Canad. Math. Bull. Vol. **49** (1), 2006 pp. 152–160

Comparison Geometry With L^1-Norms of Ricci Curvature

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Abstract. We investigate the geometry of manifolds with bounded Ricci curvature in L^1-sense. In particular, we generalize the classical volume comparison theorem to our situation and obtain a generalized sphere theorem.

1 Introduction

We shall in this paper establish some geometrical results for manifolds with bounded Ricci curvature in L^1-sense.

Let us first introduce some necessary notations: (M, g) is an n-dimensional complete Riemannain manifold with metric g. At each point x in this manifold, we denote by $\mathrm{Ric}_-(x)$ the lowest eigenvalue for the Ricci tensor at x. Let $\mathrm{S}_x \subset \mathrm{T}_xM$ denote the space of unit tangent vectors at x and $d(\theta)$ be the distance from x to the cut point in the direction $\theta \in \mathrm{S}_x = \mathrm{S}^{n-1} \subset \mathrm{T}_xM$.

Then we define $\omega(r, \theta)$ by pulling back the volume form dvol of M to $\mathrm{U}_x = \{(r, \theta) \in \mathrm{T}_xM : 0 < r < d(\theta), \theta \in \mathrm{S}_x\}$, *i.e.,*

$$\mathrm{dvol} = \omega(r, \theta) dt d\theta,$$

where $d\theta$ is the standard volume form on $\mathrm{S}_x = \mathrm{S}^{n-1}$.

For convenience, we define $\omega(r, \theta)$ to be zero for $r > d(\theta)$.

Let $\omega_\kappa(r, \theta)$ be the $\omega(r, \theta)$ of the space form $\mathbb{S}^n_\kappa$ of dimension n with constant curvature $\kappa > 0$. We then know that $\omega' = h\omega$ (resp., $\omega'_\kappa = h_\kappa \omega_\kappa$), where h (resp., h_κ) is the mean curvature of the level sets of distant function on (M, g) (resp., $\mathbb{S}^n_\kappa$).

In 1997, P. Petersen and G. Wei [PeW] generalized the classical volume comparison to a situation where the amount of Ricci curvature which lies below $(n-1)\kappa$ is small in L^p-sense for $p > \frac{n}{2}$.

Note that for some analytic reason, the condition $p > \frac{n}{2} (\geq 1)$ in the study of the geometry of manifolds with bounded Ricci curvature in L^p-sense is essential and the proof of the above result strongly relies on the condition of $p > \frac{n}{2}$, where the case $p = 1$ is excluded.

In 2000, however, some results on the geometry of manifolds with bounded Ricci curvature in L^1-sense were developed by C. Sprouse [S]. In fact, he managed to show that if one assumes the manifold has $\mathrm{Ric}_- \geq -(n-1)k (k > 0)$, then it suffices to

Received by the editors August 18, 2003; revised January 24, 2005.
This work was supported by the Korea Research Foundation Grant(KRF-2005-214-C00136).
AMS subject classification: 53C20.
Keywords: Mean curvature, Ricci curvature.

assume that the amount of Ricci curvature which lies below $(n-1)$ in L^1-norm in order to get a diameter bound close to π. Motivated by this result, the author [Y1] provided a corresponding volume structure theorem as follows.

Theorem 1.1 ([Y1]) *For given $R > \pi$, $\epsilon > 0$, $k > 0$, and an integer n, there exists a $\delta = \delta(\epsilon, R, k, n)$ such that if M is a complete n-manifold with $\int_{B(x,R)}((n-1) - \mathrm{Ric}_-)_+ \,\mathrm{dvol} < \delta$, $\mathrm{Ric}_- \geq -(n-1)k$ $(k > 0)$, then $\mathrm{vol}(B(x,R) - B(x,\pi)) < \epsilon$ for all $x \in M$.*

Here, $u_+ = \max(0, u)$ is the positive part of the function u.

By applying some results obtained while we proved Theorem 1.1, we can prove the following volume comparison theorem.

