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EXPLICIT MERTENS' THEOREMS FOR NUMBER FIELDS

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Throughout, suppose that $\mathbb{K}$ is a number field such that $\mathbb{K} \neq \mathbb{Q}$ with ring of integers $\mathcal{O}_{\mathbb{K}}$, degree $n_{\mathbb{K}}$ and discriminant $\Delta_{\mathbb{K}}$. Moreover, let $\mathfrak{a} \subset \mathcal{O}_{\mathbb{K}}$ denote *integral* ideals, $\mathfrak{p} \subset \mathcal{O}_{\mathbb{K}}$ denote *prime* ideals, $N(\mathfrak{p})$ be the *norm* of $\mathfrak{p}$ and $\kappa_{\mathbb{K}}$ be the residue of the simple pole at $s = 1$ of the Dedekind zeta-function $\zeta_{\mathbb{K}}(s)$ associated to $\mathbb{K}$. In what follows, we summarise the contributions of the author's PhD thesis [8].

To study the distribution of prime ideals in a number field, it is desirable to study the asymptotic behaviour of certain counting functions. In particular, we need the *prime ideal theorem* and *Mertens' theorems for number fields*; these are natural generalisations of the famous prime number theorem and Mertens' theorems. Landau originally proved the former in [7] and Rosen originally proved the latter in [11]. In what follows, we introduce all five of the results that this thesis proves; a conditional version (assuming the generalised Riemann hypothesis) of each result is also established to complement these unconditional results.

Explicit results. Das established the latest explicit prime ideal theorem in [1] by building upon earlier work from Lagarias and Odlyzko [6]. Throughout, an *explicit* result completely describes the order of growth of the error term therein *and* the associated implied constants. Without assuming conditions, there are significant technicalities in their results; the result only holds for an impractical range and an exceptional (or Landau–Siegel) zero may be present. Moreover, one can use the explicit prime ideal theorem to obtain explicit Mertens' theorems for number fields, although this approach would embed the same technical obstructions into the outcome. On the other hand, we were able to prove explicit Mertens' theorems for number fields with *no* technical obstructions, by making the steps in [11] completely explicit; this was joint work with Garcia (see [2]) and the main result of the thesis.

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THEOREM 1. *If $x \geq 2$ and $\mathbb{K} \neq \mathbb{Q}$, then there is a computable constant $\Upsilon_{\mathbb{K}}$ depending on $n_{\mathbb{K}}$ and $\Delta_{\mathbb{K}}$ only, such that*

$$\sum_{N(\mathfrak{p})\leq x} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})} = \log x + A_{\mathbb{K}}(x), \tag{M1}$$

$$\sum_{N(\mathfrak{p})\leq x} \frac{1}{N(\mathfrak{p})} = \log\log x + M_{\mathbb{K}} + B_{\mathbb{K}}(x), \tag{M2}$$

$$\prod_{N(\mathfrak{p})\leq x} \left(1 - \frac{1}{N(\mathfrak{p})}\right) = \frac{e^{-\gamma}}{\kappa_{\mathbb{K}} \log x}(1 + C_{\mathbb{K}}(x)), \tag{M3}$$

in which we have $|A_{\mathbb{K}}(x)| \leq \Upsilon_{\mathbb{K}}$, $|B_{\mathbb{K}}(x)| \leq 2\Upsilon_{\mathbb{K}}/\log x$, $|C_{\mathbb{K}}(x)| \leq |E_{\mathbb{K}}(x)|e^{|E_{\mathbb{K}}(x)|}$, $|E_{\mathbb{K}}(x)| \leq n_{\mathbb{K}}/2(x-1) + |B_{\mathbb{K}}(x)|$ and

$$M_{\mathbb{K}} = \gamma + \log \kappa_{\mathbb{K}} + \sum_{\mathfrak{p}} \left[\frac{1}{N(\mathfrak{p})} + \log\left(1 - \frac{1}{N(\mathfrak{p})}\right)\right]$$

satisfies $-n_{\mathbb{K}} \leq M_{\mathbb{K}} - \gamma - \log \kappa_{\mathbb{K}} \leq 0$.

Arguably the most important ingredient in our proof of Theorem 1 is the following explicit estimate for the ideal-counting function $I_{\mathbb{K}}(x) = \#\{\mathfrak{a} : N(\mathfrak{a}) \leq x\}$, which is the number fields generalisation of the floor function. This result was *by far* the most technical result in the thesis to prove, requiring an entire chapter, and it was adapted from the content of the author's published paper [9].

