\alpha})^{u\sigma, v\tau}$ given by $x \to (x + u\sigma(x)) = (x + v\tau(x))$.*

Proof. (a) From Lemma 1 (c) it is clear that $x \to (x + u\sigma(x))$ maps $\mathfrak{G}_\alpha$ into $(\mathfrak{G}_\alpha + \mathfrak{G}_{-\alpha})^{u\sigma}$. If $x + u\sigma(x) = 0$, then $x \in \mathfrak{G}_\alpha \cap \mathfrak{G}_{-\alpha}$, which is zero for $\alpha \in R$; so this map is injective. Also, if $x \in (\mathfrak{G}_\alpha + \mathfrak{G}_{-\alpha})^{u\sigma}$, say $x = x_+ + x_-$ with $x_\pm \in \mathfrak{G}_{\pm\alpha}$, then $u\sigma(x) = x$ implies that $u\sigma(x_+) = x_-$, again because $\mathfrak{G}_\alpha \cap \mathfrak{G}_{-\alpha} = 0$. Thus $x = x_+ + u\sigma(x_+)$ is in the image of the map $\mathfrak{G}_\alpha \to (\mathfrak{G}_\alpha + \mathfrak{G}_{-\alpha})^{u\sigma}$, so that this map is also surjective.

(b) follows from (a) since $\mathfrak{G}_\alpha = \mathfrak{G}_\alpha^{\sigma\tau} + \mathfrak{G}_\alpha^{-\sigma\tau}$.

The next lemma is crucial for the proof of Theorem 5 and is often a useful tool for computations in the Lie algebra $\mathfrak{G}$.

LEMMA 8. (a) *For any $x \in \mathfrak{G}_\alpha^{u\sigma\tau}$ ($\alpha \in R, u = \pm 1$)*

$$[x, \sigma(x)] = B(x, \sigma(x))z_\alpha$$
$$[x, \tau(x)] = B(x, \tau(x))z_\alpha$$

where z_α is the element in $\mathfrak{A}$ satisfying

$$B(z_\alpha, z) = \alpha(z) \text{ for all } z \in \mathfrak{A}.$$

(b) *For any $x \in (\mathfrak{G}_\alpha + \mathfrak{G}_{-\alpha})^{u\sigma, v\tau}$ ($u, v = \pm 1, \alpha \in R$) there is a unique $y \in (\mathfrak{G}_\alpha + \mathfrak{G}_{-\alpha})^{-\sigma u, -v\tau}$ so that for all $z \in \mathfrak{A}$*

$$\operatorname{ad}(z)x = \alpha(z)y$$
$$\operatorname{ad}(z)y = \alpha(z)x.$$

Such elements x, y satisfy

$$[x, y] = -B(x, x)z_\alpha = B(y, y)z_\alpha.$$

Proof. (a) If $x \in \mathfrak{G}_\alpha^{u\sigma\tau}$, then $[x, \sigma(x)] \in [\mathfrak{G}_\alpha, \mathfrak{G}_{-\alpha}] \subset \mathfrak{C}_\mathfrak{G}(\mathfrak{A})$, $\sigma[x, \sigma(x)] = [\sigma(x), x] = -[x, \sigma(x)]$, and $\tau[x, \sigma(x)] = [\tau(x), \tau\sigma(x)] = [u\sigma(x), ux] = -[x, \sigma(x)]$. So $[x, \sigma(x)] \in \mathfrak{C}_\mathfrak{G}(\mathfrak{A})^{-\sigma,-\tau} = \mathfrak{A}$. Moreover, for any $z \in \mathfrak{A}$, $B(z, [x, \sigma(x)]) = B([z, x], \sigma(x)) = \alpha(z)B(x, \sigma(x))$, which proves the first equation. The second one follows from the first one and $\tau(x) = u\sigma(x)$.

By Lemma 2(b) any $x \in (\mathfrak{G}_\alpha + \mathfrak{G}_{-\alpha})^{u\sigma,v\tau}$ can be uniquely written as $x = x_\alpha + u\sigma(x_\alpha) = x_\alpha + v\tau(x_\alpha)$ with $x_\alpha \in \mathfrak{G}_\alpha^{uv\sigma\tau}$. Let $y = x_\alpha - u\sigma(x_\alpha) = x_\alpha - v\tau(x_\alpha)$. Then

$$\begin{aligned} y &\in (\mathfrak{G}_\alpha + \mathfrak{G}_{-\alpha})^{-u\sigma,-v\tau} \quad \text{and} \\ [x, y] &= [x_\alpha + u\sigma(x_\alpha), x_\alpha - u\sigma(x_\alpha)] \\ &= -2u[x_\alpha, \sigma(x_\alpha)] \\ &= -2u\, B(x_\alpha, \sigma(x_\alpha))z_\alpha \end{aligned}$$

by (a). Also,

$$B(x, x) = B(x_\alpha + u\sigma(x_\alpha), x_\alpha + u\sigma(x_\alpha)) = 2u\, B(x_\alpha, \sigma(x_\alpha))$$

by Lemma 6(d). Similarly

$$B(y, y) = B(x_\alpha - u\sigma(x_\alpha), x - u\sigma(x_\alpha)) = -2u\, B(x_\alpha, \sigma(x_\alpha))$$

This proves the last assertion in (b), and the rest is obvious.

The proof of Theorem 5 can now be completed by an adaptation of familiar arguments [**3**, **4**, **7**]. First of all, it follows from Lemma 2 that $N_G(\mathfrak{A}) = N_K(\mathfrak{A}).\ C_G(\mathfrak{A})$, so that $N_G(\mathfrak{A})/C_G(\mathfrak{A}) \cong N_K(\mathfrak{A})/C_K(\mathfrak{A})$. This quotient group can be identified with a group W of linear transformations of $\mathfrak{A}$ (or of $\mathfrak{A}^*$, by duality), which clearly permutes the elements of R. To show that R is a root system with Weyl group W it suffices to show:

(a) For each $\alpha \in R$, W contains the B-orthogonal reflection r_α in $\alpha^\perp$ defined by

$$r_\alpha(x) = x - 2\frac{B(x, z_\alpha)}{B(z_\alpha, z_\alpha)}z_\alpha$$

(b) $2\, B(z_\alpha, z_\beta)/B(z_\alpha, z_\alpha)$ is an integer for all $\alpha, \beta \in R$.

(c) W permutes the Weyl chambers of R freely. (A *Weyl chamber of R* is a connected component of $\{x \in \mathfrak{A} : \alpha(x) \neq 0 \text{ for all } \alpha \in R\}$.)

In fact, it follows from (a), (b) and Lemma 6(a) that R is a root system in $\mathfrak{A}^*$ and that the Weyl group W_R of R (which, by definition, is generated by the reflections $r_\alpha, \alpha \in R$) is contained in W. Since W_R acts transitively on the Weyl chambers, (c) gives $W_R = W$.

Proof of (a). Given $\alpha \in R$, choose non-zero x and y as in Lemma 8(b) (for some choice of $u, v = \pm 1$). (This is always possible: $\mathfrak{G}_\alpha = \mathfrak{G}_\alpha^{\sigma\tau} + \mathfrak{G}_\alpha^{-\sigma\tau}$, so

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$\mathfrak{G}_\alpha^{\pm\sigma\tau}$ is non-zero for at least one choice of the sign.) One of them, say x, is in $\mathfrak{K} = \mathfrak{G}^\tau$. Then exp (tx), $t \in \mathbf{R}$, is a one parameter group of linear transformations of $\mathfrak{G}$ which leave the negative definite form B_τ invariant. By Lemma 8(b) the plane spanned by y and z_α is stable, but not fixed, under exp (tx). Therefore exp $(t_0x)z_\alpha = -z_\alpha$ for appropriate $t_0 \in \mathbf{R}$. (Indeed, if x is normalized by $B_\tau(x, x) = -1$, one finds by series computations using the relations of Lemma 8 that

$$\exp(tx) = \cos(t||z_\alpha||)z + \sin(t||z_\alpha||)y$$

where $||z_\alpha||^2 = B(z_\alpha, z_\alpha)$. So $t_0 = \pi||z_\alpha||^{-1}$ will do.) Moreover, since $\alpha(z) = 0$ implies ad $(x)z = -\text{ad}(z) \cdot x = -\alpha(z)y = 0$, $\alpha^\perp$ stays pointwise fixed under exp (t_0x). So exp (t_0x) gives the desired reflection r_α in $\alpha^\perp$.

Proof of (b). Fix $\alpha \in R$ and set

$$z = \frac{2}{B(z_\alpha, z_\alpha)} z_\alpha.$$

Passing to the complexified Lie algebra $\mathfrak{G}_\mathbf{C}$ of $\mathfrak{G}$ we can find $x \in i\mathfrak{G}_\alpha^{u\sigma\tau}$ (for appropriate $u = \pm 1$) satisfying $B(x, \tau(x)) = 2/B(z_\alpha, z_\alpha)$. Set $y = \tau(x)$. Lemma 8(a) shows that these elements x, y, z of $\mathfrak{G}_\mathbf{C}$ satisfy

$$[x, y] = z$$
$$[z, x] = 2x$$
$$[z, y] = -2y$$

hence generate a subalgebra of $\mathfrak{G}_\mathbf{C}$ isomorphic with $\mathfrak{S}\mathfrak{L}(2, \mathbf{C})$. For any $\beta \in R$, $(\mathfrak{G}_\beta)_\mathbf{C}$ is an eigenspace of ad (z) with eigenvalue $\beta(z) = 2B(z_\alpha, z_\alpha)/B(z_\alpha, z_\alpha)$. So $2B(z_\alpha, z_\beta)/B(z_\alpha, z_\alpha)$ corresponds to a weight of a finite dimensional representation of $\mathfrak{S}\mathfrak{L}(2, \mathbf{C})$, hence is integral [**7**, IV-**7**, Thm. 4].