Theorem 1.2 *Let $k > 0$, $n \in \mathbb{N}$, $0 < r < R$ be given. Then for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon, n, k, r, R) > 0$ such that if M is an n-dimensional Riemannian manifold with $\mathrm{Ric}_- \geq -(n-1)k$ and $\int_M((n-1) - \mathrm{Ric}_-)_+ \,\mathrm{dvol} < \delta$, then we have*

$$\frac{\mathrm{vol}\, B(x,R)}{v(n,R)} < \frac{\mathrm{vol}\, B(x,s)}{v(n,s)} + \epsilon$$

for all $x \in M$ and s with $r < s < R$, where $v(n,s)$ means the volume of metric s-ball in S^n.

As an application of Theorem 1.2, we can obtain the following volume and curvature pinching result.

Theorem 1.3 *For given $p > n$, $R > \pi$, and $C > 0$, there exists a $\delta > 0$ such that if M is an n-dimensional Riemannian manifold with*

$$\int_M |\,\mathrm{Ric}\,|^p \,\mathrm{dvol} \leq C, \quad \int_M ((n-1) - \mathrm{Ric}_-)_+ \,\mathrm{dvol} < \delta, \quad \mathrm{Ric}_- \geq -(n-1)k,$$

then M is diffeomorphic to S^n provided that $\mathrm{vol}\, B(x,R) \geq (1-\delta)\,\mathrm{vol}(\mathrm{S}^n)$ for some $x \in M$.

2 Proof of Theorem 1.2

Consider a sequence (M_i, g_i, x_i) of Riemannian n-manifolds with metrics g_i and $x_i \in M_i$ such that

$$\mathrm{Ric}_{M_i} \geq -(n-1)k\ (k > 0), \quad \int_{M_i} ((n-1) - \mathrm{Ric}_-)_+ \,\mathrm{dvol} < \delta_i,$$

where $\lim_{i\to\infty} \delta_i = 0$.

Then it suffices to show that for every $\epsilon > 0$, there exists $N = N(\epsilon, n, k, r, R) \in \mathbb{N}$ such that

$$\frac{\mathrm{vol}\, B(x_i,R)}{v(n,R)} - \frac{\mathrm{vol}\, B(x_i,s)}{v(n,s)} < \epsilon$$

https://doi.org/10.4153/CMB-2006-016-2 Published online by Cambridge University Press

for all $i \geq N$ and s with $r < s < R$.

Recall that for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\operatorname{vol}(B(x_i, R) - B(x_i, \pi)) < \epsilon$ for all $i \geq N$ by Theorem 1.1. So without loss of generality, we may assume that $R < \pi$.

We use the same notation as in [P] and repeat it here.

For any $\delta > 0$, let

$$\operatorname{vol}(E^i_\delta) := \operatorname{vol}\Big\{\, x \in B(x_i, R) \,:\, \int_{B(x_i,R)} ((n-1) - \operatorname{Ric}_-)_+ \operatorname{dvol} > \delta \,\Big\},$$

which converges to zero since

$$\begin{aligned}\int_{M_i} ((n-1) - \operatorname{Ric}_-)_+ \operatorname{dvol} &> \int_{E^i_\delta} ((n-1) - \operatorname{Ric}_-)_+ \operatorname{dvol} \\ &> \int_{E^i_\delta} \delta \operatorname{dvol} = \delta \operatorname{vol}(E^i_\delta).\end{aligned}$$

We also let

$$S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta) = \inf\{s : s > \delta_i, \theta \in (\Phi_{\sqrt[4]{\epsilon_i},\delta_i})^c, \mu(\gamma^i_\theta([\delta_i, s]) \cap E^i_\delta) \geq \sqrt[4]{\epsilon_i}\},$$

where

$$\Phi_{\sqrt[4]{\epsilon_i},\delta_i} = \{\theta \in S^{n-1} \subset T_{x_i}M_i : \mu(\gamma^i_\theta([\delta_i, \min(R, d^i(\theta))]) \cap E^i_\delta) < \sqrt[4]{\epsilon_i}\}$$

and μ is the measure on $\gamma^i_\theta(t) = \exp_{x_i} t\theta$.

