BULL. AUSTRAL. MATH. SOC. 16A70
VOL. 22 (1980), 285-289.

SOME COMMUTATIVITY RESULTS FOR RINGS

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It is proved that certain rings satisfying generalized-commutator constraints of the form $[x^m, y^n, y^n, \ldots, y^n] = 0$ must have nil commutator ideal.

Let R be an associative ring; and define generalized commutators $[x_1, x_2, \ldots, x_k]$, $k \geq 2$, as follows: $[x_1, x_2] = x_1x_2 - x_2x_1$; and for $k > 2$, $[x_1, x_2, \ldots, x_k] = [[x_1, \ldots, x_{k-1}], x_k]$. For $x_1 = x$ and $x_2 = x_3 = \ldots = x_k = y$, abbreviate $[x, y, \ldots, y]$ by $[x, y]_k$.

A few years ago it was proved independently by Herstein [2] and by Anan'in and Zyabko [1] that R has nil commutator ideal if for each $x_1, x_2 \in R$ there exist positive integers $n_1 = n_1(x_1, x_2)$ and $n_2 = n_2(x_1, x_2)$ such that $\left[x_1^{n_1}, x_2^{n_2}\right] = 0$; more recently Herstein [3] has established the same conclusion under the hypothesis that for all $x_1, x_2, x_3 \in R$ there are positive integers n_1, n_2, n_3 such that $\left[x_1^{n_1}, x_2^{n_2}, x_3^{n_3}\right] = 0$. The following conjecture arises naturally from this work.

CONJECTURE. *Let* $k > 1$ *and suppose that for each* $x, y \in R$ *, there*

Received 17 April 1980. Professor Howard E. Bell was supported by the Natural Sciences and Engineering Research Council of Canada, Grant No. A3961.

exist positive integers m, n *such that* $[x^m, y^n]_k = 0$. *Then the commutator ideal of* R *is nil.*

Given the complexity of [2] and [3], it would appear that no proof of this conjecture is in sight; indeed, even the $k = 3$ case seems difficult. Hence the following case may be of interest.

THEOREM 1. *Let* R *be a ring and let* M *be a fixed positive integer. Suppose that for each* $x, y \in R$ *there exist positive integers* $m = m(x, y) \leq M$ *and* $n = n(x, y)$ *such that* $[x^m, y^n, y^n] = 0$. *Then the commutator ideal of* R *is nil.*

Proof. By proceeding as in [3], we can reduce the problem to establishing commutativity of R under the additional hypotheses that R is prime and torsion-free, and that every element of R is either regular or nilpotent - hypotheses which we henceforth assume. Moreover, in view of the result of [1] and [2], we need only show that for each $x, y \in R$, there exist m, n for which $[x^m, y^n] = 0$.

Clearly this condition holds for nilpotent y , so we assume that y is regular, and choose $m \leq M$ and n_1 for which $[x^m, y^{n_1}, y^{n_1}] = 0$. Taking $x_1 = x^{2m}$, let w and n_2 be such that $\left[x_1^w, y^{n_2}, y^{n_2}\right] = 0$; and note that for $v = 2w$ and $n = n_1n_2$, we have $[x^m, y^n, y^n]$ and $[x^{mv}, y^n, y^n] = 0$, so that $[x^m, y^n]$ is nilpotent by [3, Lemma 1]. Thus, if a is chosen to be an appropriate power of $[x^m, y^n]$ and $z = y^n$, we have $a^2 = [a, z] = 0$.

For any $u \in R$ and $i \geq 1$, there exist $m_i \leq M$ and s_i such that $[u^{m_i}, (iz+a)^{s_i}, (iz+a)^{s_i}] = 0$. Taking $i = 1, 2, \ldots, 2M+1$ and using the pigeon-hole principle, we get i_1, i_2, i_3 with $1 \leq i_1 < i_2 < i_3 \leq 2M+1$ for which $m_{i_1} = m_{i_2} = m_{i_3}$. Denoting this common value by q and defining $s = s_{i_1}s_{i_2}s_{i_3}$, we have $\left[u^q, (i_jz+a)^s, (i_jz+a)^s\right] = 0$,

https://doi.org/10.1017/S0004972700006584 Published online by Cambridge University Press

$j = 1, 2, 3$; it now follows by use of the fact that $a^2 = [a, z] = 0$ that

(1) $$i_j^{2s}v_1 + i_j^{2s-1}v_2 + i_j^{2s-2}v_3 = 0 , \quad j = 1, 2, 3 ,$$

where v_1, v_2, v_3 are respectively defined to be $[u^q, z^s, z^s]$, $2s[u^q, z^{s-1}a, z^s]$ and $s^2[u^q, z^{s-1}a, z^{s-1}a]$. The 3×3 coefficient matrix in (1) is obtained by multiplying the rows of a Vandermonde matrix by non-zero integers, so the fact that R is torsion-free yields $v_1 = v_2 = v_3 = 0$; and since $a^2 = 0 = [a, z^{s-1}]$ and z is regular, the statement $v_3 = 0$ reduces to the result that $au^qa = 0$.