Proof of (c). Suppose $w \in W$ maps a Weyl chamber C onto itself. The convexity of C implies that it contains an element x fixed by w: one can take $x = (y + wy + \ldots + w^ny)/n$ where $y \in C$ is arbitrary and $w^{n+1} = e$. A representative for w in $N_K(\mathfrak{A})$ then centralizes the closure T of the one-parameter subgroup exp (itx), $t \in \mathbf{R}$, in Int $(\mathfrak{G}_\mathbf{C})$. Since T is a torus in the compact connected subgroup of Int $(\mathfrak{G}_\mathbf{C})$ with Lie algebra $\mathfrak{G}^\tau + i\mathfrak{G}^{-\tau}$, the centralizer of T in this subgroup is connected [**3**, VII-**2**, 2.8]. So w can be represented by exp (z) for an appropriate

$$z \in \mathfrak{C}_{\mathfrak{G}_\mathbf{C}}(x) = (\mathfrak{C}_\mathfrak{G}(x))_\mathbf{C}.$$

By Lemma 6(e) this implies that z centralizes $\mathfrak{A}$ so that w leaves $\mathfrak{A}$ pointwise fixed. This proves (b) and thereby Theorem 5.

Extend $\mathfrak{A}$ to a σ, τ-stable maximal split abelian subalgebra $\tilde{\mathfrak{A}}$ of $\mathfrak{G}$. Then R consists of the restrictions to $\mathfrak{A}$ of the roots in the root system $\tilde{R}$ of $\tilde{\mathfrak{A}}$. The Weyl group of $\tilde{R}$ is $\tilde{W} = N_G(\tilde{\mathfrak{A}})/C_G(\tilde{\mathfrak{A}}) = N_K(\tilde{\mathfrak{A}})/C_K(\tilde{\mathfrak{A}})$. The subgroup $\tilde{W}^\sigma$ of $\tilde{W}$ which leaves $\mathfrak{A} = \tilde{\mathfrak{A}}^{-\sigma}$ invariant also leaves $\tilde{\mathfrak{A}}^\sigma$ invariant ($\tilde{\mathfrak{A}}^\sigma$ being the B-orthogonal complement of $\mathfrak{A}$ in $\tilde{\mathfrak{A}}$) and can be characterized as the subgroup of $\tilde{W}$ fixed by the automorphism of $\tilde{W}$ induced by the automorphism σ of G. The action of $\tilde{W}^\sigma$ on $\mathfrak{A}$ gives a homomorphism of $\tilde{W}^\sigma$ into W. This homomorphism is in fact surjective. (Indeed, if $u \in N_K(\mathfrak{A})$, then $u\cdot\tilde{\mathfrak{A}}$ and $\tilde{\mathfrak{A}}$ are both maximal abelian in $\mathfrak{C}_{\mathfrak{G}}(\mathfrak{A})^{-\tau}$. Since $\mathfrak{C}_{\mathfrak{G}}(\mathfrak{A})$ is τ-stable, hence reductive (in $\mathfrak{G}$), $\tau|\mathfrak{C}_{\mathfrak{G}}(\mathfrak{A})$ is a Cartan involution; so $u\cdot\tilde{\mathfrak{A}}$ and $\tilde{\mathfrak{A}}$ are conjugate under $C_K(\tilde{\mathfrak{A}})$, say $\tilde{\mathfrak{A}} = vu\cdot\tilde{\mathfrak{A}}$ with $v \in K$ leaving $\mathfrak{A}$ pointwise fixed. Then $vu \in N_K(\tilde{\mathfrak{A}})$ and its action on $\tilde{\mathfrak{A}}$ coincides with that of u.) If one denotes by $\tilde{W}^{\mathfrak{A}}$ the subgroup of $\tilde{W}$ which leaves $\mathfrak{A}$ pointwise fixed and by $\tilde{R}^\sigma$ the roots in $\tilde{R}$ fixed by σ (i.e. the roots in $\tilde{R}$ vanishing on $\tilde{\mathfrak{A}}^{-\sigma} = \mathfrak{A}$), then one can summarize the situation in two short exact sequences:

(1) $0 \to \tilde{R}^\sigma \to \tilde{R} \to R \to 0$

(2) $1 \to \tilde{W}^{\mathfrak{A}} \to \tilde{W}^\sigma \to W \to 1$

Remark. In view of these circumstances it seems natural to wonder whether $\tilde{R}$ together with the involution $(-\sigma)$ is a normal sigma-system of roots in the sense of Araki [**1**], i.e. whether $\sigma\alpha + \alpha \notin \tilde{R}$ for all $\alpha \in \tilde{R}$. If so, Theorem 5 and the relations (1) and (2) above would be a consequence of general properties of normal sigma-systems. Unfortunately, this need not be the case (cf. example (b) below).

Up to isomorphism, the root system R depends only on $\mathfrak{G}$ and σ and not on the choice of $\mathfrak{A}$. (This is clear from the uniqueness of $\mathfrak{A}$ up to conjugation by H.) In fact, it is easy to identify R for the involutions of simple Lie algebras classified in [**2**] from the information given there. The method is as follows: the decomposition $\mathfrak{G}_\alpha = \mathfrak{G}_\alpha^{\sigma\tau} + \mathfrak{G}_\alpha^{-\sigma\tau}$ of the root spaces of $\mathfrak{A}$ into eigenspaces of the composite involution $\sigma\tau$ shows that $R = R_+ \cup R_-$ (not necessarily disjoint), where $R_\pm = \{\alpha \in R: \mathfrak{G}_\alpha^{\pm\sigma\tau} \neq 0\}$. R_+ is clearly just the restricted root system of the reductive Lie algebra $\mathfrak{G}^{\sigma\tau}$, while R_- consists of the weights of the representation $\mathrm{ad}_{\mathfrak{G}}$ of $\mathfrak{G}^{\sigma\tau}$ on $\mathfrak{G}^{-\sigma\tau}$. So one knows R as soon as one knows (1) the Lie algebra $\mathfrak{G}^{\sigma\tau}$ and (2) the representation of $\mathfrak{G}^{\sigma\tau}$ on $\mathfrak{G}^{-\sigma\tau}$. All of this information can be found in [**2**].

Examples. (a) *Riemannian spaces.* (1) *Compact type.* If $\mathfrak{G}$ is compact, (i.e. if B is negative definite) then the only Cartan involution of $\mathfrak{G}$ is the identity and the only split abelian subalgebra is the trivial one. So in this case the root system R is trivial for any involution σ of $\mathfrak{G}$. Conversely, suppose σ is an involution of a semisimple Lie algebra $\mathfrak{G}$ for which the corresponding root system R is trivial. Then $\mathfrak{G}^{-\sigma}$ contains no non-trivial split abelian algebra, so $\mathfrak{G}^{-\sigma,-\tau} = 0$ for any Cartan involution τ commuting with σ. Assume further that $\mathfrak{H} = \mathfrak{G}^\sigma$ contains no non-trivial ideal of $\mathfrak{G}$. (Such an ideal could have been factored out in the first place.) Then $\mathfrak{G}^\sigma = [\mathfrak{G}^{-\sigma}, \mathfrak{G}^{-\sigma}]$ (because the B-orthogonal complement of $[\mathfrak{G}^{-\sigma}, \mathfrak{G}^{-\sigma}] + \mathfrak{G}^{-\sigma}$ in $\mathfrak{G}$ is an ideal which lies in $\mathfrak{G}^\sigma$). Since $\mathfrak{G}^{-\sigma,-\tau} = 0$, $\mathfrak{G}^{-\sigma} = \mathfrak{G}^{-\sigma,\tau}$ so

$$\mathfrak{G}^\sigma = [\mathfrak{G}^{-\sigma,\tau}, \mathfrak{G}^{-\sigma,\tau}] \subset \mathfrak{G}^{\sigma,\tau} \quad \text{and}$$
$$\mathfrak{G} = \mathfrak{G}^\sigma + \mathfrak{G}^{-\sigma} \subset \mathfrak{G}^{\sigma,\tau} + \mathfrak{G}^{-\sigma,\tau} = \mathfrak{G}^\tau.$$

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This shows that τ must be the identity, hence $\mathfrak{G}$ must be compact.

(2) *Non compact type.* If σ is itself a Cartan involution of $\mathfrak{G}$, then $\tau = \sigma$ is the *only* Cartan involution commuting with σ (by the uniqueness of such Cartan involutions quoted earlier). So in this case $\mathfrak{A}$ is maximal split abelian in $\mathfrak{G}$ and R is the usual restricted root system.

(b) *Lie groups.* Let $\mathfrak{G}_0$ be any semisimple real Lie algebra. Set $\mathfrak{G} = \mathfrak{G}_0 \times \mathfrak{G}_0$ and define an involution σ of $\mathfrak{G}$ by $\sigma(x_1, x_2) = (x_2, x_1)$. The Cartan involutions τ commuting with σ are of the form

$$\tau(x_1, x_2) = (\tau_0(x_1), \tau_0(x_2))$$

where τ_0 is a Cartan involution of $\mathfrak{G}_0$; the maximal abelian algebras in $\mathfrak{G}^{-\sigma,-\tau}$ are $\mathfrak{A} = \{(x, -x); x \in \mathfrak{A}_0\}$ where $\mathfrak{A}_0$ is maximal abelian in $\mathfrak{G}_0{}^{-\tau_0}$. The root spaces of $\mathfrak{A}$ are of the form $(\mathfrak{G}_0)_\alpha \times (\mathfrak{G}_0)_\alpha$, where $(\mathfrak{G}_0)_\alpha$ is a root space of $\mathfrak{A}_0$ in $\mathfrak{G}_0$. The root system R of $\mathfrak{A}$ in $\mathfrak{G}$ is isomorphic with the restricted root system R_0 of $\mathfrak{A}_0$ in $\mathfrak{G}_0$. $\tilde{\mathfrak{A}} = \mathfrak{A}_0 \times \mathfrak{A}_0$ is a σ-stable maximal split abelian subalgebra of $\mathfrak{G}$ extending $\mathfrak{A}$. The root system $\tilde{R}$ of $\tilde{\mathfrak{A}}$ in $\mathfrak{G}$ is $\tilde{R} = \{(\alpha_1, \alpha_2): \alpha_1, \alpha_2 \in R_0\}$. σ acts on $\tilde{R}$ by sending (α_1, α_2) to (α_2, α_1), which shows that $\tilde{R}$ is (in general) not a normal sigma system of roots under $(-\sigma)$ (nor under σ itself, for that matter).