We should recall that for any $\theta \in \Phi_{\sqrt[4]{\epsilon_i},\delta_i}$ we have that $(h_i(t,\theta) - h_1(t))_+$ can be arbitrarily small on $[\sqrt[3]{\tau_i}, \min(d^i(\theta), R)]$ for sufficiently large i [Y1]. Here, τ_i is a positive number with $\lim_{i\to\infty} \tau_i = 0$.

Now, we first analyze $\operatorname{vol} B(x_i, R)$ for any $R > 0$ as follows.

$$\begin{aligned}\operatorname{vol} B(x_i, R) = \int_{S^{n-1}} \int_{B(x_i,\delta_i)} \omega_i \, dt d\theta + \int_{\Phi_{\sqrt[4]{\epsilon_i},\delta_i}} \int_{\delta_i}^{\sqrt[3]{\tau_i}} \omega_i \, dt d\theta \\ + \int_{\Phi_{\sqrt[4]{\epsilon_i},\delta_i}} \int_{\sqrt[3]{\tau_i}}^{R} \omega_i \, dt d\theta + \int_{(\Phi_{\sqrt[4]{\epsilon_i},\delta_i})^c} \int_{\delta_i}^{R} \omega_i \, dt d\theta.\end{aligned}$$

But it is easy to see that the first and the second term in the above sum converge to zero as $i \to \infty$. So we may express $\operatorname{vol} B(x_i, R)$ as follows.

$$\operatorname{vol} B(x_i, R) = \int_{\Phi_{\sqrt[4]{\epsilon_i},\delta_i}} \int_{\sqrt[3]{\tau_i}}^{R} \omega_i \, dt d\theta + \int_{(\Phi_{\sqrt[4]{\epsilon_i},\delta_i})^c} \int_{\sqrt[3]{\tau_i}}^{R} \omega_i \, dt d\theta + \eta_i \tag{2.1}$$

for some $\eta_i > 0$ with $\lim_{i\to\infty} \eta_i = 0$.

Now we recall that on $\Psi := \Psi_1 \cup \Psi_2$, where

$$\Psi_1 = \{(t,\theta) : \theta \in \Phi_{\sqrt[4]{\epsilon_i},\delta_i},\ \sqrt[3]{\tau_i} < t < R\},$$
$$\Psi_2 = \{(t,\theta) : \theta \in (\Phi_{\sqrt[4]{\epsilon_i},\delta_i})^c,\ \sqrt[3]{\tau_i} < t < S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta)\},$$

we have

$$h_i(t,\theta) - h_1(t) < \mu_i$$

for some $\mu_i > 0$ with $\mu_i \to 0$ [Y1].

Thus, from the above inequality, we have

$$(\ln \omega_i(t,\theta))' - (\ln \omega_1(t))' < \mu_i,$$

which gives $(\ln \frac{\omega_i(t,\theta)}{\omega_1(t)})' < \mu_i$.

Thus for any $(t_1,\theta), (t_2,\theta) \in \Psi$ with $t_1 < t_2$, we get

$$\int_{t_1}^{t_2} \left(\ln \frac{\omega_i(t,\theta)}{\omega_1(t)}\right)' dt < \mu_i(t_2 - t_1),$$

which implies

$$\ln \frac{\omega_i(t_2,\theta)}{\omega_1(t_2)} - \ln \frac{\omega_i(t_1,\theta)}{\omega_1(t_1)} < \mu_i(t_2 - t_1).$$

Consequently, we have

$$\frac{\omega_i(t_2,\theta)}{\omega_1(t_2)} < \exp(\nu_i)\frac{\omega_i(t_1,\theta)}{\omega_1(t_1)} \tag{2.2}$$

for some $\nu_i > 0$ with $\lim_{i\to\infty} \nu_i = 0$.

Now we consider the following lemma which is a slight modification of [Z, Lemma 3.2].

