NON-ISOMORPHIC NON-HYPERFINITE FACTORS

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Introduction. A von Neumann algebra is called *hyperfinite* if it is the weak closure of an increasing sequence of finite-dimensional von Neumann subalgebras. For a separable infinite-dimensional Hilbert space the following is known: there exist hyperfinite and non-hyperfinite factors of type II_1 (**4**, Theorem 16′), and of type III (**8**, Theorem 1); all hyperfinite factors of type II_1 are isomorphic (**4**, Theorem 14); there exist uncountably many non-isomorphic hyperfinite factors of type III (**7**, Theorem 4.8); there exist two non-isomorphic non-hyperfinite factors of type II_1 (**10**), and of type III (**11**). In this paper we will show that on a separable infinite-dimensional Hilbert space there exist three non-isomorphic non-hyperfinite factors of type II_1 (Theorem 2), and of type III (Theorem 3).

Section 1 contains an exposition of crossed product, which is developed mainly for the construction of factors of type III in § 3. The second half of § 1 contains a "cutting" lemma, important for our final result.

In § 2 we introduce a new algebraic property of von Neumann algebra: property C. We construct a non-hyperfinite factor of type II_1 which has properties C and Γ (**4**, Definition 6.1.1). Then we establish the non-isomorphism of three non-hyperfinite factors of type II_1 by showing that C does not hold (Γ does) for a non-hyperfinite factor of type II_1 used by Schwartz (**10**, Corollary 12).

Section 3 contains a similar but more complicated construction of three non-isomorphic non-hyperfinite factors of type III.

In this paper, all Hilbert spaces are complex and we use the following notation: $B(H)$ denotes the algebra of all bounded linear operators on a Hilbert space H, I the identity operator, S' the von Neumann algebra of operators which are permutable with the elements in $S \subset B(H)$, $T_i \to T$ strong operator convergence, $||T||_2 = (\mathrm{tr}(T^*T))^{1/2}$ the trace norm of an operator in a factor of type II_1. Isomorphism (automorphism) of von Neumann algebras will mean *-isomorphism (*-automorphism). R denotes a von Neumann algebra on H, a vector x in H is called separating for R if $t \in R$, $tx = 0$ implies $t = 0$, cyclic for R (equivalently, separating for R') if the closed linear subspace generated by Rx is H. G denotes a group with identity e. G is called ICC (infinite class of conjugates) if $\{hgh^{-1}|h \in G\}$ is infinite for each $e \neq g \in G$; $H \otimes G$ the Hilbert space of all functions x on G with all $x(g) \in H$ and

$$||x||^2 = \sum_g ||x(g)||^2 < \infty;$$

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$u\colon G \to B(H)$ a unitary representation such that

$$u(g)Ru(g^{-1}) = R, \quad P_g\colon H \otimes G \to H$$

the partial isometry with $P_g x = x(g)$, α^g (for any vector or operator α) the function on G with value α at g and value 0 elsewhere. Each $T \in B(H \otimes G)$ has a matrix representation: $T = (T_{g,h})$, $T_{g,h} = P_g T P_h \in B(H)$ for $g, h \in G$ such that

$$(Tx)(g) = \sum_h T_{g,h} x(h).$$

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1. The crossed product $R \otimes u$**.** Suppose that H, R, G, u, and $H \otimes G$ are as described in the Introduction.

Definition 1. A bounded linear operator on $H \otimes G$, to be denoted $L(t)$, is called an *R-shifter* if it is determined by the formula

$$(L(t)x)(g) = \sum_h t(h)u(h)x(gh)$$

for some $t\colon G \to B(H)$ with the property that the sum $\sum_h t(h)u(h)x(gh)$ converges in the strong topology of H for all $x \in H \otimes G$, $g \in G$ (it is easily verified that s^g is such a function t for all $s \in R$, $g \in G$, and $||L(s^g)|| = ||s||$).

Definition 2. The set of all R-shifters, to be denoted $R \otimes u$, is called the *crossed product* of R by u.

LEMMA 1. *$T \in B(H)$ is of the form $L(t)$ (with t necessarily unique) if and only if: $T_{g,h} = T_{e,g^{-1}h}$ and $T_{e,g}u(g^{-1}) \in R$ for all g, h (then $t(g) = T_{e,g}u(g^{-1})$).*

Proof. This is easily verified.

COROLLARY 1. *If $L(t)$ and $L(s)$ are R-shifters and c is a complex number, then $L(I^e)$ is the identity operator on $H \otimes G$ and*

$$cL(t) = L(ct), \qquad L(t) + L(s) = L(t + s),$$
$$L(t)L(s) = L(t * s), \qquad (L(t))^* = L(t^*),$$

where

$$(ct)(g) = ct(g), \qquad (t + s)(g) = t(g) + s(g),$$

(1) $$(t * s)(g) = \sum_h t(h)u(h)s(h^{-1}g)u(h^{-1}),$$

(2) $$t^*(g) = u(g)(t(g^{-1}))^* u(g^{-1}).$$

Proof. This is easily verified (use Lemma 1 and matrix representations).

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COROLLARY 2. *Suppose that $x \in H$ is separating for R. Then $x^e \in H \otimes G$ is separating for $R \otimes u$.*

Proof. Suppose that $T \in R \otimes u$ and $Tx^e = 0$. Then we have:

$$T_{g,h}x = T_{h^{-1}g,e}x = (Tx^e)(h^{-1}g) = 0,$$

and hence $T_{g,h} = 0$ for all g, h. Thus $T = 0$.

THEOREM 1. *$R \otimes u$ is a von Neumann algebra on $H \otimes G$.*

Proof. $R \otimes u$ is a *-subalgebra of $B(H \otimes G)$ containing the identity operator by Corollary 1 to Lemma 1. To show that $R \otimes u$ is strongly closed, we let (T_i) be a net in $R \otimes u$ with $T_i \to T$. Then

$$(T_i)_{g,h} = P_g T_i P_h \to P_g T P_h = T_{g,h} \quad \text{and} \quad (T_i)_{e,g} u(g^{-1}) \to T_{e,g} u(g^{-1});$$

since R is strongly closed, Lemma 1 shows that $T \in R \otimes u$. Thus $R \otimes u$ is strongly closed and hence it is a von Neumann algebra.

Corollary 1 to Lemma 1 shows that $(R \otimes u)_0 = \{L(t) | t \text{ of finite support}\}$ is a *-subalgebra of $R \otimes u$. As in (**3**, Lemma 12.3.4),

$$(R \otimes u)' = \{L(t^g) | t \in R, g \in G\}'$$

and $(R \otimes u)_0$ is strongly (and weakly) dense in $R \otimes u$.