(c) *Real forms.* Suppose $\mathfrak{G}$ is a complex Lie algebra, σ the conjugation with respect to a real form $\mathfrak{G}_0 = \mathfrak{H}$ of $\mathfrak{G}$. The Cartan involutions τ of $\mathfrak{G}$ ($\mathfrak{G}$ regarded as a real Lie algebra) are the conjugations with respect to compact real forms. If τ commutes with σ, then $\tau_0 = \tau|\mathfrak{G}_0$ is a Cartan involution of $\mathfrak{G}_0$. The maximal abelian subspaces of $\mathfrak{G}^{-\sigma,-\tau}$ are therefore of the form $\mathfrak{A} = i\mathrm{A}_0$ where A_0 is maximal abelian in the maximal compactly embedded subalgebra $\mathfrak{K}_0 = \mathfrak{G}_0{}^{\tau_0}$ of $\mathfrak{G}_0$. The σ-stable maximal abelian algebras in $\mathfrak{G}^{-\tau}$ extending $\mathfrak{A}$ are of the form $\tilde{\mathfrak{A}} = \mathfrak{A}_0 + i\mathrm{A}_0$ with $\mathfrak{A}_0$ maximal abelian in $\mathfrak{C}_{\mathfrak{G}_0}(\mathrm{A}_0)^{-\tau}$. Note that $\mathfrak{A}_0 + \mathrm{A}_0$ is a maximally compact ("fundamental") Cartan subalgebra of $\mathfrak{G}_0$, and every maximally compact Cartan subalgebra of $\mathfrak{G}_0$ can be obtained in this way. The root system $\tilde{R}$ of $\tilde{\mathfrak{A}}$ is the usual root system of a complex Lie algebra. Theorem 5 shows that the restriction R of $\tilde{R}$ to $\mathfrak{A} = i\mathrm{A}_0$ is also a root system. (I do not know whether this fact has been previously observed.)

Theorem 1 and its corollaries are also of some interest in this case. In fact, the σ-stable maximal split abelian subalgebras of $\mathfrak{G}$ are in one-to-one correspondence with the Cartan subalgebras of $\mathfrak{G}_0$ (via $\mathfrak{A} \leftrightarrow \mathfrak{A}^\sigma + i\mathfrak{A}^{-\sigma}$). The theorem and its corollaries correspond to well-known conjugacy statements about real Cartan subalgebras: for example, Corollary 3 says that there are only finitely many conjugacy classes of Cartan subalgebras in a real semisimple Lie algebra.

(d) *Pseudo-Grassmannian spaces.* $SO(p, q)/SO(p_1, q_1) \times SO(p_2, q_2)$; $SU(p, q)/S\{U(p_1, q_1) \times U(p_2, q_2)\}$; $Sp(p, q)/Sp(p_1, q_1) \times Sp(p_2, q_2)$. This example is intended as an illustration of the recipe for computing the root system R given above.

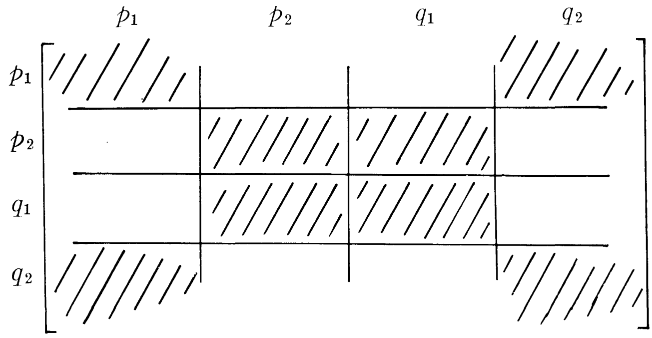
Let $F = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$ and $\mathfrak{G}$ the Lie algebra of matrices with entries from F of the form

$$\begin{array}{c|cc} & p & q \\ \hline p & x_1 & x_2 \\ q & x_2^* & x_3 \end{array} \quad x_1 = -x_1, x_3 = -x_3^*.$$

where $x^* = \bar{x}^t$, and with the additional restriction that $\mathrm{Tr}\,(x_1) + \mathrm{Tr}\,(x)_2 = 0$ if $F = \mathbf{C}$. Thus $\mathfrak{G} = \mathfrak{SO}(p, q)$, $\mathfrak{SU}(p, q)$ or $\mathfrak{SP}(p, q)$ according as $F = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$. Conjugation by a diagonal matrix of the form

$$\begin{array}{c|cccc} & p_1 & p_2 & q_1 & q_2 \\ \hline p_1 & +1 & & & \\ p_2 & & -1 & & \\ q_1 & & & +1 & \\ q_2 & & & & -1 \end{array} \quad p_1 + p_2 = p, q_1 + q_2 = q,$$

defines an involution σ of $\mathfrak{G}$, whose fixed point set $\mathfrak{G}^\sigma$ consists of matrices of the form

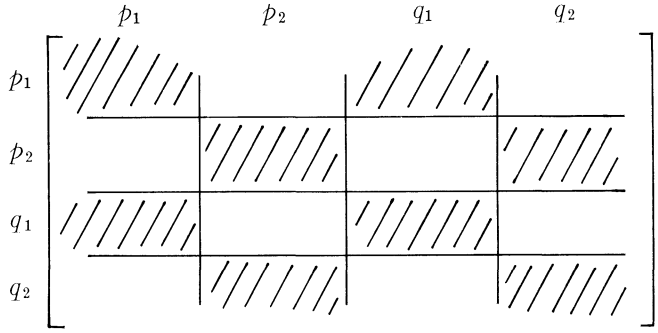


and is isomorphic with $\mathfrak{SO}(p_1, q_1) \times \mathfrak{SO}(p_2, q_2)$, $\mathfrak{S}\{\mathfrak{U}(p_1, q_1) \times \mathfrak{U}(p_2, q_2)\}$ or $\mathfrak{SP}(p_1, q_1) \times \mathfrak{SP}(p_2, q_2)$. A Cartan involution τ commuting with σ is defined by $\tau(x) = -x^*$. A maximal abelian subspace of $\mathfrak{G}^{-\sigma,-\tau}$ is given by the matrices of the form

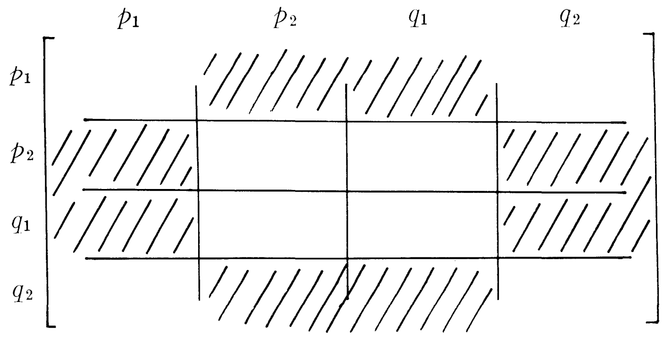
$$\begin{array}{c|cccc} & p_1 & p_2 & q_1 & q_2 \\ \hline p_1 & & & & a' \\ p_2 & & & a'' & \\ q_1 & & a'^* & & \\ q_2 & a''^* & & & \end{array} \quad \begin{array}{l} a_{ij}' = 0 \text{ if } i \neq j, \text{ and } a_{ii}' \in \mathbf{R}; \\ \\ a_{ij}'' = 0 \text{ if } i \neq j, \text{ and } a_{ii}'' \in \mathbf{R}. \end{array}$$

Thus $\dim\,(\mathfrak{A}) = \min\,(p_1, q_2) + \min\,(p_2, q_1)$.

The fixed point set $\mathfrak{G}^{\sigma\tau}$ of the composite involution $\sigma\tau$ consists of matrices of the form



and is isomorphic with $\mathfrak{SO}(p_1, q_2) \times \mathfrak{SO}(p_2, q_1)$, $\mathfrak{S}\{\mathfrak{U}(p_1, q_2) \times \mathfrak{U}(p_2, q_1)\}$, or $\mathfrak{SP}(p_1, q_2) \times \mathfrak{SP}(p_2, q_1)$. $\mathfrak{G}^{-\sigma\tau}$ consists of matrices of the form



and the representation of $\mathfrak{G}^{\sigma\tau}$ on $\mathfrak{G}^{-\sigma\tau}$ is equivalent to the natural one on $F^{p_1+q_2} \otimes_{\mathbf{R}} F^{p_2+q_1}$. As remarked above, the decomposition $\mathfrak{G} = \mathfrak{G}^{\sigma\tau} + \mathfrak{G}^{-\sigma\tau}$ is stable under the adjoint action of $\mathfrak{A}$. Denote by α_i', α_i'' the linear functionals on $\mathfrak{A}$ defined by $\alpha_i'(a) = a_{ii}'$, $\alpha_i''(a) = a_{ii}''$ for $a \in \mathfrak{A}$ as above. Then the weights of the representation of $\mathfrak{A}$ on $\mathfrak{G}^{\sigma\tau}$ are the restricted roots of $\mathfrak{G}^{\sigma\tau}$, which are [**4**, vol. II, pp. 109-112]:

$$\pm\alpha_i', \pm\alpha_i' \pm \alpha_j'', i, j \leqq \min (p_1, q_2), i \neq j,$$

$$\pm\alpha_i'', \pm\alpha_i'' \pm \alpha_j'', i, j \leqq \min (p_2, q_1), i \neq j,$$

if $F = \mathbf{R}$, and these together with

$$\pm 2\alpha_i', \pm 2\alpha_i''$$

if $F = \mathbf{C}$ or $\mathbf{H}$. The weights of the representation of $\mathfrak{A}$ on $\mathfrak{G}^{-\sigma\tau} \cong F^{p_1+q_2} \otimes_{\mathbf{R}} F^{p_2+q_1}$ are

$$\pm\alpha_i' \pm \alpha_j'', i \leqq \min (p_1, q_2), j \leqq \min (p_2, q_1).$$

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So the root system R is (in the notation of Tits):

B_n, if $F = \mathbf{R}$

BC_n if $F = \mathbf{C}$ or $\mathbf{H}$,

where $n = \min(p_1, q_1) + \min(p_2, q_2)$.