Lemma 2.1 *Let f, g be two positive continuous functions defined on $[0,\infty]$. If $\frac{f(b)}{g(b)} \le \exp(\nu)\frac{f(a)}{g(a)}$ for some $\nu > 0$ and for all a, b with $0 < a < b$, then for any given $R > 0$, $r > 0$ and $a > 0$ with $R > r > a$ we have*

$$\frac{\int_a^R f(t)\,dt}{\int_a^R g(t)\,dt} \le \frac{\int_a^s f(t)\,dt}{\int_a^s g(t)\,dt} + \tau(\nu)$$

for all $s > 0$ with $R \ge s \ge r > a$ and for some $\tau(\nu) > 0$ satisfying $\lim_{\nu\to 0} \tau(\nu) = 0$.

Proof It suffices to show that the function

$$F(y) = \frac{\int_a^y f(t)\,dt}{\int_a^y g(t)\,dt}$$

https://doi.org/10.4153/CMB-2006-016-2 Published online by Cambridge University Press

is almost nonincreasing with respect to $y \in [r, R]$. Specifically, we first compute

$$F'(y) = \frac{1}{(\int_a^y g(t)\,dt)^2}\Big\{ f(y)\int_a^y g(t)\,dt - g(y)\int_a^y f(t)\,dt \Big\}$$
$$= \frac{g(y)\int_a^y g(t)\,dt}{(\int_a^y g(t)\,dt)^2}\left\{ \frac{f(y)}{g(y)} - \frac{\int_a^y f(t)\,dt}{\int_a^y g(t)\,dt} \right\}.$$

But

$$\frac{f(y)}{g(y)} \leq \exp(\nu)\frac{f(t)}{g(t)}$$

for $a \leq t \leq y$.

Thus $\int_a^y f(t)\,dt \geq \exp(-\nu)\frac{f(y)}{g(y)}\int_a^y g(t)\,dt$, that is,

$$\frac{f(y)}{g(y)} \leq \exp(\nu)\frac{\int_a^y f(t)\,dt}{\int_a^y g(t)\,dt}.$$

Consequently, we have

$$F'(y) \leq \frac{g(y)\int_a^y g(t)\,dt}{(\int_a^y g(t)\,dt)^2}\frac{\int_a^y f(t)\,dt}{\int_a^y g(t)\,dt}(\exp(\nu) - 1) \tag{2.3}$$

for all y with $a < r \leq y \leq R$.

Since the right-hand side of the above inequality tends to zero as $\nu \to 0$, we can express $F'(y) \leq \mu(\nu)$ for some $\mu(\nu) > 0$ satisfying $\lim_{\nu\to 0}\mu(\nu) = 0$. Then by integrating this inequality from s to R, we get $F(R) - F(s) \leq (R - s)\mu(\nu)$.

So if we let $\tau(\nu) := (R-s)\mu(\nu) < R\mu(\nu)$, then we have $F(R) \leq F(s)+\tau(\nu)$, which is our desired result. ■

We can now estimate the volume ratio for the case $(t, \theta) \in \Psi_1$ using (2.2) and the above lemma.

For $\nu_i > 0$ in (2.2), we define $y_i(> \sqrt[3]{\tau_i})$ so that $\int_{\sqrt[3]{\tau_i}}^{y_i} \omega_1 dt = \sqrt{\nu_i}$.

Then from (2.3) in the proof of Lemma 2.1 and (2.2), it is easy to check

$$\left(\frac{\int_{\sqrt[3]{\tau_i}}^{y} \omega_i\,dt}{\int_{\sqrt[3]{\tau_i}}^{y} \omega_1 dt} \right)'\Bigg|_{y_i \leq y \leq R} \leq \frac{\exp(\nu_i) - 1}{\sqrt{\nu_i}} C(k, n, R),$$

which converges to zero as $i \to \infty$.

So we have

$$\frac{\int_{\sqrt[3]{\tau_i}}^{R} \omega_i\,dt}{\int_{\sqrt[3]{\tau_i}}^{R} \omega_1\,dt} \leq \frac{\int_{\sqrt[3]{\tau_i}}^{s} \omega_i\,dt}{\int_{\sqrt[3]{\tau_i}}^{s} \omega_1\,dt} + \tau(\nu_i)$$

for some $\tau(\nu_i) > 0$ satisfying $\lim_{i\to\infty}\tau(\nu_i) = 0$ and for all s with $y_i \leq s \leq R$.