LEMMA 2. *Suppose that R is a factor, and G is ICC. Then $R \otimes u$ is also a factor.*

Proof. Let $L(t)$ be in the centre of $R \otimes u$. Then:

$$L(t)L(I^h) = L(I^h)L(t) \quad \text{for all } h \in G;$$

$$L(t)L(s^e) = L(s^e)L(t) \quad \text{for all } s \in R.$$

By (1), we have

(3) $$t(g)u(g) = u(g)t(h^{-1}gh) \quad \text{for all } h, g \in G;$$

(4) $$t(g)s = st(g) \quad \text{for all } s \in R, g \in G.$$

Suppose that $t(g) \neq 0$ for some $g \neq e$. Then for every $x \in H$:

(5) $$||L(t)x^e||^2 = \sum_h ||(L(t)_{h,e}x||^2 = \sum_h ||t(h^{-1})u(h^{-1})x||^2 = \sum_h ||t(h)x||^2.$$

In this sum there are infinitely many summands equal to $||t(g)x||^2 \neq 0$ since G is ICC. Hence $t(g)x = 0$ for all $x \in H$. Thus $t(g) = 0$ for all $g \neq e$, or $L(t) = t(e)^e$. Since R is a factor, (4) implies that $t(e)$, hence $L(t)$, is a scalar multiple of the identity operator.

Remark 1. In the special case that H (hence R) is the complex field, u the identity representation of G, $R \otimes u$ is just the group algebra associated with G, which we shall denote by $A(G)$. $A(G)$ is a factor of type II_1 if G is ICC. I^e is a trace vector of $A(G)$, and $||L(t)||_2 = (\sum_g |t(g)|^2)^{1/2}$.

LEMMA 3. *$R \otimes u$ is purely infinite if R is purely infinite (in the case that $R \otimes u$ is a factor on a separable Hilbert space, this is equivalent to $R \otimes u$ is of type* III).

Proof. First we show, assuming $0 \neq L(t) \geqq 0$, that $0 \neq t(e) \geqq 0$. We have $L(t) = L(s)(L(s))^*$ for some s,

$$t(e) = (s \otimes s^*)(e) = \sum_h s(h)u(h)s^*(h^{-1})u(h^{-1})$$
$$= \sum_h s(h)u(h)u(h^{-1})(s(h))^*u(h)u(h^{-1}) = \sum_h s(h)(s(h))^* \geqq 0$$

and $t(e) = 0$ would imply: $s(h) = 0$ for all h, $s = 0$, $L(s) = 0$, $L(t) = 0$.

Now we use an argument of Sakai (**9**, § 3). Suppose, if possible, that there exists a non-zero finite projection $L(p)$. Then $p(e) \in R$ and $0 \neq p(e) \geqq 0$. Hence, for some non-zero projection $q \in R$: $\lambda p(e) \geqq q \geqq 0$ for some $\lambda > 0$, thus $q = q_1 p(e)$ for some $q_1 \in R$. To complete the proof, it is sufficient to show that q is finite. By Sakai's proposition (**9**, Proposition 2′), we may suppose that $t_n \in qRq$, $t_n \to 0$, and we need only to show that $t_n^* \to 0$.

Let $L_n = L(t_n^e)$. Then $L_n \to 0$ since $\sup_n ||L_n|| = \sup_n ||t_n|| < \infty$ and $L_n(x^g) = (t_n x)^g \to 0$ for all $x \in H$, $g \in G$. Hence $L_n L(p) \to 0$. Then by Sakai's proposition (**9**, Proposition 2): $L(p)L_n^* \to 0$, hence $p(e)t_n^* = (L(p)L_n^*)_{e,e} \to 0$. Thus $t_n^* = qt_n^* = q_1 p(e) t_n^* \to 0$ as required, and the proof is complete.

Let u_1: $K \to B(H \otimes K)$ be a unitary representation of a group K such that $u_1(k)(R \otimes u)u_1(k^{-1}) = R \otimes u$ for all $k \in K$. We make the convention that $R \otimes u \otimes u_1$, $H \otimes G \otimes K$, and $\alpha^{g,k}$ shall mean $(R \otimes u) \otimes u_1$, $(H \otimes G) \otimes K$, and $(\alpha^g)^k$, respectively. We still write $L(t)$ for an element of $R \otimes u \otimes u_1$, but where t is an R-valued function on the Cartesian product $G \times K$ such that $L(t(\cdot, k)) \in R \otimes u$ for each $k \in K$. Let x be a separating vector for R, then by Corollary 2 to Lemma 1, $\xi = x^{e,e}$ is a separating vector for $R \otimes u \otimes u_1$. Suppose that $||Tu_1(k)x^e|| = ||Tx^e||$ for all $T \in R \otimes u$. Then applying (5) twice, we have

$$||L(t)\xi||^2 = \sum_k \sum_g ||t(g, k)u(g)x||^2. \tag{6}$$

For any function t on $G \otimes K$, let $\bar{t}$ denote the function: $\bar{t}(e, k) = t(e, k)$, if $k \in \Delta$, $\bar{t}(g, k) = 0$ if $g \neq e$ or $k \notin \Delta$, where Δ is a subgroup of K. Let R_1 be the set of all elements of $R \otimes u \otimes u_1$ of the form $L(\bar{t})$. R_1 is certainly a vector space. Suppose, further, that $u_1(k)R^e u_1(k^{-1}) = R^e$ for all $k \in K$, where $R^e = \{L(s^e) | s \in R\}$. Then a computation based on (1) shows that R_1 is a $*$-subalgebra of $R \otimes u \otimes u_1$. Now suppose $L(\bar{t}_\alpha) \to L(t)$. By (6), we have

$$||(\bar{t}_\alpha(g, k)u(t) - t(g, k)u(g))x|| \to 0 \quad \text{for each } (g, k) \in G \times K.$$

Hence $t(g, k) = 0$ if $g \neq e$ or $k \notin \Delta$. This shows that R_1 is strongly closed, i.e. a von Neumann subalgebra of $R \otimes u \otimes u_1$. We note that the set R_F of all finite sums of $s^{e,k}$, $s \in R$, $k \in \Delta$, form a strongly dense *-subalgebra of R_1.

LEMMA 4. *Let $R_2 = R \otimes u \otimes u_1$, $H \otimes G \otimes K$, x, ξ, $\Delta \subset K$, R_1 be as described in the preceding discussion, i.e. $||T_0 u_1(k) x^e|| = ||T_0 x^e||$ for all $T_0 \in R \otimes u$, $k \in K$, and $u_1(k) R^e u_1(k^{-1}) = R^e$ for all $k \in K$. Suppose that the positive linear functional $f(S) = (S\xi|\xi)$ on R_2 is such that $f(TS) = f(ST)$ for all $T \in R_2$, $S \in R_1$. Then there exists a projection P of norm one from the Banach space R_2 (with the operator norm) onto its subspace R_1 such that*

$$P(L(t)) = L(\bar{t}) \quad \text{for all } L(t) \in R_2. \tag{7}$$

Proof. Let A^+ denote the positive part of an operator algebra A. For each $T \in R_2^+$, define $f_T(S) = f(TS)$, $S \in R_1$. Then f_T is a positive linear functional on R_1 satisfying: $f_T(S) \leqq ||T|| f(S)$ for all $S \in R_1^+$. Also, the trace $f(S) = (S\xi|\xi)$ on R_1 is regular in the sense that if E is a projection, $f(E) = 0$ implies $E = 0$. In fact, $||E\xi||^2 = (E\xi|\xi) = 0$ implies $E = 0$, since ξ is a separating vector for R_2. By (**12**, Lemma 14.1), there exists a unique positive operator T' in R_1 such that $f(TS) = f(T'S)$ for all $S \in R_1$. This mapping $T \mapsto T'$ of R_2^+ to R_1^+ can be uniquely extended (via the canonical decomposition of an operator) to a linear mapping P: $T \mapsto T'$ from R_2 onto R_1 such that $f(TS) = f(T'S)$ for all $S \in R_1$.