3. Decompositions of symmetric spaces. Let G be a connected, semi-simple, real Lie group, σ an involutive automorphism of G. σ is called a *Cartan involution of* G if the corresponding involution of $\mathfrak{G}$ (also denoted by σ) is Cartan. One knows that there always is a Cartan involution τ of G which commutes with σ [**4**, vol. I, p. 155]. Its fixed point group $K = G^\tau$ is connected, and compact modulo the center of G. Let H be an open subgroup of the fixed point group G^σ of σ. (So H is a—necessarily closed—subgroup of G between G^σ and its identity component $(G^\sigma)_0$). According to M. Berger's generalization of a theorem of E. Cartan on Riemannian symmetric spaces one has [**2**, § 55]:

THEOREM 9 (Berger-Cantan). $K\backslash G/H \cong \mathrm{Ad}_G(H \cap K)\backslash \mathfrak{G}^{-\sigma,-\tau}$.

More precisely, the map $K \times \mathfrak{G}^{-\sigma,-\tau} \times H \rightarrow G$, $(k, x, h) \rightarrow k \exp(x)h$, is surjective with fiber $\{(kl^{-1}, \mathrm{Ad}_G(l)x, lh): l \in H \cap K\}$ above $k \exp(x)h$.

In terms of the symmetric space G/H, this means that G/H is in one-to-two correspondence with the vector bundle $K \times_{(H \cap K)} \mathfrak{G}^{-\sigma,-\tau}$ over the compact symmetric space $K/H \cap K$. (Of course, all of the maps involved are actually real analytic).

Choose a maximal abelian algebra $\mathfrak{A}$ in $\mathfrak{G}^{-\sigma,-\tau}$ and let $A = \exp(\mathfrak{A})$ be the corresponding subgroup of G. Set $W_{H \cap K} = N_{H \cap K}(A)/C_{H \cap K}(A)$, which can be regarded as a subgroup of the Weyl group $W = W_K$ of the root system of $\mathfrak{A}$ in $\mathfrak{G}$. Since $\exp: \mathfrak{A} \rightarrow A$ is bijective we can also think of $W_{H \cap K}$ as a transformation group of A. With this understanding:

THEOREM 10. $K\backslash G/H \cong W_{H \cap K}\backslash A$.

More precisely, the multiplication map $K \times A \times H \rightarrow G$ is surjective, and the A-components of any two elements in the same fiber are conjugate under $N_{H \cap K}(A)$. The map is regular at (k, a, h) if and only if $a^\alpha \neq 1$ whenever $\mathfrak{G}_\alpha{}^{\sigma\tau} \neq 0$.

Proof. We know that the maximal abelian algebras in $\mathfrak{G}^{-\sigma,-\tau}$ are $H \cap K$ conjugate to $\mathfrak{A}$. Since every element of $\mathfrak{G}^{-\sigma,-\tau}$ lies in (at least) one of these, $\mathfrak{G}^{-\sigma,-\tau} = \mathrm{Ad}_G(H \cap K) \cdot \mathfrak{A}$. Suppose two elements x and x' of $\mathfrak{A}$ are conjugate under $H \cap K$, say $x = \mathrm{Ad}(u)x'$ with $u \in H \cap K$. Then $\mathfrak{A}$ and $\mathrm{Ad}(u)\mathfrak{A}$ are two maximal abelian algebras in $\mathfrak{C}_\mathfrak{G}(x)^{-\sigma,-\tau} = \mathfrak{C}_\mathfrak{G}(x)^{\sigma\tau,-\tau}$. Since $\tau|\mathfrak{C}_\mathfrak{G}(x)^{\sigma\tau}$ is a Cartan involution of the reductive Lie algebra $\mathfrak{C}_\mathfrak{G}(x)^{\sigma\tau}$ there is an element v in the identity component of $C_G(x)^{\sigma\tau,\tau} = C_G(x)^{\sigma,\tau}$ so that $\mathrm{Ad}(u) \cdot \mathfrak{A} = \mathrm{Ad}(v) \cdot \mathfrak{A}$. So $u^{-1}v$ lies in $H \cap K$, normalizes $\mathfrak{A}$, and $\mathrm{Ad}(u^{-1}v)x = \mathrm{Ad}(u^{-1})x = x'$, which shows that x and x' are also conjugate under $N_{N \cap K}(A)$. In view of Theorem 9 this proves that $K\backslash G/H \cong W_{H \cap K}\backslash A$.

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It remains to prove the regularity assertion. Since the multiplication map $K \times A \times H \to G$ is compatible with left multiplications from K and right multiplications from H, it suffices to determine when its tangent map at a point of the form (e, a, e) is surjective. This tangent map: $\mathfrak{K} \times \mathfrak{A} \times \mathfrak{H} \to \mathfrak{G}$ sends (x, y, z) to $\mathrm{Ad}\,(a)x + y + z$, so its image is $\mathrm{Ad}\,(a)\mathfrak{K} + \mathfrak{A} + \mathfrak{H}$. Comparing dimensions one finds that $\mathrm{Ad}\,(a)\mathfrak{K} + \mathfrak{A} + \mathfrak{H} = \mathfrak{G}$ if and only if $\mathrm{Ad}\,(a)\mathfrak{K} \cap (\mathfrak{A} + \mathfrak{H}) = \mathfrak{C}_{\mathfrak{H} \cap \mathfrak{K}}(\mathfrak{A})$. Now every element in $\mathfrak{K}$ can be written as

$$x_0 + \sum_{\alpha \in R} x_\alpha + \tau(x_\alpha)$$

with $x_0 \in \mathfrak{C}_{\mathfrak{K}}(\mathfrak{A})$ and $x_\alpha \in \mathfrak{G}_\alpha$; every element in $\mathfrak{A} + \mathfrak{H}$ as

$$y_0 + \sum_{\alpha \in R} y_\alpha + \sigma(y_\alpha)$$

with $y_0 \in \mathfrak{A} + \mathfrak{C}_{\mathfrak{H}}(\mathfrak{A})$ and $y_\alpha \in \mathfrak{G}_\alpha$. So for an element in $\mathrm{Ad}\,(a)\mathfrak{K} \cap (\mathfrak{A} + \mathfrak{H})$ one has an equation

$$\mathrm{Ad}\,(a)(x_0 + \textstyle\sum_\alpha x_\alpha + \tau(x_\alpha)) = y_0 + \textstyle\sum_\alpha y_\alpha + \sigma(y_\alpha),$$

which gives

$$\begin{aligned} x_0 &= y_0 \\ a^\alpha x_\alpha &= y_\alpha \\ a^{-\alpha}\tau(x_\alpha) &= \sigma(y_\alpha). \end{aligned}$$

Thus $x_0 = y_0$ must be in $\mathfrak{C}_{\mathfrak{K}}(\mathfrak{A}) \cap (\mathfrak{A} + \mathfrak{C}_{\mathfrak{H}}(\mathfrak{A})) = \mathfrak{C}_{\mathfrak{H} \cap \mathfrak{K}}(\mathfrak{A})$; and if $\alpha \in R$, then

$$y_\alpha = a^{-\alpha}\sigma\tau(x_\alpha) = a^{-2\alpha}\sigma\tau(y_\alpha).$$

Since the eigenvalues of $\sigma\tau$ are ± 1 this equation can be satisfied for a non-zero y_α if and only if $\mathfrak{G}_\alpha^{\sigma\tau} \neq 0$ and $a^\alpha = 1$.

As remarked earlier, $R_+ = \{\alpha \in R: \mathfrak{G}_\alpha^{\sigma\tau} \neq 0\}$ is the root system of the reductive Lie algebra $\mathfrak{G}^{\sigma\tau}$, and if H is connected, then $W_{H \cap K}$ can be identified with the Weyl group of this root system. In that case the closure of a Weyl chamber of R_+ in A (i.e. the closure of a connected component of $\{a \in A: a^\alpha \neq 1$ for all $\alpha \in R_+\}$) provides a fundamental domain for the action of $W_{H \cap K}$ on A, hence a parametrization of the $K - H$ double cosets in G.

From Theorem 10 one gets a system of spherical coordinates on the symmetric space G/H:

COROLLARY 11. *The map $K/C_{H \cap K}(A) \times A \to G/H$, $(k, a) \to kaH$, is surjective and is locally invertible (by an analytic map) at every point (k, a) for which $a^\alpha \neq 1$ whenever $\mathfrak{G}_\alpha^{\sigma\tau} \neq 0$.*

The geometry of the situation is nicely illustrated by the following:

Example. Real hyperbolic spaces. $SO_0(p, q)/SO_0(p, q-1)$. $G = SO_0(p, q)$ is

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the group of the matrices which leave the quadratic form

$$(x, x) = -x_1^2 - \ldots - x_p^2 + x_{p+1}^2 + \ldots + x_{p+q}^2$$

invariant and whose upper left hand $p \times p$ submatrix has a positive determinant. $H = SO_0(p, q - 1)$ is embedded in G as the subgroup fixing $e_{p+q} = (0, \ldots 0, 1) \in \mathbf{R}^{p+q}$; $K = SO(p) \times SO(q)$ as the subgroup leaving the subspaces $\mathbf{R}^{p+0}$ and $\mathbf{R}^{0+q}$ of $\mathbf{R}^{p+q}$ (spanned by $e_1, \ldots e_p$ and by $e_{p+1}, \ldots e_{p+q}$) invariant. For A one can take the one-parameter subgroup of G consisting of the matrices $a = a_t$, $t \in \mathbf{R}$, defined by

$$\begin{aligned} a \cdot e_1 &= \cosh(t)e_1 + \sinh(t)e_{p+q} \\ a \cdot e_{p+q} &= \sinh(t)e_1 + \cosh(t)e_{p+q} \\ a \cdot e_i &= e_i \text{ for } i \neq 1, p + q, \end{aligned}$$

It is easy to check that $W_K \cong \mathbf{Z}_2$; also $W_K = W_{H \cap K}$, except when $p = 1$, in which case $W_{H \cap K}$ is trivial.