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From the above inequality, we can easily obtain the following.

$$\frac{\int_{\Phi_{\sqrt[4]{\epsilon_i},\delta_i}} \int_{\sqrt[3]{\tau_i}}^{R} \omega_i \, dt d\theta}{\int_{S^{n-1}} \int_{\sqrt[3]{\tau_i}}^{R} \omega_1 \, dt d\theta} \leq \frac{\int_{\Phi_{\sqrt[4]{\epsilon_i},\delta_i}} \int_{\sqrt[3]{\tau_i}}^{s} \omega_i \, dt d\theta}{\int_{S^{n-1}} \int_{\sqrt[3]{\tau_i}}^{s} \omega_1 \, dt d\theta} + \tau(\nu_i). \tag{2.4}$$

Here we used $\tau(\nu_i)$ as a generic constant with the property $\lim_{i\to\infty} \tau(\nu_i) = 0$, and we always use $\tau(\nu_i)$ in such a way afterwards.

Next, we shall estimate the volume ratio for the case $(t, \theta) \in \Psi_2$ in the similar way.

Note first that $(\Phi_{\sqrt[4]{\epsilon_i},\delta_i})^c$ can be divided into the following three subsets:

$$\begin{aligned}
(\Phi^1_{\sqrt[4]{\epsilon_i},\delta_i})^c &= \{\theta \in (\Phi_{\sqrt[4]{\epsilon_i},\delta_i})^c : S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta) < y_i < R\},\\
(\Phi^2_{\sqrt[4]{\epsilon_i},\delta_i})^c &= \{\theta \in (\Phi_{\sqrt[4]{\epsilon_i},\delta_i})^c : y_i < S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta) < R\},\\
(\Phi^3_{\sqrt[4]{\epsilon_i},\delta_i})^c &= \{\theta \in (\Phi_{\sqrt[4]{\epsilon_i},\delta_i})^c : y_i < R < S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta)\}.
\end{aligned}$$

For the case $(t, \theta) \in \Psi_2$ and $\theta \in (\Phi^1_{\sqrt[4]{\epsilon_i},\delta_i})^c$, we get, for all s with $y_i \leq s \leq R$,

$$\frac{\int_{(\Phi^1_{\sqrt[4]{\epsilon_i},\delta_i})^c} \int_{\delta_i}^{S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta)} \omega_i \, dt d\theta}{\int_{S^{n-1}} \int_{\sqrt[3]{\tau_i}}^{R} \omega_1 \, dt d\theta} \leq \frac{\int_{(\Phi^1_{\sqrt[4]{\epsilon_i},\delta_i})^c} \int_{\delta_i}^{s} \omega_i \, dt d\theta}{\int_{S^{n-1}} \int_{\sqrt[3]{\tau_i}}^{s} \omega_1 \, dt d\theta}, \tag{2.5}$$

which is evident because $\int_{S^{n-1}} \int_{\sqrt[3]{\tau_i}}^{R} \omega_1 \, dt d\theta > \int_{S^{n-1}} \int_{\sqrt[3]{\tau_i}}^{s} \omega_1 \, dt d\theta$ and $S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta) < s$.

For the case $(t, \theta) \in \Psi_2$ and $\theta \in (\Phi^2_{\sqrt[4]{\epsilon_i},\delta_i})^c$, we use Lemma 2.1 and (2.2) to get

$$\frac{\int_{(\Phi^2_{\sqrt[4]{\epsilon_i},\delta_i})^c} \int_{\sqrt[3]{\tau_i}}^{S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta)} \omega_i \, dt d\theta}{\int_{S^{n-1}} \int_{\sqrt[3]{\tau_i}}^{S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta)} \omega_1 \, dt d\theta} \leq \frac{\int_{(\Phi^2_{\sqrt[4]{\epsilon_i},\delta_i})^c} \int_{\sqrt[3]{\tau_i}}^{s} \omega_i \, dt d\theta}{\int_{S^{n-1}} \int_{\sqrt[3]{\tau_i}}^{s} \omega_1 \, dt d\theta} + \tau(\nu_i)$$

for all s with $y_i \leq s \leq S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta)$.