It is clear that P is a projection. Now, for any $T_1, T_2 \in R_2$,

$$f((T_1'T_2)'S = f(T_1'T_2S) = f(T_2'ST_1') = f(T_1'T_2'S) = f((T_1T_2')'S)$$

for all $S \in R_1$. Hence, $(T_1'T_2)' = T_1'T_2' = (T_1T_2')'$ for $T_1, T_2 \in R_2$. $(T^*)' = (T')^*$ since $T \geqq 0$ implies $T' \geqq 0$. Moreover, for any $T \in R_2$,

$$0 \leqq ((T - T')^*(T - T'))' = (T^*T)' - T^{*\prime}T',$$

i.e. $T^{*\prime}T' \leqq (T^*T)'$. For any $T \in R_2$, $0 \leqq T^*T \leqq ||T^*T||I$. Thus we have

$$0 \leqq T^{*\prime}T' \leqq (T^*T)' \leqq ||T^*T||I,$$
$$||T'|| = (||T'^*T'||)^{1/2} \leqq ||T^*T||^{1/2} = ||T||.$$

Hence the projection P from the Banach space R_2 onto its subspace R_1 is of norm one.

We know that P satisfies (7). Let R_0 denote the dense (weakly, strongly) *-subalgebra of R_2 consisting of all operators $L(t)$ with t of finite support on $G \times K$. For an arbitrary $L(t_0) \in R_0$, $L(\bar{t}_0) = \sum_{k \in \Delta} L(t(e, k)^{e,k})$ is a finite sum of bounded operators, hence a well-defined element in R_1. We have

$$(L(t_0)L(s^{e,k})\xi|\xi) = (t_0(e, k^{-1})x_0|x_0) = (L(\bar{t}_0)L(s^{e,k})\xi|\xi)$$

for all $k \in K$. Since R_F is dense in R_1, we conclude that

$$(L(t_0)S\xi|\xi) = (L(\bar{t}_0)S\xi|\xi) \quad \text{for all } S \in R_1.$$

Hence $P(L(t)) = L(\bar{t})$ for all $L(t) \in R_0$.

Now, let $T = L(t)$ be an arbitrary operator in R_2. And let $T_i = L(t_i) \to T$, where $T_i \in R_0$, $||T_i|| \leqq ||T||$, $i = 1, 2, \ldots$ (such a sequence exists because of Kaplansky's density theorem and the metrizability of the unit ball in strong operator topology (**1**, § 3)). For each $S \in R_1$,

$$(T_i'\xi|S\xi) = (T_i\xi|S\xi) \to (T\xi|S\xi) = (T'\xi|S\xi).$$

Consequently, $(T_i'\xi|y) \to (T'\xi|y)$ for all $y \in H \otimes G \otimes K$. Since ξ is cyclic for R_2', we have $(T_i'z|y) \to (T_i'z|y)$ for all $z, y \in H \otimes G \otimes K$. Hence P is continuous from the unit ball of R_2 with strong operator topology to the unit ball of R_1 with weak operator topology. $||(T - T_i)\xi||^2 \to 0$ and (6) imply that

$$||(t(g, k)u(t) - t_i(g, k)u(g))x|| \to 0$$

for each $(g, k) \in G \times K$. Since the norms of all

$$t(g, k), \quad t_i(g, k), \qquad i = 1, 2, \ldots, (g, k) \in G \times K,$$

are bounded by $||T||$, this implies that $t_i(g, k)u(g) \to t(g, k)u(g)$ for each $(g, k) \in G \times K$. In particular,

$$t_i(e, k) \to t(e, k) \quad \text{weakly for each } k \in \Delta. \tag{8}$$

On the other hand, suppose that $P(T) = L(\bar{s}) \in R_1$. Then

$$L(\bar{s}) = P(T) = \text{weak lim } P(T_i) = \text{weak lim } L(\bar{t}_i);$$

$$((s(e, k) - t_i(e, k))x|s'x) = ((L(\bar{s}) - L(\bar{t}_i))\xi|(s'x)^{e,k^{-1}}) \to 0$$

for each $k \in \Delta$, $s' \in R'$. Hence $t_i(e, k) \to s(e, k)$ weakly for each $k \in \Delta$, since x is cyclic for R'. In view of (8), we have $s(e, k) = t(e, k)$ for all $k \in \Delta$. Thus, $P(L(t)) = L(\bar{t})$. This completes the proof of the lemma.

Remark 2. For the special case that R is the complex field, $G = \{e\}$, $\Delta \subset K$, then P: $L(t) \to L(t|_\Delta)$, where $t|_\Delta(k) = t(k)$ if $k \in \Delta$, $t|_\Delta(k) = 0$, if $k \notin \Delta$. This case has already been proved in (**4**, Appendix).

LEMMA 5 (Pukanszky (**8**, Lemma 10)). *Let G be a group and B a subset of G. Suppose that there exists a subset $S \subset B$ and two elements $g_1, g_2 \in G$ such that* (i) *$S \cup h_1Sg_1 = B$ and* (ii) *the sets S, $g_2^{-1}Sg_2$, $g_2Sg_2^{-1} \subset B$ are pairwise disjoint. Let $f(g)$ be a complex-valued function on G such that $\sum_{g \in G} |f(g)|^2 < \infty$, and*

$$\Big(\sum_{g \in G} |f(g_i g g_i^{-1}) - f(g)|^2\Big) < \epsilon \qquad (i = 1, 2).$$

Then $(\sum_{g \in B} |f(g)|^2) < 14\epsilon$.

2. Non-isomorphic factors of type II$_1$. The following definitions describe the properties we shall use to distinguish between factors.

Definition 3 (**4**, Definition 6.1.1). A factor R, of type II$_1$, is said to have property Γ if for any given finite set of elements $T_1, T_2, \ldots, T_n \in R$ and any

$\epsilon > 0$, there exists a unitary $U \in R$ with $\mathrm{tr}(U) = 0$ and $||U^*T_iU - T_i||_2 < \epsilon$, $i = 1, 2, \ldots, n$.

Definition 4. A von Neumann algebra R is said to have property C, if for each sequence U_k $(k = 1, 2, \ldots)$ of unitary operators in R with the property that strong $\lim U_k^*TU_k = T$ for each $T \in R$ there exists a sequence V_k $(k = 1, 2, \ldots)$ of mutually commuting operators in R such that

$$\text{strong}\lim(U_k - V_k) = 0.$$

Since algebraic isomorphism between two von Neumann algebras preserves the strong convergence of sequences of operators (**6**), it preserves property C as well as property Γ.