The symmetric space G/H can be identified with

$$X = \{x \in \mathbf{R}^{p+q}: -x_1^2 - \ldots - x_p^2 + x_{p+1}^2 + \ldots + x_{p+q}^2 = 1\},$$

and $K/C_{H \cap K}(A)$ with

$$Y = \{y \in \mathbf{R}^{p+q}: y_1^2 + \ldots + y_p^2 = 1 = y_{p+1}^2 + \ldots + y_{p+q}^2\} \\ \cong S^{p-1} \times S^{q-1},$$

except when $q = 1$, in which case one has to add the restriction $x_{p+q} > 0$ to insure the connectedness of X and Y. The map $K/C_K(A) \times \mathfrak{A} \rightarrow G/H$, $(kC_K(A), x) \rightarrow k \exp(x)H$, is then identified with $Y \times \mathbf{R} \rightarrow X$, $(y, t) \rightarrow \cosh(t)y' + \sinh(t)y''$, if $y = y' + y''$ with $y' \in \mathbf{R}^{p+0}$ and $y'' \in \mathbf{R}^{0+q}$. If $p > 1$, this map has critical points along $t = 0$; if $p = 1$ the map is regular everywhere. This agrees with Theorem 10 and its corollary since $W_{H \cap K}$ and R_+ are trivial precisely when $p = 1$. If $p + q = 3$ the distinction between the cases $p = 1$ and $p > 1$ is geometrically obvious.

We now go on to describe the decomposition of the symmetric space G/H into orbits of a parabolic subgroup P of G. This is equivalent to a description of (1) the decomposition of G into $P - H$ double cosets, (2) the decomposition of the real flag manifold $P \backslash G$ into orbits of H, or (3) the decomposition of G-conjugates of P into H-conjugates. (The last reformulation comes from the fact that P is its own normalizer, so that $Pg \rightarrow g^{-1}Pg$ is bijection between $P \backslash G$ and the set of G-conjugates of P.) The method used here is an extension of the one used by Wolf [**10**], who treats the special case when G is complex and H a real form of G. The relevant facts about parabolics can be found in [**9**]. We start with the case of a minimal parabolic and observe:

LEMMA 12. *Every minimal parabolic subgroup P of G contains a σ-stable maximal split abelian subgroup A of G, unique up to conjugation by an element from $H \cap N$ (N being the unipotent radical of P).*

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Proof. First we note that any two minimal parabolics P and P' have a maximal split abelian subgroup of G in common. In fact, in virtue of the conjugacy of the minimal parabolics we can write $P' = gPg^{-1}$ for some $g \in G$. Let A be a maximal split abelian subgroup of G in P. Then $G = P \cdot N_G(A) \cdot P$, by the Bruhat decomposition. If $g = pwp'$ with $p, p' \in P$ and $w \in N_G(A)$ then $P \cap P' = P \cap (pwPw^{-1}p^{-1}) = p(P \cap wPw^{-1})p^{-1}$, hence $P \cap P'$ contains pAp^{-1}, which is also maximal split abelian. In particular, $P \cap \sigma(P)$ contains a maximal split abelian subgroup A of G. $C_G(A)$ is then a Levi subgroup of $P \cap \sigma(P)$, hence $N \cap \sigma(N)$—conjugate to a σ-stable Levi subgroup of $P \cap \sigma(P)$ [**5**], which must be of the form $C_G(A')$, where A' is a σ-stable $N \cap \sigma(N)$—conjugate of A.

Now suppose that A_1 and A_2 are two σ-stable maximal split abelian subgroups in P. By what has just been shown we can write $A_2 = nA_1n^{-1}$ with $n \in N \cap \sigma(N)$. Then $\sigma(nA_1n^{-1}) = nA_1n^{-1}$ and also $\sigma(nA_1n^{-1}) = \sigma(n)A_2\sigma(n^{-1})$, which shows that $n^{-1}\sigma(n) \in N_G(A)$. This can happen only if $n^{-1}\sigma(n) = e$ (because of the uniqueness properties of the Bruhat decomposition $G = N \cdot N_G(A) \cdot N$). The equation $\sigma(n) = n$ implies that $n = \exp(x)$ where $x \in \mathfrak{N} \cap \sigma(\mathfrak{N})$ satisfies $\sigma(x) = x$ (because of the well known fact that $\exp: \mathfrak{N} \cap \sigma(\mathfrak{N}) \to N \cap \sigma(N)$ is bijective). Hence $n \in H \cap N$.

Let $\mathscr{A}$ be the set of σ-stable maximal split abelian subgroups of G. H acts on $\mathscr{A}$ on the right: $(A)h = h^{-1}Ah$, $\mathscr{A}/H$ is finite (by Corollary 4), and the lemma gives a map $P\backslash G/H \to \mathscr{A}/H$ sending PgH to the H-conjugacy class $[A]$ of a σ-stable A in $g^{-1}Pg$. The fiber of this map above $[A] \in \mathscr{A}/H$ is in one-to-one correspondence with $W_G(A)/W_H(A)$ (where $W_G(A) = N_G(A)/C_G(A)$ and $W_H(A) = N_H(A)/C_H(A)$), because any two minimal parabolics containing A are conjugate under $N_G(A)$.

Fix a σ-stable A in P. Any other σ-stable maximal split abelian subgroup of G is then of the form $g^{-1}Ag$ where $g \in G$ satisfies $\sigma(g^{-1})A\sigma(g) = g^{-1}Ag$, i.e. $g\sigma(g^{-1}) \in N_G(A)$. From the above remarks we see that every P-H double coset has a representative g satisfying $g\sigma(g^{-1}) \in N_G(A)$, and that g is unique up to left translations from $C_G(A)$ and right translations from H. To summarize these observations, set

$$N_{G/H}(A) = \{gH \in G/H: g\sigma(g^{-1}) \in N_G(A)\},$$
$$W_{G/H}(A) = C_G(A)\backslash N_{G/H}(A) = \{C_G(A)gH: g\sigma(g^{-1}) \in N_G(A)\},$$

and denote by $W_{G^\sigma}(A)$ the subgroup of $W_G(A)$ fixed by σ (i.e. the subgroup of $W_G(A)$ which leaves $\mathfrak{A}^{\pm\sigma}$ invariant). Note the inclusions

$$W_H(A) \subset W_{G^\sigma}(A) \subset W_G(A)$$
$$W_{G^\sigma}(A)/W_H(A) \subset W_G(A)/W_H(A) \subset W_{G/H}(A).$$

THEOREM 13. *Let P be a minimal parabolic in G, A a σ-stable maximal split abelian subgroup in P.*

(1) *$W_{G/H}(A)$ is finite, and there is a one-to-one correspondence*

$$P\backslash G/H \cong W_{G/H}(A)$$

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so that

$$PgH \leftrightarrow C_G(A)gH$$

if $g\sigma(g^{-1}) \in N_G(A)$.

(2) *The union of the open P-H double cosets is dense in G. If* $P \cdot H$ *itself is open, then there is a one-to-one correspondence*

$$(P\backslash G/H)_{\text{open}} \cong W_{G^\sigma}(A)/W_H(A)$$

so that

$$PgH \leftrightarrow \bar{g}W_H(A)$$

if $g \in N_G(A)$ *represents* $\bar{g} \in W_{G^\sigma}(A)$.

Remark. If $P \cdot H$ itself is *not* open, we can find $g_0H \in N_{G/H}(A)$ so that Pg_0H is open, and we can apply (2) with P replaced by $g_0^{-1}Pg_0$ and A replaced by $g_0^{-1}Ag_0$ to see that the open P-H double cosets are then in one-to-one correspondence with $W_{G^\sigma}(g_0^{-1}Ag_0)/W_H(g_0^{-1}Ag_0)$ so that $Pg_0gH \leftrightarrow \bar{g}W_H(g_0^{-1}Ag_0)$.

Proof. The first part of the theorem is a summary of the preceding observations. The fact that the union of the open orbits is dense is clear from the finiteness of $P\backslash G/H$. To prove the rest we need a lemma and some more notation:

$$\begin{aligned}
R &= \text{the roots of } \mathfrak{A} \text{ in } \mathfrak{G} \\
R(P) &= \text{the roots of } \mathfrak{A} \text{ in } \mathfrak{P} \\
&= \text{the system of positive roots for } R \text{ corresponding to } P \\
R^\sigma &= \{\alpha \in R : \sigma \cdot \alpha = \alpha\} \\
R_\sigma &= \{\alpha \in R : \sigma \cdot \alpha \neq \alpha\} \\
R^\sigma(P) &= R^\sigma \cap R(P) \\
R_\sigma(P) &= R_\sigma \cap R(P)
\end{aligned}$$

Recall that if $\mathfrak{A}^{-\sigma}$ is maximal split abelian in $\mathfrak{G}^{-\sigma}$, then $R_\sigma|\mathfrak{A}^{-\sigma}$ is a root system whose Weyl group can be identified with the quotient of $W_{G^\sigma}(A)$ by the subgroup which leaves $\mathfrak{A}^{-\sigma}$ pointwise fixed.

LEMMA 14. *The following conditions on P are equivalent:*

(1) $P \cdot H$ *is open in G.*

(2) *If* $\alpha \in R(P) \cap \sigma \cdot R(P)$, *then* $\mathfrak{G}_\alpha \subset \mathfrak{H}$.

(3) $\mathfrak{A}^{-\sigma}$ *is maximal split abelian in* $\mathfrak{G}^{-\sigma}$ *and* $R(P) \cap \sigma R(P) = R^\sigma(P)$.