If $b \in R$ and $b^2 = 0$, we claim that $aba = 0$. For if $v \in R$, there exists $q \leq M$ for which $a(avab+b)^qa = 0$, which yields $ab(avab)^{q-1}a = 0 = (abav)^q$. Thus $abaR$ is a nil right ideal of bounded index, which by the Nagata-Higman Theorem [4, p. 274] must be nilpotent; and the primeness of R forces $aba = 0$.

Now if $c, d \in R$ with $cd = 0$, $(dvc)^2 = 0$ for arbitrary $v \in R$, and hence $advca = 0$. Since R is prime, we have $ad = 0$ or $ca = 0$, so $cad = 0$. Thus, insertion of a as a factor preserves triviality of products; and from $au^qa = 0$ we can conclude $(au)^{q+1} = 0$. Therefore $(au)^{M+1} = 0$ for all $u \in R$, and another appeal to the Nagata-Higman Theorem gives $a = 0$. Thus we have that any power of the nilpotent element $[x^m, y^n]$ whose square is 0 must also be 0 , so $[x^m, y^n] = 0$. The proof of Theorem 1 is now complete.

The following theorem, except of having its own interest, shows that the conjecture is implied by the Köthe Conjecture.

THEOREM 2. *Let R be a ring with no non-zero nil right ideals, and let $k > 1$. Suppose that for each $x, y \in R$ there exist $m, n \geq 1$ such that $[x^m, y^n]_k = 0$. Then R is commutative.*

Proof. Let a be in R with $a^2 = 0$, and let x be an arbitrary

https://doi.org/10.1017/S0004972700006584 Published online by Cambridge University Press

element of R. Take $m, n > 1$ such that $[(a+ax)^m, (ax)^n]_k = 0$. This condition reduces to $(ax)^t a = 0$ where $t = m - 1 + (k-1)n$, hence aR is nil, so $a = 0$. Consequently, R has no non-zero nilpotent elements; and by a well-known result it is a subdirect product of domains. Our proof will be complete once we establish that each of these domains must be commutative. This is easily verified as in [3] for such a domain of prime characteristic; and such a torsion-free domain is commutative by the following lemma.

LEMMA. *Let R be a torsion-free domain, and let $k > 1$. Suppose that for each $x, y \in R$, there exist $m, n \geq 1$ such that $[x^m, y^n]_k = 0$. Then R is commutative.*

Proof. Assume $k \geq 3$ and let $x, y \in R$. Then there exist m, r_1 such that $[x^m, y^{r_1}]_k = 0$ and there exist m', r_2 such that $[x^{(2m)m'}, y^{r_2}]_k = 0$. It can easily be verified that

$$[x^m, y^r]_k = [x^{(2m)m'}, y^r]_k = 0$$

for $r = r_1 r_2$. Taking $x_0 = x^m$ and $t = 2m'$ and letting δ be the derivation defined by $u\delta = [u, y^r]$, we have $x_0\delta^{k-1} = \left(x_0^t\right)\delta^{k-1} = 0$. Now $t \geq 2$ and $k \geq 3$, so $t(k-2) \geq k - 1$ and therefore $\left(x_0^t\right)\delta^{t(k-2)} = 0$. Expanding this last equation and using the fact that $x_0\delta^{k-1} = 0$, we obtain a non-zero integer s for which $s\left(x_0\delta^{k-2}\right)^t = 0$ and our hypotheses on R yield $x_0\delta^{k-2} = 0$, which we may express as $[x^m, y^r]_{k-1} = 0$. Thus we work back to the $k = 2$ case of [1] and [2].

The entire problem becomes much more tractable for rings with 1. Indeed, we can establish the following theorem.

THEOREM 3. *For all $k > 1$ the conjecture is true for rings R with* 1.

We omit the details of the proof. The computational details are similar to the ones already presented. We merely note that it suffices to establish commutativity of R under the additional hypotheses that R is prime and torsion-free. A Vandermonde argument is used to prove that if these additional hypotheses hold, then R has no non-zero nilpotent elements so it is a domain, and it is commutative by the result of the lemma.

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