Let Π denote the group of all finite permutations on the set of all natural numbers, Φ_2 the free group with two generators, and $\Pi \times \Phi_2$ their direct product. Then it is known that $A(\Pi)$ is hyperfinite, but $A(\Phi_2)$ and $A(\Pi \times \Phi_2)$ are non-hyperfinite; $A(\Pi)$ and $A(\Pi \times \Phi_2)$ have property Γ, but $A(\Phi_2)$ does not (**10**).

We construct below a factor $A(\Phi \otimes \Delta)$ of type II_1 for which we shall prove the following lemmata.

LEMMA 6. *$A(\Phi \otimes \Delta)$ has property* Γ.

LEMMA 7. *$A(\Phi \otimes \Delta)$ has property* C.

LEMMA 8. *Neither $A(\Pi)$ nor $A(\Pi \times \Phi_2)$ has property* C.

In view of the above lemmata, we have the following theorem.

THEOREM 2. *$A(\Pi)$, $A(\Phi_2)$, $A(\Pi \times \Phi_2)$, and $A(\Phi \otimes \Delta)$ are four pairwise non-isomorphic factors of type* II_1.

Construction of $A(\Phi \otimes \Delta)$. Let Φ be a free group with an infinite system of generators $\{a_0, b_0, a_1, b_1, a_2, b_2, \ldots\}$. Let ρ^i be the permutation on the set of free generators of Φ which permutes a_i with b_i, and leave all other generators fixed, $i = 1, 2, \ldots$. Let Δ be the group of permutations on the set of free generators of Φ which is generated by ρ_i, $i = 1, 2, \ldots$. Δ is abelian. It is clear that each $\lambda \in \Delta$ induces an automorphism $g \to \lambda g$ of Φ in an obvious way, i.e. via the word representation of $g \in \Phi$. Hence, Δ can be regarded as an abelian group of automorphisms of Φ.

Let $\Phi \otimes \Delta = \{(g, \lambda) | \, g \in \Phi, \ \lambda \in \Delta\}$. Define $(g, \lambda)(h, \mu) = (g\lambda h, \lambda\mu)$ for (g, λ), $(h, \mu) \in \Phi \otimes \Delta$. Then, it is easy to check that $\Phi \otimes \Delta$ under this multiplication is a countable ICC group. Therefore, $A(\Phi \otimes \Delta)$ is a factor of type II_1 on a separable Hilbert space.

Proof of Lemma 6. Let S be a finite subset of $\Phi \otimes \Delta$. Let q be the largest natural number j such that a_j or b_j appears in the reduced word representation of the first coordinate of some element in S. Then, $(e, e) \neq (e, \rho_{q+1}) \in \Phi \otimes \Delta$

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clearly satisfies $(e, \rho_{q+1})k = k(e, \rho_{q+1})$ for all $k \in S$. By (**4**, Lemma 6.1.1), we conclude that $A(\Phi \otimes \Delta)$ has property Γ.

Proof of Lemma 7. Let $u_k = L(t_k)$ $(k = 1, 2, \ldots)$ be a sequence of unitary operators in $A(\Phi \otimes \Delta)$ with the property that $\lim \|U_k^* T U_k - T\|_2 = 0$ for each $T \in A(\Phi \otimes \Delta)$ (this is equivalent to strong lim $U_k^* T U_k = T$). Let Δ denote the subgroup (e, Δ) of $\Phi \otimes \Delta$. We claim that

$$V_k = L(t_k|_\Delta) \in A(\Delta) \subset A(\Phi \otimes \Delta) \qquad (k = 1, 2, \ldots)$$

is a bounded (by Lemma 4 and Remark 2, $\|V_k\| \leqq \|U_k\| = 1$) sequence of mutually commuting operators (since Δ is abelian) required for having property C.

Let $T_i = L(I^{g_i})$ $(i = 1, 2)$, where $g_1 = (a_0, e)$, $g_2 = (b_0, e)$. Let S be the subset $\{(g, \lambda) | \lambda \in \Delta, g \in \Phi, g$ in reduced word representation ends in a non-zero power of $a_0\}$ of $\Phi \times \Delta$. Put $B = \Phi \otimes \Delta \setminus (e, \Delta)$. We note that $B = S \cup g_1 S g_1^{-1}$, and $S, g_2 S g_2^{-1}, g_2^{-1} S g_2$ are pairwise disjoint subsets of B. Given any $\epsilon > 0$, there is an $N = N(\epsilon)$ such that $k > N$ implies

$$\|U_k^* T_i U_k - T_i\|_2 = \|T_i^* U_k T_i - U_k\|_2$$
$$= \left(\sum_{g \in (\Phi \otimes \Delta)} |t_k(g_i g g_i^{-1}) - t_k(g)|^2\right)^{1/2} < \epsilon \qquad (i = 1, 2).$$

By Lemma 5, we have

$$\|U_k - V_k\|_2 = \|L(t_k) - L(t_k|_\Delta)\|_2 = \left(\sum_{g \in B} |t_k(g)|^2\right)^{1/2} < 14\epsilon$$

for all $k > N$. Hence strong $\lim(U_k - V_k) = 0$.

Proof of Lemma 8. Let g_i be the element in Π which permutes i with $i + 1$ and leaves all other natural numbers fixed, for each $i = 1, 2, \ldots$. Given any operator $T = L(t)$ in $A(\Pi)$, let $T' = L(t') \in A(\Pi)$ be such that $t'(g) = 0$ for all $g \in \Pi$ except on a finite subset S of Π, and $\|T - T'\|_2 < \epsilon/2$. Let N be the largest natural number which is permuted by some element in S. It is easy to see that U_i commutes with T' for all $i > N$. Thus, $i > N$ implies

$$\|U_i^* T U_i - T\|_2 \leqq \|U_i^*(T - T')U_i\|_2 + \|T - T'\|_2 < \epsilon.$$

Hence $\lim \|U_i^* T U_i - T\|_2 = 0$, or equivalently, strong lim $U_i^* T U_i = T$ for each $T \in A(\Pi)$.

Suppose that $A(\Pi)$ has property C. Then there exists a sequence V_i $(i = 1, 2, \ldots)$ of mutually commuting operators in $A(\Pi)$ such that strong $\lim(U_i - V_i) = 0$. Now, since $g_i g_{i+1} \neq g_{i+1} g_i$ for $i = 1, 2, \ldots$, we have

$$\sqrt{2} = \|L(I^{g_i g_{i+1}} - I^{g_{i+1} g_i}) I^e\| = \|U_i U_{i+1} - U_{i+1} U_i\|_2$$
$$\leqq \|(U_i - V_i) U_{i+1}\|_2 + \|V_i(U_{i+1} - V_{i+1})\|_2 + \|(V_{i+1} - U_{i+1}) V_i\|_2$$
$$+ \|U_{i+1}(V_i - U_i)\|_2 \leqq 2\|U_i - V_i\|_2 + 2\|V_i\| \, \|U_{i+1} - V_{i+1}\|_2,$$

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the last step follows since the trace is unitary invariant and

$$|\mathrm{tr}(ST)| \leqq ||S|| \cdot |\mathrm{tr}(T)|.$$

By the uniform boundedness principle, the strong convergence of $(U_i - V_i)$ implies that $\{||V_i - U_i||\}$, $i = 1, 2, \ldots$, and $\{||V_i||\}$, $i = 1, 2, \ldots$, are bounded by some positive number M. Hence, each term in the last expression of the above inequality approaches 0 as $i \to \infty$. This contradiction shows that $A(\Pi)$ does not have property C. Replace all Π by $\Pi \times \Phi_2$ and g_i by (g_i, e) in the preceding proof, we also conclude that $A(\Pi \times \Phi_2)$ does not have property C.