THEOREM 2. *If $x > 0$ and $\mathbb{K} \neq \mathbb{Q}$, then there is a computable constant $\Lambda_{\mathbb{K}}(n_{\mathbb{K}})$ depending on $n_{\mathbb{K}}$ only, such that*

$$|I_{\mathbb{K}}(x) - \kappa_{\mathbb{K}} x| < \Lambda_{\mathbb{K}}(n_{\mathbb{K}})|\Delta_{\mathbb{K}}|^{1/(n_{\mathbb{K}}+1)}(\log |\Delta_{\mathbb{K}}|)^{n_{\mathbb{K}}-1} x^{(n_{\mathbb{K}}-1)/(n_{\mathbb{K}}+1)}.$$

To prove Theorem 1, we applied Theorem 2, so the definition of $\Upsilon_{\mathbb{K}}$ will also depend on $\Lambda_{\mathbb{K}}(n_{\mathbb{K}})$. The explicit description for the constant $\Lambda_{\mathbb{K}}(n_{\mathbb{K}})$ in Theorem 2 that we obtain significantly refines the previous best, which was established by Sunley in her thesis [12, Theorem 3.3.5]. To see the margin of our improvement, refer to Table 1. Further, Theorem 2 is independently interesting, because it has potential applications in studying the zeros, size and value-distribution of L-functions defined over number fields.

Applications. By circumventing the technical issues that would be present in an explicit prime ideal theorem, Theorem 1 unlocks three new applications, which are presented below. Note that Corollary 3 is an explicit version of Bertrand's postulate for number fields (which was originally established inexplicitly in [5, Section 3]), Corollary 4 gives explicit versions of a result Nagell originally proved in [10] and Corollary 5 was jointly established with Garcia, Suh and Yu in [3].

COROLLARY 3. *For $x \geq 2$, there exists at least one prime ideal $\mathfrak{p}$ in $\mathbb{K}$ such that $N(\mathfrak{p}) \in [x, Ax]$ when $\log A \geq 2\Upsilon_{\mathbb{K}}$.*

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TABLE 1. Comparison between values of $\Lambda_{\mathbb{K}}(n_{\mathbb{K}})$ using the explicit definitions proved in [12, Theorem 3.3.5] and Theorem 2 for several choices of $n_{\mathbb{K}}$.

	$\Lambda_{\mathbb{K}}(n_{\mathbb{K}})$	
$n_{\mathbb{K}}$	[12, Theorem 3.3.5]	Theorem 2
2	1.75425×10^{30}	2.49133×10^{10}
3	8.57799×10^{44}	8.45088×10^{11}
4	7.88887×10^{59}	9.84482×10^{13}
5	1.20023×10^{75}	1.41763×10^{16}
10	1.90904×10^{153}	9.65555×10^{26}
15	1.10367×10^{234}	5.27930×10^{38}

COROLLARY 4. *Let $g \in \mathbb{Z}[X]$ be irreducible with degree $d \geq 1$, leading coefficient c, discriminant D_g and weighted discriminant $\mathbf{D}_g = |c|^{(d-1)(d-2)}|D_g|$. If $x \geq \max\{2, \sqrt{\mathbf{D}_g}\}$, then there are computable constants $Q_g = O(1)$ and $\widetilde{Q}_g(x) = o(1)$ which depend on c, d and D_g such that*

$$\left|\sum_{p\leq x}\frac{\omega_g(p)\log p}{p} - \log x\right| \leq Q_g \quad \text{and} \quad \left|\sum_{p\leq x}\frac{\omega_g(p)}{p} - \log\log x\right| \leq \widetilde{Q}_g(x),$$

where $\omega_g(p)$ denotes the number of solutions to the congruence $g(X) \equiv 0 \pmod p$.

COROLLARY 5. *Let $f = f_1 f_2 \cdots f_k \in \mathbb{Z}[X]$ be a product of distinct, irreducible non-constant polynomials $f_1, f_2, \ldots, f_k \in \mathbb{Z}[X]$ such that f has degree $d \geq 1$, leading coefficient c and discriminant D_f. If $x \geq \max\{2, |D_f|, \sqrt{\mathbf{D}_f}\}$, then there exist computable constants $\mathbf{A}_f = O(1)$ and $\mathbf{B}_f(x) = o(1)$ which depend on c, d and D_f such that*

$$\left|\frac{1}{\log\log x}\sum_{p\leq x}\frac{\omega_f(p)}{p} - k\right| \leq \frac{\mathbf{A}_f + \mathbf{B}_f(x)}{\log\log x}.$$

The broad-strokes description of our proof of Corollary 4 is that we observed that each of the objective sums is approximately equal to one of the sums in (M1) or (M2), then applied Theorem 1. The significance of Corollary 4 is that ω_g is an important multiplicative function that arises in sieve methods. In particular, Halberstam and Richert tell us how to use the sums in Corollary 4 to give upper bounds on the number of primes representable by a polynomial in [4]. Corollary 5 is another consequence of Corollary 4; this result tells us that there is a finite list of primes that certifies the number of irreducible factors of a polynomial $f \in \mathbb{Z}[X]$.

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