(4) $\mathfrak{A}^{-\sigma}$ *is maximal split abelian in* $\mathfrak{G}^{-\sigma}$ *and* $R_\sigma(P)|\mathfrak{A}^{-\sigma}$ *is a system of positive roots for* $R_\sigma|\mathfrak{A}^{-\sigma}$.

Proof (of the lemma). Choose a Cartan involution τ commuting with σ so that $\mathfrak{A} \subset \mathfrak{G}^{-\tau}$.

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(1) $\Leftrightarrow$ (2). $P \cdot H$ is open in G if and only if $\mathfrak{G} = \mathfrak{H} + \mathfrak{P}$. Moreover,

$$\mathfrak{G} = \mathfrak{G}_0 + \textstyle\sum_R \mathfrak{G}_\alpha$$
$$\mathfrak{H} = \mathfrak{G}_0{}^\sigma + \textstyle\sum_R (\mathfrak{G}_\alpha + \mathfrak{G}_{\sigma\cdot\alpha})^\sigma$$
$$\mathfrak{P} = \mathfrak{G}_0 + \textstyle\sum_{R(P)} \mathfrak{G}_\alpha$$

so that $\mathfrak{G} = \mathfrak{H} + \mathfrak{P}$ if and only if: $\alpha \in R(P)$ and $\sigma \cdot \alpha \in R(P)$ implies $(\mathfrak{G}_\alpha + \mathfrak{G}_{\sigma\cdot\alpha})^{-\sigma} = 0$, i.e. $\alpha \in R(P) \cap \sigma \cdot R(P)$ implies $\mathfrak{G}_{-\alpha} \subset \mathfrak{H}$, i.e. $\alpha \in R(P) \cap \sigma R(P)$ implies $\mathfrak{G}_\alpha = \tau \mathfrak{G}_{-\alpha} \subset \mathfrak{H}$.

(2) $\Rightarrow$ (3). If $\mathfrak{A}^{-\sigma}$ is *not* maximal abelian in $\mathfrak{G}^{-\sigma,-\tau}$ then there is $x \in \mathfrak{G}^{-\sigma,-\tau}$ so that $[\mathfrak{A}^{-\sigma}, x] = 0$ but $x \notin \mathfrak{A}^{-\sigma}$. Write $x = \sum x_\alpha$ with $\alpha \in R \cup \{0\}$ and $0 \neq x_\alpha \in \mathfrak{G}_\alpha$. Since $[\mathfrak{A}^{-\sigma}, x] = 0$, $\alpha|^{-\sigma} = 0$, i.e. $\sigma \cdot \alpha = \alpha$; since $x \notin \mathfrak{A}^{-\sigma}$ at least one such α is non-zero. Then $\alpha \neq 0$, $x_\alpha \neq 0$, and $x_{-\alpha} = \tau(x_\alpha) \neq 0$, so we can choose such an α in $R(P)$. If so, $\alpha = \sigma \cdot \alpha \in R(P) \cap \sigma R(P)$ and $\sigma(x_\alpha) = -x_{\sigma\cdot\alpha} = -x_\alpha$ is a non-zero element in $\mathfrak{G}^{-\sigma}$ contradicting (2).

(3) $\Rightarrow$ (2). Assume (3). So if $\alpha \in R(P) \cap \sigma \cdot R(P)$ then $\sigma \cdot \alpha = \alpha$, hence $\sigma(\mathfrak{G}_\alpha) = \mathfrak{G}_\alpha$ and $\mathfrak{G}_\alpha = \mathfrak{G}_\alpha{}^\sigma + \mathfrak{G}_\alpha{}^{-\sigma}$. If $x \in \mathfrak{G}_\alpha{}^{-\sigma}$, then $x - \tau(x) \in \mathfrak{G}^{-\sigma,-\tau}$ and $[\mathfrak{A}^{-\sigma}, x - \tau(x)] = 0$ (because $\alpha|\mathfrak{A}^{-\sigma} = 0$), so $x - \tau(x) \in \mathfrak{A}^{-\sigma}$ (because $\mathfrak{A}^{-\sigma}$ is assumed to be maximal abelian in $\mathfrak{G}^{-\sigma,-\tau}$). Thus $x \in \mathfrak{A}^\sigma \cap (\mathfrak{G}_\alpha + \mathfrak{G}_{-\alpha}) = 0$. Since x was arbitrary in $\mathfrak{G}_\alpha{}^{-\sigma}$, $\mathfrak{G}_\alpha{}^{-\sigma} = 0$ and $\mathfrak{G}_\alpha = \mathfrak{G}_\alpha{}^\sigma \subset \mathfrak{H}$.

(3) $\Rightarrow$ (4). Let $>$ be an order in $\mathfrak{A}^*$ so that $R(P) > 0$ and denote the restriction of this order to the subspace $(\mathfrak{A}^*)^{-\sigma}$ by the same symbol. Suppose $\alpha \in R_\sigma$ and $\alpha > 0$ (i.e. $\alpha \in R_\sigma(P)$). Then $\sigma \cdot \alpha < 0$ by (3), so $\alpha|\mathfrak{A}^{-\sigma} = \frac{1}{2}(\alpha - \sigma \cdot \alpha) > 0$ (identifying $(\mathfrak{A}^{-\sigma})^*$ with $(\mathfrak{A}^*)^{-\sigma}$ via $\alpha|\mathfrak{A}^{-\sigma} = \frac{1}{2}(\alpha - \sigma \cdot \alpha)$). Conversely, suppose $\alpha \in R_\sigma$ and $\alpha|\mathfrak{A}^{-\sigma} > 0$. Then $\frac{1}{2}(\alpha - \sigma \cdot \alpha) > 0$, i.e. $\alpha > \sigma \cdot \alpha$. In view of (3) this can happen only if $\alpha > 0$ (and $\sigma \cdot \alpha < 0$). So $R_\sigma(P)|\mathfrak{A}^{-\sigma}$ consists of the roots in $R_\sigma|\mathfrak{A}^{-\sigma}$ which are positive with respect to the order $>$ in $(\mathfrak{A}^{-\sigma})^*$.

(4) $\Rightarrow$ (3). Suppose $\alpha \in R(P) \cap \sigma \cdot R(P)$, i.e. $\alpha > 0$ and $\sigma \cdot \alpha > 0$. If $\alpha \in R_\sigma$, then (4) implies that $\alpha|\mathfrak{A}^{-\sigma} > 0$ and $\sigma \cdot \alpha|\mathfrak{A}^{-\sigma} > 0$, which is impossible (because $\alpha|\mathfrak{A}^{-\sigma} = -\sigma \cdot \alpha|\mathfrak{A}^{-\sigma}$). Thus $\alpha \notin R_\sigma$, i.e. $\alpha \in R^\sigma$.

Returning now to the proof of the theorem, assume that $P \cdot H$ is open, i.e. that the conditions of the lemma are satisfied. If PgH is also open for some $g \in G$, then the conditions of the lemma also hold with P replaced by $P' = g^{-1}Pg$ and A replaced by a σ-stable A' in P'. In particular $\mathfrak{A}^{-\sigma}$ and $(\mathfrak{A}')^{-\sigma}$ are both maximal split abelian in $\mathfrak{G}^{-\sigma}$. In view of Corollary 3 we may therefore assume that $\mathfrak{A} = \mathfrak{A}'$ (by replacing g by gh for some $h \in H$). The theorem will now follow if we can show that the systems $R(P)$ and $R(P')$ of positive roots for R are $W_{G^\sigma}(A)$-related.

By Lemma 14(4) and the remarks about the Weyl group of $R_\sigma|\mathfrak{A}^{-\sigma}$ preceding the lemma we can find $u \in W_{G^\sigma}(A)$ so that $R_\sigma(P')|\mathfrak{A}^{-\sigma} = u \cdot R_\sigma(P)|\mathfrak{A}^{-\sigma}$. Therefore

$$R_\sigma(P') = \{\alpha \in R_\sigma \colon \alpha|\mathfrak{A}^{-\sigma} \in R_\sigma(P')|\mathfrak{A}^{-\sigma}\}$$
$$= \{\alpha \in R_\sigma \colon \alpha|\mathfrak{A}^{-\sigma} \in u \cdot R_\sigma(P)|\mathfrak{A}^{-\sigma}\} = u \cdot R_\sigma(P).$$

Also, since $R^\sigma(P')$ and $u \cdot R^\sigma(P)$ are two systems of positive roots for R^σ ($=$ the system of roots of $\mathfrak{A}^\sigma$ in $\mathfrak{C}_\mathfrak{H}(\mathfrak{A}^{-\sigma})$) there is a v in the Weyl group of this root system (which is contained in the subgroup of $W_{G^\sigma}(A)$ which leaves $\mathfrak{A}^{-\sigma}$ pointwise fixed) so that $R^\sigma(P') = vu \cdot R^\sigma(P)$. Since v leaves $\mathfrak{A}^{-\sigma}$ pointwise fixed and since $R_\sigma(P')$ is uniquely determined by $R_\sigma(P')|\mathfrak{A}^{-\sigma}$ we also have $R_\sigma(P') = v \cdot R_\sigma(P') = vu \cdot R_\sigma(P)$. Thus $R(P') = R^\sigma(P') \cup R_\sigma(P') = uv \cdot R^\sigma(P) \cup uv \cdot R_\sigma(P) = uv \cdot R(P)$, which shows that $R(P)$ and $R(P')$ are indeed $W_{G^\sigma}(A)$ related.

Remarks. (1) The fact that $P\backslash G/H$ is finite is also proved in [**11**]. For the case when G is complex and H a real form of G, the description of $(P\backslash G/H)_{\text{open}}$ coincides with the one given in [**10**]. Other special cases have of course long been known (cf. the examples below).