3. Non-isomorphic factors of type III. The following algebraic property of von Neumann algebras was introduced by Pukanszky (**8**) to distinguish a pair of factors of type III.

Definition 5. A von Neumann algebra is said to have property L, if there exists a sequence U_k $(k = 1, 2, \ldots)$ of unitary operators in R such that weak lim $U_k = 0$ and strong lim $U_k T U_k^* = T$ for each $T \in R$.

Our construction of non-isomorphic factors of type III follows the construction in Pukanszky (**8**) and the construction of the new factor of type II_1 in § 2. R_1 is the factor M_1 in (**8**) and R_2 is the factor M_2 in (**8**).

Construction of R_1. Let G be an infinite group and let $x_0 = \{0, 1\}$. Let μ_0 be the measure on X_0 with $\mu_0(\{0\}) = p$, $\mu_0 = (\{1\}) = q$, $p + q = 1$, $0 < p < q$. Let $X = \prod_{g \in G} X_g$ be the Cartesian product of $\{X_g\}$, $g \in G$, where all $X_g = X_0$, and let μ be the completion of the product measure $\mu' = \prod_{g \in G} \mu_g$ on X, where all $\mu_g = \mu_0$. Let $H = L^2(X, \mu)$ be the Hilbert space of all μ-square-integrable functions f on X. Let $M(X, \mu)$ be the abelian von Neumann algebra consisting of all multiplication operators on H, i.e. $M(X, \mu) = \{m_{f_0} | f_0$ a bounded μ-measurable function on X and $(m_{f_0})(x) = f_0(x) f(x)$ for all $f \in H\}$. We shall simply write f_0 for m_{f_0} hereafter. The function $f(x) \equiv 1$ on X is a separating cyclic vector for $M(X, \mu)$ and we denote it by I.

Next, let K be the subset of X consisting of those elements of X which take the value 1 only at finitely many points of G. Define $(x + y)(g) = x(g) + y(g)$ (mod 2) for all $x, y \in X$. Then K is an abelian group with identity $e(g) \equiv 0$. Each $\alpha \in K$ defines a transformation α: $x \to x + \alpha$ on X; and the measure μ is quasi-invariant under K (**8**, Corollary to Lemma 3). Define $\mu_\alpha(E) = \mu(E + \alpha)$ for each μ-measurable subset E of X, and let $(d\mu_\alpha/d\mu)(x)$ be the Radon-Nikodym derivative of μ_α for each $\alpha \in K$. Define

$$(u(\alpha)f)(x) = \left(\frac{d\mu_\alpha}{d\mu}(x)\right)^{1/2} f(x + \alpha)$$

for all $f \in H$. Then u: $\alpha \to u(\alpha)$ is a faithful unitary representation of K on H such that $u(\alpha) f(x) u(\alpha^{-1}) = f(x + \alpha) \in M(X, \mu)$ for all $f(x) \in M(X, \mu)$. By

(**8**, Lemma 7), the transformation group K is (i) free, (ii) ergodic, (iii) non-measurable on X; hence the crossed product $R_1 = M(X, \mu) \otimes u$ on $H_1 = H \otimes K$ is a factor of type III with I^e as a separating vector (**5**, Lemmas 3.6.5, 4.3.5). Glimm (**2**, § 2) has shown that R_1 is hyperfinite. An arbitrary element in R_1 is denoted by $L(f(x, \alpha))$, where for each $\alpha \in K, f(x, \alpha)$ is a bounded measurable function on X.

Construction of R_2. Let the group G in the construction of R_1 be Φ_2, the free group with two generators. For each $g \in \Phi_2$, define

$$(u_1(g)f)(x, \alpha) = f(gx, g\alpha) \quad \text{for all } f(x, \alpha) \in H_1, \tag{9}$$

where $gx(h) = x(hg)$ for $x \in X \supset K$. u_1: $g \to u_1(g)$ is a faithful unitary representation of Φ_2 on H_1, and $u_i(g)R_1u_1(g^{-1}) = R_1$ for all $g \in \Phi_2$. Also, it is easily verified that for each $g \in \Phi_2$, we have

$$||Tu_1(g)I^e|| = \left(\sum_{\alpha \in K} \int_X |f(x, \alpha)|^2 \, d\mu\right)^{1/2} = ||TI^e||$$

for all $T = L(f(x, \alpha)) \in R_1$. Since Φ_2 is an ICC group, the crossed product $R_2 = R_1 \otimes u_1$ on the Hilbert space $H_2 = H_1 \otimes \Phi_2$ is a factor by Lemma 2. By Lemma 3, R_2 is a factor of type III since R_1 is purely infinite. Indeed, R_2 can be identified with M_2 in (**8**) by the isomorphism i: $R_2 \to M_2$ such that $i(f^{e,e}) = \bar{L}_f$, $i(I^{\alpha,e}) = \bar{U}_{(\alpha,e)}$, $\alpha \in K$ (Δ in (**8**)), $i(I^{e,g}) = \bar{U}_{(e,g)}$, $g \in \Phi_2$. As shown in (**5**, Theorem VIII), $R_2' = WR_2W$, where W is an involuntary on H_2 defined by

$$(Wf)(x, \alpha, g) = \left(\frac{d\mu_{g^{-1}\alpha}}{d\mu}(x)\right)^{1/2} f(g^{-1}(x + \alpha), g^{-1}\alpha, g^{-1})$$

for all $f(x, \alpha, g) \in H_2$.

Construction of R_3. Let Φ be a free group with an infinite system of generators $\{a_{-1}, a_0, a_1, a_2, \ldots\}$, and let the group G in the construction of R_1 be the subgroup Φ_2 of Φ generated by a_{-1} and a_2. Let Π be the group of all finite permutations on the set of natural numbers. Put $\pi(a_{-1}) = a_{-1}$, $\pi(a_0) = a_0$, and $\pi(a_i) = a_{\pi(i)}$, $i = 1, 2, \ldots$, for each $\pi \in \Pi$. Π is a group of permutations on the set of free generators of Φ, and naturally, a group of automorphisms of Φ. Let $\Phi \otimes \Pi = \{(g, \pi) | g \in \Phi, \pi \in \Pi\}$, and define $(g, \pi)(h, \pi_1) = (g\pi(h), \pi\pi_1)$ for (g, π), $(h, \pi_1) \in \Phi \otimes \Pi$. It is easily seen that $\Phi \otimes \Pi$ is an ICC group under this multiplication. The mapping ϕ: $a_{-1} \to a_{-1}$, $a_0 \to a_0$, $a_i \to e$, $i = 1, 2, \ldots$, $\pi \to e$, $\pi \in \Pi$, between generators of $\Phi \otimes \Pi$ and that of Φ_2 clearly induces a homomorphism ϕ': $g \to g'$ of $\Phi \otimes \Pi$ onto Φ_2. The free group Φ_2 has a unitary representation u_1 on H_1 defined by (9) which induces a group of automorphisms of R_1. Put $v_1 = u_1 \circ \phi'$. v_1 is obviously a unitary representation of $\Phi \otimes \Pi$ on H_1 such that $v_1(g)R_1v_1(g^{-1}) = R_1$ for all $g \in \Phi \otimes \Pi$. By Lemmas 2 and 3, the crossed product $R_3 = R_1 \otimes v_1$ on $H_3 = H_1 \otimes \Phi \otimes \Pi$ is a factor of type III with a separating vector $\xi = I^{e,e}$. As shown in (**5**, Theorem VIII), $R_3' = W_1 R_3 W_1$, where W is an involuntary on H_3 defined by

