PARTITION FUNCTIONS AND SPIRALLING IN PLANE RANDOM WALK

Z. A. Melzak

(received September 12, 1962)

1. Consider the plane symmetric random walk on a square lattice: a particle is initially at the origin in the xy-plane, it makes n consecutive steps of unit length, and each step is made with the probability 1/4 in each one of the four directions parallel to the axes. We call the path of the particle a spiral if the following conditions are met: a) the particle never occupies the same position twice, b) the path of the particle, whenever it turns, either turns always clockwise or always counter-clockwise throughout the path, and c) for every $m > n$ the given n-step path can be continued in at least one way to give an m-step path meeting the conditions. Conditions a) and b) are natural for spirals; c) is necessary to eliminate such paths as (0, 0)-(1, 0)-(1, 1)-(1, 2)-(0, 2)-(0, 1) or (0, 0)-(1, 0)-(2, 0)-(2, 1)-(1, 1)-(0, 1)-(-1, 1)-(-1, 0)-(-1, -1)-(-1, -2)-(0, -2)-(1, -2)-(1, -1). We shall calculate the probability p_n that the path of the particle is a spiral; it will turn out that the answer is given in terms of the partition function $p(n)$ for unrestricted partitions, and that other partition functions also enter into the problem.

2. The total number of paths is 4^n and each one of them occurs with the probability 4^{-n}. Let us consider only the paths starting along the positive x-axis. Observing that a spiral is then either the straight segment $[0, n]$ or it has k turns, $k \geq 1$, either clockwise or counter-clockwise, and that any clockwise spiral is mirrored into a counter-clockwise one by reflexion in the x-axis, we have

Canad. Math. Bull. vol. 6, no. 2, May 1963.

https://doi.org/10.4153/CMB-1963-021-3 Published online by Cambridge University Press

(1) $$p_n = 4^{1-n}\left[1 + 2 \sum_{k=1}^{\infty} N(n,k)\right]$$

where $N(n,k)$ is the number of different n-step spirals with k counter-clockwise turns, along which the first step takes the particle from (0,0) to (1,0). We use here the natural convention that $N(n,k) = 0$ if k is so large, relative to n, that there are no required spirals at all. For instance, $N(1,k) = 0$ if $k \geq 1$, $N(2,k) = 0$ if $k \geq 2$, $N(3,k) = N(4,k) = 0$ if $k \geq 3$, and so on. Under these conditions we have to count only the spirals for which the successive rectilinear displacements are due east, north, west, south, east, etc. For a spiral with k turns there must be $k+1$ such displacements; let their lengths be $a_1, a_2, \ldots, a_k, a_{k+1}$. These numbers determine the spiral uniquely, and we have

Lemma 1. A sequence $(a_1, a_2, \ldots, a_k, a_{k+1})$ of $k+1$ positive integers determines a spiral if and only if

(2) $$\sum_{i=1}^{k+1} a_i = n, \qquad a_{k+1} \geq 1,$$

and the first k integers satisfy

(3) $$a_1 < a_3 < a_5 < \ldots, \qquad a_2 < a_4 < a_6 < \ldots.$$

$N(n,k)$ is therefore the number of distinct solutions of (2) and (3).

To prove the lemma we observe that (2) must obviously hold since the length of a spiral is the sum of the lengths of successive displacements, and the length of the tail (= the last displacement) can be any positive integer. (3) must hold since by the conditions a), b) and c), except for the tail, each displacement parallel to the x-axis must be longer than the previous one, and the same is true for the displacements parallel to the y-axis.

3. To calculate $N(n,k)$ we introduce two combinatorial functions: $Q(m,j)$ - the number of distinct representations of m as a sum of j increasing positive integers, and $Q(m,j,s)$,

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which is defined in the same way, subject to the condition that the smallest integer is s. We have

$$(4)\quad \begin{cases} Q(m, j, s) = 0 & \text{if} \quad m < sj + j(j-1)/2 \\ Q(m, j) = 0 & \text{if} \quad m < j(j+1)/2 \end{cases}$$

$$(5)\quad Q(m, j) = \sum_{s=1}^{\infty} Q(m, j, s) .$$

We obtain next a recursion formula for $Q(m, j)$. Consider any representation

$$(6)\quad m = s + m_1 + m_2 + \ldots + m_{j-1}$$

of m as a sum of j positive integers which are increasing and start with s. Subtracting sj from each side we have

$$(7)\quad m - sj = (m_1 - s) + (m_2 - s) + \ldots + (m_{j-1} - s) ,$$

which is a representation of $m - sj$ as a sum of $j - 1$ increasing integers. Conversely, from any representation of the type (7) we get a representation of the type (6). Therefore

$$(8)\quad Q(m-js, j-1) = Q(m, j, s) .$$

Elimination of $Q(m, j, s)$ from (5) and (8) gives

$$Q(m, j) = \sum_{s=1}^{\infty} Q(m-js, j-1)$$

or

$$(9)\quad Q(m, j+1) = \sum_{s=1}^{\infty} Q(m-(j+1)s, j) .$$

By a simple calculation we have

(10) $Q(m, 1) = 1$ for $m \geq 1$, $Q(m, 2) = -1 + m/2$ for even $m \geq 4$. $Q(m, 2) = (m-1)/2$ for odd $m \geq 3$.