(2) $N_{G/H}(A) = \{gH \in G/H: g\sigma(g^{-1}) \in N_G(A)\}$ is a symmetric subspace of G/H. i.e. if $\bar{g} = gH \in N_{G/H}(A)$ then the symmetry $\sigma_g : G/H \to G/H$ about $\bar{g}$ (defined by $\sigma_{\bar{g}}(\overline{g'}) = \overline{g\sigma(g^{-1})\sigma \cdot (g')}$ leaves $N_{G/H}(A)$ invariant. Note incidentally that $N_{G/H}(A)$ can be directly characterized in terms of the symmetric space structure of G/H: $N_{G/H}(A)$ consists of those points of G/H whose symmetries normalize A (A being regarded as a transformation group of G/H)). Note also that $W_{G/H}(A)$ is a finite symmetric space (in the sense of Loos [**4**]) with the symmetry $\sigma_{\bar{g}}$ about $\bar{g} = C_G(A)gH$ ($g\sigma(g^{-1}) \in N_G(A)$) defined by

$$\sigma_{\bar{g}}(g') = \overline{g\sigma(g^{-1})g'}.$$

(3) It seems natural to wonder whether in fact $G = P \cdot N_G(A) \cdot H$, which would amount to $W_{G/H}(A) = W_G(A)/W_H(A)$. But this is false, in general. (It happens if and only if the σ-stable maximal split abelian subgroups of G are all conjugate under H.)

The above theorem becomes particularly simple if H is the full fixed point set of σ. In that case one easily checks that G/H can be embedded into G by the map $gH \to g\sigma(g^{-1})$. The image G_σ of this map is just the connected component of the identity in the set of σ-symmetric elements, i.e. elements $g \in G$ satisfying $\sigma(g^{-1}) = g$ [**4**, vol. I, p. 182]. The action of G on G/H by left translations corresponds to the action of G on G_σ defined by $g(x) = gx\sigma(g^{-1})$ for $g \in G$ and $x \in G_\sigma$.

COROLLARY 15. *Let P be a minimal parabolic in G, A a σ-stable maximal split abelian subgroup in P. Let G act on $G_\sigma = \{g\sigma(g^{-1}): g \in G\}$ by $g(x) = gx\sigma(g^{-1})$ and set*

$$N_{G_\sigma}(A) = N_G(A) \cap G_\sigma,\ C_{G_\sigma}(A) = C_G(A) \cap G_\sigma.$$

Then the following quotients of this action are isomorphic:

(1) $P\backslash G_\sigma \cong C_G(A)\backslash N_{G_\sigma}(A)$.

(2) *If the P-orbit of e is open in G_σ, then*

$$(P\backslash G_\sigma)_{\text{open}} \cong C_G(A)\backslash C_{G_\sigma}(A).$$

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Proof. (1) is immediate from Theorem 13(1). (2) will follow from Theorem 13(2) if we can show that $g \in N_G(A)$ leaves $\mathfrak{A}^{\pm\sigma}$ invariant if and only if $g\sigma(g^{-1}) \in C_G(A)$, i.e. if and only if the action of g on A commutes with the action of σ on A, and this is clear.

Remark. In general, when H is a possibly proper subgroup of G^σ, the map $G/H \to G_\sigma$, $gH \to g\sigma(g^{-1})$ is a finite covering whose fibers are in one-to-one correspondence with G^σ/H, which is a group isomorphic to a finite direct product of cyclic groups of order two [**4**, vol. I, p. 171].

We now drop the assumption that P be minimal. If A is a σ-stable maximal split abelian subgroup in P, set $W_P(A) = N_P(A)/C_G(A)$ and $W_{P^\sigma}(A) = W_P(A) \cap W_{G^\sigma}(A)$. With this notation we have:

COROLLARY 16. (1) *$P\backslash G/H$ is finite for any parabolic P in G.*

(2) *The union of the open P-H double cosets is dense in G. If $P \cdot H$ itself is open then there is a σ-stable maximal split abelian subgroup A in P so that*

$$(P\backslash G/H)_{\text{open}} \cong W_{P^\sigma}(A)\backslash W_{G^\sigma}(A)/W_H(A).$$

Proof. The fact that $P\backslash G/H$ is finite and that the union of the open P-H double cosets is dense is a trivial consequence of Theorem 13(1).

Let P_0 be a minimal parabolic contained in P. Clearly, a P-H double coset will be open if and only if it contains an open P_0-H double coset. If $P \cdot H$ itself is open we can choose P_0 so that $P_0 \cdot H$ is also open. Assume that this is so and choose a σ-stable A in P_0. The open P_0-H double cosets are then of the form P_0gH with $g \in N_G(A)$ and $g\sigma(g^{-1}) \in C_G(A)$ (by Theorem 13(2)). Suppose two of these, say P_0g_1 and P_0g_2H, lie in the same P-H double coset: $Pg_1H = Pg_2H$. We have to show that $P_0g_1H = P_0pg_2H$ for some $p \in N_p(A)$. Replacing P by $g_1^{-1}Pg_1$ and P_0 by $g_1^{-1}P_0g_1$ one sees that it suffices to show that P_0gH is contained in $P \cdot H$ for some $g \in N_G(A)$, $g\sigma(g^{-1}) \in C_G(A)$ only if $P_0gH = P_0pH$ for some $p \in N_P(A)$. So suppose that $P_0gH \subset PH$, $g \in N_G(A)$, and $g\sigma(g^{-1}) \in C_G(A)$. Write $g = ph$ with $p \in P$ and $h \in H$. Then $A = gAg^{-1} = phAh^{-1}p^{-1}$, so $A' = p^{-1}Ap = hAh^{-1}$ lies in P and is σ-stable. After replacing A' by a conjugate under $P \cap H$ we may even assume that A and A' lie in a common σ-stable Levi subgroup M of $P \cap \sigma(P)$. (This follows from the fact that any two σ-stable Levi subgroups of $P \cap \sigma(P)$ are conjugate under a σ-stable element from the radical of $P \cap \sigma(P)$, cf. the proof of Lemma 12.) The intersections of $\mathfrak{A}$ and $\mathfrak{A}'$ with the center of $\mathfrak{M}$ coincide, being the largest subalgebra of the center of $\mathfrak{M}$ which splits in $\mathfrak{G}$; and modulo the center of $\mathfrak{M}$, $\mathfrak{A}$ and $\mathfrak{A}'$ are maximal split abelian in $\mathfrak{M}$ while $\mathfrak{A}^{-\sigma}$ and $(\mathfrak{A}')^{-\sigma}$ are maximal split abelian in $\mathfrak{M}^{-\sigma}$. ($\mathfrak{A}^{-\sigma}$ and $(\mathfrak{A}')^{-\sigma}$ are maximal split abelian in $\mathfrak{G}^{-\sigma}$ (by Lemma 14.)) Applying Corollary 3 to the (semisimple) quotient of $\mathfrak{M}$ by its center one gets that $\mathfrak{A}$ and $\mathfrak{A}'$ are conjugate under $M \cap H$, say $\mathfrak{A}' = \mathrm{Ad}\ (m) \cdot \mathfrak{A}$ with $m \in M \cap H$. Thus $A = pA'p^{-1} = pmAm^{-1}p^{-1}$, $pm \in N_P(A)$, and $P_0gH = P_0pH = P_0pmH$, as required.

Remark. It follows from Theorem 13 that any P-H double coset (open or not) has a representative g such that $g\sigma(g^{-1}) \in N_G(A)$. However, the description of all such representatives of the same P-H double coset seems too complicated to be of much use.

There is a class of parabolic subgroups of G which are of special significance for the symmetric space G/H. They are constructed as follows. Suspending previous conventions we now denote by $\mathfrak{A}$ a maximal split abelian subalgebra in $\mathfrak{G}^{-\sigma}$ and by $\mathfrak{G}_\alpha (\alpha \in R)$ the root spaces of $\mathfrak{A}$ in $\mathfrak{G}$. Choose a linear order in $\mathfrak{A}^*$ and set

$$\mathfrak{N} = \sum_{\alpha > 0} \mathfrak{G}_\alpha.$$

From the relation $[\mathfrak{G}_\alpha, \mathfrak{G}_\beta] \subset \mathfrak{G}_{\alpha+\beta}$ it follows that $\mathfrak{N}$ is a nilpotent subalgebra of $\mathfrak{G}$ normalized by $\mathfrak{G}_0 = \mathfrak{C}_\mathfrak{G}(\mathfrak{A})$. So $\mathfrak{P} = \mathfrak{C}_\mathfrak{G}(\mathfrak{A}) + \mathfrak{N}$ is a subalgebra of $\mathfrak{G}$. Similarly, the connected subgroup N of G with Lie algebra $\mathfrak{N}$ is normalized by $C_G(A)$, so that $P = C_G(A) \cdot N$ is a subgroup of G with Lie algebra $\mathfrak{P}$. In fact, P is a *parabolic* subgroup of G, N is its unipotent radical, and $C_G(A)$ a Levi subgroup. To see this, extend $\mathfrak{A}$ to a σ-stable maximal split abelian subalgebra $\tilde{\mathfrak{A}}$ of $\mathfrak{G}$, and denote the root spaces of $\tilde{\mathfrak{A}}$ in $\mathfrak{G}$ by $\mathfrak{G}_{\tilde{\alpha}} (\tilde{\alpha} \in \tilde{R})$. So $R = \tilde{R}|\mathfrak{A}$ and for any $\alpha \in R$,

$$\mathfrak{G}_\alpha = \sum \{\mathfrak{G}_{\tilde{\alpha}} : \tilde{\alpha}|\mathfrak{A} = \alpha\}.$$

Choose a linear order in $\tilde{\mathfrak{A}}^*$ compatible with the order in $\mathfrak{A}^*$ defining $\mathfrak{N}$ (i.e. if $\tilde{\alpha} \in \mathfrak{A}^*$ and $\tilde{\alpha}|\mathfrak{A} > 0$, then $\tilde{\alpha} > 0$). Set

$$\tilde{\mathfrak{N}} = \sum_{\tilde{\alpha} > 0} \mathfrak{G}_{\tilde{\alpha}}.$$

Since $\tilde{\alpha} > 0$ implies $\tilde{\alpha}|\mathfrak{A} \geqq 0$,

$$\tilde{\mathfrak{N}} = \mathfrak{C}_{\tilde{\mathfrak{N}}}(\mathfrak{A}) + \mathfrak{N}.$$

So $\tilde{N} \subset C_G(A) \cdot N$ and $P = C_G(A) \cdot N$ contains the minimal parabolic $\tilde{P} = C_G(\tilde{A}) \cdot \tilde{N}$, which shows that P itself is parabolic.