$$(W_1 f)(x, \alpha, g) = \left(\frac{d\mu_{g^{-1}\alpha}}{d\mu}(x)\right)^{1/2} f((g')^{-1}(x + \alpha), (g')^{-1}\alpha, g^{-1})$$

for all $f(x, \alpha, g) \in H_3$.

Construction of R_4. Let $\Phi \otimes \Delta$ be the group constructed in § 2, and let the group G in the construction of R_1 be the subgroup Φ_2 of $\Phi \otimes \Delta$ generated by (a_0, e) and (b_0, e). The free group Φ_2 has a unitary representation u_1 on H defined by (9). Now, the mapping ϕ_1: $(a_0, e) \to (a_0, e)$, $(b_0, e) \to (b_0, e)$, $(a_i, e) \to (e, e)$, $(b_i, e) \to (e, e)$, $i = 1, 2, \ldots$, $(e, \lambda) \to (e, e)$, $\lambda \in \Delta$, clearly induces a homomorphism ϕ_1': $g \to g'$ of $\Phi \otimes \Delta$ onto Φ_2. Then $v = u_1 \circ \phi_1'$ is a unitary representation of $\Phi \otimes \Delta$ on H_1 such that $v(g) R_1 v(g^{-1}) = R_1$ and $||Tv(g)I^e|| = ||TI^e||$ for all $g \in \Phi \otimes \Delta$, $T \in R_1$. By Lemmas 2 and 3, the crossed product $R_4 = R_1 \otimes v = M(x, \mu) \otimes u \otimes v$ on $H_4 = H_1 \otimes \Phi \otimes \Delta$ is a factor of type III with a separating vector $\xi = I^{e,e}$. As in (**5**, Theorem VIII), it can be verified that $R_4' = W_2 R_4 W_2$, where W_2 is an involuntary on H_4 defined by

$$(W_2 f)(x, \alpha, g) = \left(\frac{d\mu_{(g')^{-1}\alpha}}{d\mu}(x)\right)^{1/2} f((g')^{-1}(x + \alpha), (g')^{-1}\alpha, g^{-1})$$

for all $f(x, \alpha, g) \in H_4$.

We shall prove the following lemmata for the factors of type III we constructed on separable Hilbert spaces H_i, $i = 1, 2, 3, 4$.

LEMMA 9. *R_2, R_3, R_4 are non-hyperfinite.*

LEMMA 10. *Both R_3 and R_4 have property* L.

LEMMA 11. *R_3 does not have property* C.

LEMMA 12. *R_4 has property* C.

Since R_2 is just the factor M_2 in (**8**) which does not have property L (**8**, Lemma 13), the above lemmata imply the following theorem.

THEOREM 3. *R_2, R_3, R_4 are three pairwise non-isomorphic non-hyperfinite factors of type III on a separable infinite-dimensional Hilbert space.*

Proof of Lemma 9. Suppose that R_4 is hyperfinite. Since R_4' and R_4 are isomorphic by an involuntary W_2, R_4' is also hyperfinite. Let

$$M_1 \subset M_2 \subset \ldots \subset M_n \subset \ldots R_4'$$

be an increasing sequence of finite-dimensional von Neumann subalgebras of R_4' which generates it weakly. For any $x, y \in H_4$ and $T \in B(H_4)$, define

$$(\theta_n(T)x|y) = \int_{U_n} (UTU^* x|y)\mu_n\,(dU),$$

where U_n is the compact group of all unitary operators in M_n, and μ_n is the normalized Haar measure on U_n, $n = 1, 2, \ldots$. Let

$$(\theta(T)x|y) = \text{Banach}\lim_{n\to\infty} (\theta_n(T)x|y).$$

Then θ: $T \to \theta(T)$ is a linear mapping from $B(H_4)$ onto $(R_4')' = R_4$ such that (i) $\theta(T^*) = \theta(T)^*$, (ii) $\theta(I) = I$, (iii) $\theta(AT) = A\theta(T)$, $\theta(TA) = \theta(T)A$ for all $A \in R_4$, and (iv) $T \geqq 0$ implies $\theta(T) \geqq 0$ (see **10**).

The Hilbert space H_4 is the space of all complex functions $F(x, \alpha, g)$ on $X \times K \times \Phi \otimes \Delta$ such that

$$\sum_{g\in\Phi\otimes\Delta} \sum_{\alpha\in K} \int_X |F(x, \alpha, g)|^2 \, d\mu < +\infty.$$

Put $\tau(T) = (T\xi|\xi)$ for $T \in R_4$. We shall prove that for each $h \in \ker v$ (kernel of v) $\subset \Phi \otimes \Delta$,

$$\tau(L(I^{e,h})^* TL(I^{e,h})) = \tau(T) \quad \text{for all } T \in R_4. \tag{10}$$

Since the linear span of all operators of the form $L(f^{\alpha,g})$ is weakly dense in R_4, we only need to verify that

$$(L(I^{e,h})^* L(f^{\alpha,g})L(I^{e,h})\xi|\xi) = (L(f^{\alpha,g})\xi|\xi),$$

where $f \in M(x, \mu)$, $\alpha \in K$, $g \in \Phi \otimes \Delta$, for each $h \in \ker v$. In fact, if $g \neq e$, both sides equal 0; if $g = e$, both sides equal $(L(f^{\alpha,e})\xi|\xi)$ since $L(I^{e,h})$ commutes with $L(f^{\alpha,e})$ when $h \in \ker v$.