But since $S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta) < R$ in this case, we can rewrite the above inequality as follows:

$$\frac{\int_{(\Phi^2_{\sqrt[4]{\epsilon_i},\delta_i})^c} \int_{\sqrt[3]{\tau_i}}^{S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta)} \omega_i \, dt d\theta}{\int_{S^{n-1}} \int_{\sqrt[3]{\tau_i}}^{R} \omega_1 \, dt d\theta} \leq \frac{\int_{(\Phi^2_{\sqrt[4]{\epsilon_i},\delta_i})^c} \int_{\sqrt[3]{\tau_i}}^{s} \omega_i \, dt d\theta}{\int_{S^{n-1}} \int_{\sqrt[3]{\tau_i}}^{s} \omega_1 \, dt d\theta} + \tau(\nu_i) \tag{2.6}$$

for all s with $y_i \leq s \leq S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta)$.

Furthermore, in case $S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta) < s \leq R$ we clearly have

$$\frac{\int_{(\Phi^2_{\sqrt[4]{\epsilon_i},\delta_i})^c} \int_{\sqrt[3]{\tau_i}}^{S_{\sqrt[4]{\epsilon_i},\delta_i}(\theta)} \omega_i \, dt d\theta}{\int_{S^{n-1}} \int_{\sqrt[3]{\tau_i}}^{R} \omega_1 \, dt d\theta} \leq \frac{\int_{(\Phi^2_{\sqrt[4]{\epsilon_i},\delta_i})^c} \int_{\sqrt[3]{\tau_i}}^{s} \omega_i \, dt d\theta}{\int_{S^{n-1}} \int_{\sqrt[3]{\tau_i}}^{s} \omega_1 \, dt d\theta}.$$

So we may say that (2.6) holds for any s with $y_i \leq s \leq R$.

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Thirdly, we obtain the similar estimate for the case $(t, \theta) \in \Psi_2$ and $\theta \in (\Phi^3_{\sqrt[4]{\epsilon_i},\delta_i})^c$ using the same method as above.

$$(2.7) \qquad \frac{\int_{(\Phi^3_{\sqrt[4]{\epsilon_i},\delta_i})^c} \int_{\sqrt[3]{\tau_i}}^{R} \omega_i \, dt d\theta}{\int_{S^{n-1}} \int_{\sqrt[3]{\tau_i}}^{R} \omega_1 \, dt d\theta} \leq \frac{\int_{(\Phi^3_{\sqrt[4]{\epsilon_i},\delta_i})^c} \int_{\sqrt[3]{\tau_i}}^{s} \omega_i \, dt d\theta}{\int_{S^{n-1}} \int_{\sqrt[3]{\tau_i}}^{s} \omega_1 \, dt d\theta} + \tau(\nu_i).$$

for all s with $y_i \leq s \leq R$.

Now we sum the above four inequalities (2.4)–(2.7) and use (2.1) together with [Y1, Lemma.2.1] to show that, for every $\epsilon > 0$, there exits $N \in \mathbb{N}$ such that

$$\frac{\operatorname{vol} B(p_i, R)}{v(n, R)} < \frac{\operatorname{vol} B(p_i, s)}{v(n, s)} + \epsilon$$

for all $i \geq N$ and for all s with $y_i \leq s \leq R$.

Since $y_i \to 0$, we complete the proof of Theorem 1.2.

3 Proof of Theorem 1.3

Let (M_i, g_i, x_i) be a sequence of manifolds such that

$$(3.1) \qquad \begin{gathered} \int_{M_i} |\operatorname{Ric}_{M_i}|^p \operatorname{dvol} \leq C, \quad \int_{M_i} ((n-1) - \operatorname{Ric}_-)_+ \operatorname{dvol} < \delta_i, \\ \operatorname{Ric}_{M_i} \geq -(n-1)k, \quad \text{and} \quad \operatorname{vol} B(x_i, R) \geq (1 - \delta_i) \operatorname{vol}(S^n), \end{gathered}$$

where δ_i tends to zero as i goes to infinity.