Remark. It is easy to identify P in terms of the classification of parabolic subgroups of G by sets of simple roots in $\tilde{R} : P$ is the parabolic corresponding to the set S of simple roots vanishing on $\mathfrak{A} = \tilde{\mathfrak{A}}^{-\sigma}$ (i.e. the set of simple roots fixed by σ). However, $\mathfrak{A}$ need not coincide with the split component $S^\perp$ of the center of $\mathfrak{C}_\mathfrak{G}(\mathfrak{A})$. ($\mathfrak{A}$ may be properly contained in $S^\perp$.)

Corollary 16 applies in the present situation (with $\tilde{A}$ playing the role of the A there):

$$(P\backslash G/H)_{\text{open}} \cong W_P{}^\sigma(\tilde{A})\backslash W_G{}^\sigma(\tilde{A})/W_H(\tilde{A}).$$

Now $W_P{}^\sigma(\tilde{A})$ is precisely the (normal) subgroup of $W_G{}^\sigma(\tilde{A})$ which leaves the subspace $\mathfrak{A}$ of $\mathfrak{A}$ pointwise fixed. According to the discussion after the proof of Theorem 5

$$W_G^\sigma(\tilde{A})/W_P^\sigma(\tilde{A}) \cong W_G(A)$$

and

$$W_H^\sigma(\tilde{A})/W_P^\sigma(\tilde{A}) \cong W_G(A),$$

where $W_G(A) = N_G(A)/C_G(A)$ is the Weyl group of R and $W_H(A) = N_H(A)/C_H(A)$. So in the present situation Corollary 16 becomes:

COROLLARY 17. *$P\backslash G/H$ is finite, and there is a one-to-one correspondence*

$$(P\backslash G/H)_{\text{open}} \cong W_G(A)/W_H(A)$$

so that

$$PgH \leftrightarrow \bar{g}W_H(A)$$

if $g \in N_G(A)$ represents $\bar{g} \in W_G(A)$.

Remark. It follows from Theorem 13 that any P-H double coset (open or not) has a representative g such that $g\sigma(g^{-1}) \in N_{\mathfrak{G}}(\tilde{A})$. However, one may not be able to choose g so that $g\sigma(g^{-1}) \in N_G(A)$.

Examples. (a) *Riemannian space.* (1) *Compact type.* If G is compact then the only parabolic subgroup of G is G itself and the only split abelian subgroup of G is $\{e\}$. So in this case Theorem 12 is trivial.

(2) *Non compact type.* Suppose σ is a Cartan involution of G so that $H = K$ is maximal compact modulo the center of G. The σ-stable maximal split abelian subalgebras of $\mathfrak{G}$ are then precisely the maximal abelian algebras in $\mathfrak{G}^{-\sigma}$. If A is a σ-stable maximal split abelian subgroup of G and $g\sigma(g^{-1}) \in N_G(A)$ for some $g \in G$, write $g = l \cdot k$ with $l \in \exp(\mathfrak{G}^{-\tau})$ and $k \in K$. Then $g\sigma(g^{-1}) = l^2 \in N_G(A) \cap \exp(\mathfrak{G}^{-\tau}) = A$, which shows that

$$W_{G/K}(A) = \{C_G(A)gK : g\sigma(g^{-1}) \in N_G(A)\}$$

has only one element. So $G = P \cdot K$ for any (minimal) parabolic P of G (by Theorem 13). This is essentially Iwasawa's decomposition theorem. The "special" parabolics are in this case just the minimal ones.

(b) *Lie groups.* Let G_0 be a connected, semisimple, real Lie group. Set $G = G_0 \times G_0$ and let H be the diagonal in G. So H is the fixed point set of the involution σ of G defined by $\sigma(g_1, g_2) = (g_2, g_1)$ and the map $G/H \to G_0$, $gH \to g_1 \cdot g_2^{-1}$, is a one-to-one correspondence. The (minimal) parabolics of G are of the form $P = P_1 \times P_2$ where P_1 and P_2 are (minimal) parabolics in G. The σ-stable maximal split abelian subgroups in such a parabolic P are of the form $A = A_0 \times A_0$ where A_0 is a maximal split abelian subgroup of G in $P_1 \cap P_2$.

The bijection $G/H \to G_0$ maps $N_{G/H}(A)$ onto $N_{G_0}(A_0)$ and induces a bijection of $W_{G/H}(A)$ onto $W_{G_0}(A_0)$. So the relation $P\backslash G/H \cong W_{G/H}(A)$ of Theorem 13(1) becomes $P_1\backslash G_0/P_2 \cong W_{G_0}(A_0)$, which is essentially Bruhat's decomposition theorem. (Of course, this is not an independent proof of Bruhat's theorem, since that was used in the proof of Lemma 12.) The "special" parabolics are in this case of the form $P = P_1 \times P_2$, where P_1 and P_2 are opposite minimal parabolics of G_0.

(c) *Real forms.* Let G be a connected semisimple complex Lie group, $G_0 = H$ the connected real form of G defined by a conjugation σ in G. (So σ is an involutive automorphism of the real Lie group G satisfying $\sigma(ix) = -i\sigma(x)$ for all $x \in \mathfrak{G}$). Let B be a minimal parabolic (=Borel) subgroup of G, A a σ-stable, maximal split abelian subgroup of G in B. Then $\mathfrak{A}_0 = \mathfrak{A}^\sigma + i\mathfrak{A}^{-\sigma}$ is a Cartan subalgebra of $\mathfrak{G}_0$ with split component $\mathfrak{A}^\sigma$ and compact component $i\,\mathfrak{A}^{-\sigma}$. $\mathfrak{A}^{-\sigma}$ is maximal split abelian in $\mathfrak{G}^{-\sigma}$ if and only if $\mathfrak{A}_0$ is a maximally compact Cartan subalgebra of $\mathfrak{G}_0$, and if this is the case then no root of $\mathfrak{A}$ vanishes on $\mathfrak{A}^{-\sigma}$ [**9**, 1.3.3.4]. It therefore follows from Lemma 14, (1) $\Leftrightarrow$ (2), that $B \cdot G_0$ is open in G if and only if $\mathfrak{A}_0$ is a maximally compact Cartan subalgebra of $\mathfrak{G}_0$; and if this is the case, then $(P\backslash G/G_0)_{\text{open}} \cong W_{P^\sigma}(A)\backslash W_{G^\sigma}(A)/W_{G_0}(A)$ for any parabolic P containing B (Corollary 16). This is essentially Theorem 4.9 in [**11**]. The "special" parabolics are in this case the Borel subgroups of G which contain a maximally compact Cartan subgroup of G_0.

(d) *Real hyperbolic spaces.* Let G, H, and A be as in the example after Corollary 11. For one of the two possible choices of an order in $\mathfrak{A}^* \cong \mathbf{R}$ the "special" parabolic $P = C_G(A) \cdot N$ defined above is then the subgroup of G which leaves the one-dimensional subspace of $\mathbf{R}^{p+q}$ spanned by $e_1 + e_{p+q}$ invariant (the other order corresponds to the subgroup $\bar{P}$ of G which leaves the subspace spanned by $e_1 - e_{p+q}$ invariant). Note that P is not minimal, except when $q = 1$ (Riemannian case). We know that $W_G(A) \cong \mathbf{Z}_2$, and that $W_H(A) = W_G(A)$ unless $p = 1$, in which case $W_H(A)$ is trivial. It therefore follows from Corollary 17 that there is only one open P-H double coset, unless $p = 1$ in which case there are two. This can be seen directly as follows. Identify G/H with the connected component X of e_{p+q} in the hypersurface

$$\{x \in \mathbf{R}^{p+q} : (x, x) = -x_1^2 \ldots -x_p^2 + \ldots + x_{p+q}^2 = 1\}.$$

Suppose $x \in X$ is in the P-orbit of e_{p+q}, say $x = p \cdot e_{p+q}$ with $p \in P$. Since $p \cdot (e_1 + e_{p+q}) = c(e_1 + e_{p+q})$ for some $c \neq 0$, $p \cdot e_1 = -x + c(e_1 + e_{p+q})$. So $-1 = (p \cdot e_1, p \cdot e_1) = 1 - c(e_1 + e_{p+q}, x)$, i.e. $(e_1 + e_{p+q}, x) = 1/c \neq 0$. Conversely, if $c = (e_1 + e_{p+q}, x) \neq 0$ for some $x \in X$, one can find $p \in 0(p, q)$ so that $p \cdot e_{p+q} = x$ and $p \cdot e_1 = -x + c(e_1 + e_{p+q})$ (because $(x, x) = 1 = (e_{p+q}, e_{p+q})$ and $(-x + c(e_1 + e_{p+q}), -x + c(e_1 + e_{p+q}) = -1 = (e_1, e_1))$. Moreover, only the first and the last columns of p are determined by x, so that one can even choose p to lie in $SO_0(p, q)$ provided $p > 1$. (Recall that matrices

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in $SO_0(p, q)$ must have an upper left hand $p \times p$ submatrix of positive determimant.) If $p = 1$, one may not be able to choose p in $SO_0(p, q)$: for example, if $x = -e_{p+q}$, then $c = -1$ and $(p \cdot e_1, p \cdot e_1) = (-x + c(e_1 + e_{p+q}), e_1) = -1$, thus p must have a negative entry in the upper left hand corner. So if $p = 1$, then there are two open orbits of P on $X = G/H$, namely the orbits of e_{p+q} and of $-e_{p+q}$; if $p > 1$ there is only one orbit, in agreement with the theory. The argument shows further that the singular orbits of P on X make up the intersection of X with the hyperplane $\{x \in \mathbf{R}^{p+q} : (x, e_1 + e_{p+q}) = 0\}$.

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McMaster University,
Hamilton, Ontario