The mapping η': $(a_0, e) \to e$, $(b_0, e) \to e$, $(a_i, \lambda) \to (a_i, \lambda)$, $(b_i, \lambda) \to (b_i, \lambda)$, $i = 1, 2, \ldots, \lambda \in \Delta$, obviously induces a homomorphism η of $\Phi \otimes \Delta$ onto $\ker v \subset \Phi \otimes \Delta$. Now, for each subset σ of $\ker v$, let T_σ be the non-negative operator on the Hilbert space H_4 defined by

$$(T_\sigma F)(x, \alpha, g) = \begin{cases} F(x, \alpha, g) & \text{if } g \in \eta(\sigma), \\ 0 & \text{if } g \notin \eta(\sigma). \end{cases}$$

$\sigma \to T_\sigma$ is a finitely additive operator-valued function of all subsets of $\ker v$, and $T_{\ker v} = I$. Put $\nu(\sigma) = \tau(\theta(T_\sigma))$. Then $\nu(\sigma)$ is a non-negative finitely additive function defined for all subsets of $\ker v$ with $\nu(\ker v) = 1$ by (ii) and (iv) of the mapping θ. An elementary computation shows that

$$L(I^{e,g})^* T_\sigma L(I^{e,g}) = T_{g^{-1}\sigma}$$

for each $g \in \ker v$. Then it follows from (iii) and (10) that

$$\begin{aligned}\nu(g^{-1}\sigma) = \tau(\theta(L(I^{e,g})^* T_\sigma L(I^{e,g}))) &= \tau(L(I^{e,g})^* \theta(T_\sigma) L(I^{e,g})) \\ &= \tau(\theta(T_\sigma)) = \nu(\sigma) \quad \text{for each } g \in \ker v.\end{aligned}$$

Hence ν is a Banach mean on the group $\ker v$. But $\ker v$ obviously contains a subgroup isomorphism to the free group with two generators, consequently, ker v cannot be amenable. This contradiction shows that R_4', and hence R_4, are non-hyperfinite. The proof for R_3 or R_2 is exactly the same. We omit the repetition here.

Proof of Lemma 10. We first note that strong lim $U_kTU_k^* = T$ is equivalent to $\lim_{k\to\infty}||(U_kTU_k^* - T)\xi|| = 0$ for each $T \in R_3$ (R_4), and weak lim $U_k = 0$ is equivalent to $\lim|(U_k\xi|\xi)| = 0$ for any sequence of unitary operators U_k $(k = 1, 2, \ldots)$ since ξ is a cyclic vector for R_3' (R_4'). Let λ_k be the element of Π which permutes k with $k + 1$ and leaves all others fixed $(\lambda_k = \rho_k \in \Delta)$ and let $U_k = L(I^{(e,\lambda_k)})$ for $k = 1, 2, \ldots$. U_k is unitary and $(U_k\xi|\xi) = 0$, $k = 1, 2, \ldots$. Hence weak lim $U_k = 0$.

For any given operator $T = L(t)$ in $R_3 = R_1 \otimes v_1$ $(R_4 = R_1 \otimes v)$, where t is an R_1-valued function on $\Phi \otimes \Pi$ $(\Phi \otimes \Delta)$, and $\epsilon > 0$, let $T' = L(t') \in R_3$ (R_4) be such that $t'(g) = 0$ for all g in $\Phi \otimes \Pi$ $(\Phi \otimes \Delta)$ except on a finite subset S, and $||(T - T')|| < \epsilon/2$. Let p denote the largest natural number j for which there is a $(g, \pi) \in S$ with $\pi(j) \neq j$, q denote the largest natural number j such that a_j (a_j or b_j) appears in the reduced word representation of the first coordinate of some element in S. Let $N = \max(p, q)$. At this point, we note that $L(s^e)L(I^h) = L(I^h)L(s^e)$ for all $h \in (e, \Pi)$ $((e, \Delta))$, $s \in R_1$. Clearly, for all $k > N$, U_k commutes with $L(I^g)$ if $g \in S$. In short, $T'U_k = U_kT'$ for all $k > N$. Hence $k > N$ implies

$$||(U_kTU_k^* - T)\xi|| \leqq ||U_k(T - T')U_k^*\xi|| + ||(T - T')\xi|| = 2||(T - T')\xi|| < \epsilon.$$

The last step in the above expression is justified since for each $h \in (e, \Pi)$ $((e, \Delta))$ we have:

$$(11) \qquad (L(I^h)TL(I^h)^*\xi|\xi) = (T\xi|\xi) \quad \text{for all } T \in R_3\ (R_4).$$

To verify this, we only need to show that

$$(L(I^{e,h})L(f^{\alpha,g})L(I^{e,h})\xi|\xi) = (L(f^{\alpha,g})\xi|\xi)$$

for arbitrary $f \in M(x, \mu)$, $\alpha \in K$, $g \in \Phi \otimes \Pi$ $(\Phi \otimes \Delta)$. In fact, both sides are equal to zero if $g \neq e$ or $\alpha \neq e$, and equal to $\int_X f(x)\,d\mu$ if $g = e$, $\alpha = e$. Hence $\lim_{k\to\infty}||(UTU^* - T)\xi|| = 0$, i.e. strong lim $U_kTU_k^* = T$. Therefore, R_3 and R_4 have property L.

Proof of Lemma 11. Assume, on the contrary, that R_3 has property C. Then, for the unitary sequence U_k $(k = 1, 2, \ldots)$ in the proof of Lemma 10, there exists a sequence V_k $(k = 1, 2, \ldots)$ of mutually commuting operators in R_3 such that strong $\lim(U_k - V_k) = 0$. Since $\lambda_{k+1}\lambda_k \neq \lambda_k\lambda_{k+1}$, for $k = 1, 2, \ldots$, we have:

$$\begin{aligned}\sqrt{2} = ||(U_{k+1}U_k - U_kU_{k+1})\xi|| &\leqq ||(U_{k+1} - V_k)U_k\xi|| + ||V_{k+1}(U_k - V_k)\xi|| \\ &\quad + ||(V_k - U_k)(V_{k+1} - U_{k+1})\xi|| + ||(V_k - U_k)U_{k+1}\xi|| \\ + ||U_k(V_{k+1} - U_{k+1})\xi|| &\leqq 2||(U_{k+1} - V_{k+1})\xi|| + ||(V_k - U_k)\xi|| \\ &\quad + ||V_{k+1}||\,||(U_k - V_k)\xi|| + ||V_k - U_k||\,||(V_{k+1} - U_{k+1})\xi||,\end{aligned}$$

by (11). Since strong $\lim(U_k - V_k)$ exists, $\{||U_k - V_k||\}$, $k = 1, 2, \ldots$, and $\{||V_k||\}$, $k = 1, 2, \ldots$, are bounded by some positive number M by the uniform boundedness principle. Therefore, each term in the last expression in the above inequality approaches 0 as $k \to \infty$. This contradiction proves that R_3 does not have property C.