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Introduce the generating function

$$f_j(x) = \sum_{m=1}^{\infty} Q(m,j)\, x^m ;$$

then from (10)

$$f_1(x) = x/(1-x), \quad f_2(x) = x^3/(1-x)(1-x^2).$$

The recurrence relation (9) is equivalent to

$$f_{j+1}(x) = f_j(x)\, x^{j+1}/(1-x^{j+1}) ;$$

this allows us to determine $f_j(x)$:

$$(11) \quad f_j(x) = x^{j(j+1)/2} / \prod_{i=1}^{j} (1-x^i).$$

The above formula shows that $Q(m,j)$, as could be expected, is related to some partition functions, and we remark further that by a theorem of Euler, [1] p. 275, we have

$$1 + \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} Q(m,j)\, x^m z^j = \prod_{n=1}^{\infty} (1 + z x^n).$$

4. We proceed now to the evaluation of $N(n,k)$. Suppose first that $k = 2p$ is even, that the length of the tail is t, and that

$$(12) \quad a_1 + a_3 + \dots + a_{2p-1} = A$$

so that

$$(13) \quad a_2 + a_4 + \dots + a_{2p} = n - t - A .$$

By Lemma 1 $N(n,k)$ is the number of distinct solutions of (2) and (3). In the terminology of the previous section the

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numbers of distinct solutions of (12) and (13) are $Q(A, p)$ and $Q(n-t-A, p)$ respectively. Therefore the number of distinct solutions of (12) and (13) together is $Q(A, p)\, Q(n-t-A, p)$. Since $t \geq 1$ is arbitrary, we get

$$(14) \quad N(n, k) = \sum_{t=1}^{\infty} \sum_{A=1}^{\infty} Q(A, k/2)\, Q(n-t-A, k/2) .$$

When k is odd an entirely similar procedure gives

$$(15) \quad N(n, k) = \sum_{t=1}^{\infty} \sum_{A=1}^{\infty} Q(A, (k+1)/2)\, Q(n-t-A, (k-1)/2) .$$

We obtain now the generating function of $N(n, k)$. Consider the convolution

$$C(u, j) = \sum_{q=1}^{\infty} Q(q, j)\, Q(u-q, j) .$$

Its generating function is obtained by squaring $f_j(x)$ in (11):

$$(16) \quad f_j^2(x) = \sum_{u=1}^{\infty} C(u, j)\, x^u = x^{j(j+1)} / \prod_{i=1}^{j} (1-x^i)^2 .$$

The inner sum in (14) is $C(n-t, k/2)$ and

$$N(n, k) = \sum_{t=1}^{\infty} C(n-t, k/2) ;$$

multiplying (16) by $x/(1-x)$ and putting $j = k/2$ we get the generating function of $N(n, k)$ for k even:

$$(17) \quad \sum_{n=1}^{\infty} N(n, 2k)\, x^n = x^{k(k+1)+1} /(1-x) \prod_{i=1}^{k} (1-x^i)^2 .$$

To get the generating function of $N(n, k)$ for k odd we consider first the convolution

$$D(u,k) = \sum_{q=1}^{\infty} Q(q,(k+1)/2)\, Q(u-q,(k-1)/2) .$$

Its generating function is obtained by multiplying $f_{(k+1)/2}$ and $f_{(k-1)/2}$: $f_{(k+1)/2}(x)\, f_{(k-1)/2}(x) = \sum_{u=1}^{\infty} D(u,k)\, x^u =$

$$x^{(k+1)^2/4}/(1 - x^{(k+1)/2}) \prod_{i=1}^{(k-1)/2} (1-x^i)^2 .$$

Since the inner sum in (15) is $D(n-t,k)$, multiplying the above formula by $x/(1-x)$ we get the generating function of $N(n,k)$ for k odd:

$$(18) \qquad \sum_{n=1}^{\infty} N(n,2k-1)\, x^n = x^{k^2+1}/(1-x)(1-x^k) \prod_{i=1}^{k-1} (1-x^i)^2 .$$

To obtain the generating function for $\sum_{k=1}^{\infty} N(n,k)$ we simply add the generating functions in (17) and (18):

$$(19) \qquad \sum_{n=1}^{\infty} \left[\sum_{k=1}^{\infty} N(n,k) \right] x^n = x/(1-x) \sum_{k=1}^{\infty} x^{k^2} / \prod_{i=1}^{k} (1-x^i)^2 .$$

5. Recalling (1) we get from (19) the generating function of p_n

$$(20) \qquad \sum_{n=1}^{\infty} p_n x^n = 4x \left[1 + 2 \sum_{k=1}^{\infty} 4^k x^{k^2} / \prod_{i=1}^{k} (4^i - x^i)^2 \right] / (4-x) .$$

We develop now the connection between p_n and the unrestricted partition function $p(n)$. The generating function of $p(n)$ is

$$F(x) = 1 + \sum_{n=1}^{\infty} p(n)\, x^n = \prod_{i=1}^{\infty} (1-x^i)^{-1} ;$$

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by a theorem of Euler, [1] p. 278,

$$F(x) = 1 + \sum_{k=1}^{\infty} x^{k^2} / \prod_{i=1}^{\infty} (1 - x^i)^2 .$$

Comparing this with (19) we have

$$\sum_{n=1}^{\infty} p_n x^n = 8x\,[F(x/4) - 1/2]/(4 - x)$$

and therefore

$$(21) \quad p_n = 4^{1-n} [1 + 2 \sum_{i=1}^{n-1} p(i)] ,$$

$$(22) \quad p(n) = 2^{2n-3} (4\, p_{n+1} - p_n) .$$

Formula (22) suggests a theoretical possibility of calculating p(n) by a Monte-Carlo type of a method. We imagine that our plane random walk is simulated on an automatic computer and run off N times for n steps, and also N times for n + 1 steps. Let f_n and f_{n+1} be the frequencies of the occurence of spirals in these two runs. Then as $N \to \infty$, we have

$$2^{2n-3} (4\, f_{n+1} - f_n) \to p(n)$$

with probability 1. However, the required number N of runs for a close approximation is by far too large for any practical application, even for only moderately large n.

REFERENCE

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McGill University

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