We first show that

$$\sup\{d(x_i, q_i) : q_i \in M_i\} < 3R.$$

To obtain this, suppose that it were not true and find $q_i \in M_i$ such that $d(x_i, q_i) = 3R$ for each large i. Then we easily see that $B(x_i, R) \subset B(q_i, 4R) - B(q_i, \pi)$, which implies $\operatorname{vol}(B(q_i, 4R) - B(q_i, \pi)) \geq \operatorname{vol} B(x_i, R) \geq (1 - \delta_i) \operatorname{vol}(S^n)$.

By letting $i \to \infty$, the above inequality gives a contradiction by Theorem 1.1. Consequently, we have

$$\sup\{d(x_i, q_i) : q_i \in M_i\} < 3R,$$

which means that $B(x_i, 3R) = M_i$ for all i. Now we show an analogue of [Y2, Lemma 3.1].

Lemma 3.1 *For sufficiently small δ_i, the class of all complete Riemannian manifolds satisfying* (3.1) *is precompact in the $C^{1+\alpha}$ topology* $(1 + \alpha < 2 - \frac{n}{p})$.

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Proof The proof is similar to that of [Y2, Lemma 3.1] and the argument depends on the proof of [Pe, Theorem 5.1].

To obtain the necessary volume growth condition, we first claim that for any given $\eta > 0$, there exists a $D \in (0, \pi)$ such that

$$\frac{\operatorname{vol}(B(x_i, D))}{v(n, D)} \geq 1 - \eta$$

for all sufficiently large i. Indeed, if this were not true, we may choose $D_i < \pi$ with $D_i \to \pi$ such that

$$\frac{\operatorname{vol}(B(x_i, D_i))}{v(n, D_i)} < 1 - \eta$$

for each i.

Then we have

$$\begin{aligned}\eta - \delta_i &= (1 - \delta_i) - (1 - \eta) \\ &< \frac{\operatorname{vol} B(x_i, R)}{\operatorname{vol}(S^n)} - \frac{\operatorname{vol} B(x_i, D_i)}{v(n, D_i)} \\ &= \frac{v(n, D_i) \operatorname{vol} B(x_i, R) - \operatorname{vol}(S^n) \operatorname{vol} B(x_i, D_i)}{\operatorname{vol}(S^n) v(n, D_i)}.\end{aligned}$$

By Theorem 1.1, we know that $\operatorname{vol} B(x_i, R) - \operatorname{vol} B(x_i, D_i)$ converges to zero. So the last quantity in the above inequalities tends to zero as i goes to infinity. Consequently $\eta - \delta_i$ tends to zero, which is a contradiction.

Next, by Theorem 1.2, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\frac{\operatorname{vol}(B(x_i, R))}{v(n, R)} - \epsilon \leq \frac{\operatorname{vol}(B(x_i, s))}{v(n, s)}$$

for all s with $y_i < s < R$ and $i \geq N$. So if we choose η and ϵ so that $\eta + \epsilon = \eta_n$, where η_n is the universal constant appearing in [An, Lemma 3.1], then we obtain that

$$\frac{\operatorname{vol}(B(x_i, s))}{v(n, s)} \geq 1 - \eta_n$$

for all s with $y_i < s < R$.

Since $y_i \to 0$ as $i \to \infty$, there is no problem in applying the same arguments as in [Y2, Lemma 3.1] and we easily arrive at the desired result by the standard metric rescaling argument. ■

By Lemma 3.1, we have a $C^{1+\alpha}$-manifold (N, g) and $(M_i, g_i) \to (N, g)$ in the $C^{1+\alpha}$ topology. Since the same argument in [Y2, Lemma 3.2] can be used for our situation, we can show that (N, g) is a $C^{1+\alpha}$-Wiedersehens manifold and we know that it is isometric to S^n(See [Y2, Lemma 3.2] for details). Thus we have established the theorem.

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