Proof of Lemma 12. Let $U_k = L(f_k(x, \alpha, g))$ $(k = 1, 2, \ldots)$ be a sequence of unitary operators in $R_4 = M(x, \mu) \otimes u \otimes v$ such that strong $\lim U_k^* T U_k = T$ for each $T \in R_4$, where for each $(\alpha, g) \in K \times \Phi \otimes \Delta$, $f_k(x, \alpha, g)$ is a bounded μ-measurable function on X. Let R_{41} denote the von Neumann subalgebra of R_4 consisting of all $L(f(x, \alpha, g))$ with $f(x, \alpha, g) = 0$ if $\alpha \neq e$ or $g \notin \Delta = (e, \Delta)$. Note that $L(I^{e,h})L(f^{e,e}) = L(f^{e,e})L(I^{e,h})$ for all $h \in \Delta \subset \ker v$, $f \in M(x, \mu)$. Since $M(X, \mu)$ and Δ are abelian, R_{41} is an abelian von Neumann subalgebra of R_4. By (11), $(L(I^{e,h})T\xi|\xi) = (TL(I^{e,h})\xi|\xi)$ for all $h \in \Delta$, $T \in R_4$. Also, for each $f_0 \in M(X, \mu)$, we have:

$$(L(f_0^{e,e})T\xi|\xi) = (TL(f_0^{e,e})\xi|\xi) \quad \text{for all } T \in R_4.$$

To verify this, we only need to show that

$$(L(f_0^{e,e})L(f^{\alpha,g})\xi|\xi) = (L(f^{\alpha,g})L(f_0^{e,e})\xi|\xi)$$

for any $f \in M(X, \mu)$, $\alpha \in K$, $g \in \Phi \otimes \Delta$. In fact, both sides are non-zero only if $\alpha = e$, $g = e$, and in this case both sides are equal to $\int_X f_0(x)f(x)\, d\mu$. Hence, for any $T \in R_4$, $S \in R_{41}$, we have $(TS\xi|\xi) = (ST\xi|\xi)$, since the linear span of $L(f_0^{e,e})L(I^{e,h})$, $f_0 \in M(X, \mu)$, $h \in \Delta$, is weakly dense in R_{41}. Now, by Lemma 4, there exists a projection P of norm one from R_4 into R_{41}. We claim that $V_k = P(U_k) = L(\bar{f}_k(x, \alpha, g))$ $(k = 1, 2, \ldots)$ (where $\bar{f}_k(x, e, g) = f_k(x, e, g)$ if $g \in \Delta$, $\bar{f}_k(x, \alpha, g) = 0$ if $\alpha \neq e$ or $g \notin \Delta$) is a sequence of mutually commuting (since R_{41} is abelian) operators required for having property C.

Let $G = \Phi \otimes \Delta$, g_1, g_2, S, B as described in the proof of Lemma 7. Let $T_i = L(I^{e,g_i})$, $i = 1, 2$. Note that $v(g_i) = I$ $(i = 1, 2)$. For given $\epsilon > 0$, suppose that $N = N(\epsilon)$ is such that for $i = 1, 2$, $k > N$ implies

$$\begin{aligned}\epsilon &> ||(U_k^* T_i U_k - T_i)\xi|| \\ &= ||(L(I^{e,g_i})L(f_k(x, \alpha, g)) - L(f_k(x, \alpha, g))L(I^{e,g_i}))\xi|| \\ &= \left(\sum_{g \in G} \sum_{\alpha \in K} \int_X |f_k(g_i x, g_i \alpha, g) - f_k(x, \alpha, g_i g g_i^{-1})|^2\right)^{1/2}.\end{aligned}$$

Put $F(\alpha, g) = (\int_X |f_k(x, \alpha, g)|^2\, d\mu)^{1/2}$, where k is an arbitrary integer greater than N. We observe that $x \to g_i x$ $(i = 1, 2)$ is a measure-preserving transformation on X; thus by an application of the triangle inequality, for $i = 1, 2$, we have:

$$\begin{aligned}&\sum_{g \in G} \sum_{\alpha \in K} |F(\alpha, g_i g g_i^{-1}) - F(g)|^2 \\ &= \sum_{g \in G} \sum_{\alpha \in K} \left| \left(\int_X |f_k(x, \alpha, g_i g g_i^{-1})|^2\, d\mu \right)^{1/2} - \left(\int_X |f_k(g_i x, g_i \alpha, g)|^2\, d\mu \right)^{1/2} \right| \\ &\leqq ||(T_i U_k - U_k T_i)\xi||^2 < \epsilon^2.\end{aligned}$$

By Lemma 5, we have:

(12) $$\sum_{g\in B}\sum_{\alpha\in K}\int_X |f_k(x,\alpha,g)|^2 = \sum_{g\in B}\sum_{\alpha\in K}|F(\alpha,g)|^2 < 196\epsilon^2.$$

For α, $\beta \in K$, we write $\alpha\sim\beta$ if there exists a $g\in\Phi_2$ such that $g\alpha=\beta$, where Φ_2 is the subgroup of G generated by g_1, g_2. It is easy to see that in this way we obtain an equivalence relation on K. We denote by Ω the totality of the equivalence classes not containing the identity e of K. In each $\omega\in\Omega$, choose an element α_ω. Then every element of K can be written uniquely in the form $g\alpha_\omega$ $(g\in\Phi_2)$. We introduce the function

$$f_\omega(g) = \left(\sum_{h\in\Delta}|F(g\alpha_\omega,h)|^2\right)^{1/2}$$

on Φ_2 for each $\omega\in\Omega$. Let

$$c_\omega = \left(\sum_{g\in\Phi_2}|f_\omega(g)|^2\right)^{1/2}, \qquad b_\omega = \sup_{i=1,2}\left(\sum_{g\in\Phi_2}|f_\omega(gg_i)-f_\omega(g)|^2\right)^{1/2}.$$

We remark that by (**8**, Lemma 11), we have $c_\omega \leqq 20\, d_\omega$. Hence

(13) $$\sum_{\substack{\alpha\in K;\\ \alpha\neq e}}\sum_{h\in\Delta}\int_X |f_k(x,\alpha,h)|^2\,d\mu$$
$$= \sum_{g\in\Phi_2}\sum_{\omega\in\Omega}\sum_{h\in\Delta}|F(g\alpha_\omega,h)|^2 = \sum_{\omega\in\Omega}c_\omega^{\,2} \leqq \sum_{\omega\in\Omega}400\,d_\omega^{\,2}$$
$$= 400\sup_{i=1,2}\sum_{\omega\in\Omega}\sum_{g\in\Phi_2}\sum_{h\in\Delta}|F(gg_i\alpha_\omega,h)-F(g\alpha,h)|^2$$
$$\leqq 400\sup_{i=1,2}\sum_{\substack{\alpha\in\Delta;\\ \alpha\neq e}}\sum_{h\in\Delta}\int_X |f_k(g_ix,g_i\alpha,h)-f_k(x,\alpha,g)|^2\,d\mu$$
$$\leqq 400\sup_{i=1,2}||(T_iU_k-U_kT_i)\xi||^2 \leqq 400\epsilon^2.$$

By (12) and (13), we have, for $k>N$,

$$||(U_k-V_k)\xi||^2 = \sum_{g\in B}\sum_{\alpha\in K}\int_X |f_k(x,\alpha,g)|^2$$
$$+\sum_{\substack{\alpha\in K;\\ \alpha\neq e}}\sum_{h\in\Delta}\int_X |f_k(x,\alpha,h)|^2\,d\mu < (196+400)\epsilon^2.$$

Hence $||(U_k-V_k)\xi||\to 0$ as $k\to\infty$. Since $||V_k|| = ||P(U_k)|| \leqq ||U_k|| = 1$, $k=1,2,\ldots$, $\{||V_k-U_k||\}$, $k=1,2,\ldots$, is also bounded. Thus, strong lim $(U_k-V_k)=0$, since ξ is a cyclic vector for R_4'. This completes the proof that R_4 has property C.

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