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H-EQUIVALENCE CLASSES OF MULTIPLICATIONS ON CERTAIN FIBER SPACES

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The enumeration of the H-equivalence classes of multiplications on a space is a topic of current interest. In this paper we try to study the H-equivalence classes of multiplications on a CW complex X with finitely many non-vanishing homotopy groups, by using the Postnikov decomposition of X and multiplier arguments [**1**; **4**]. This paper presents a way to compute the set of H-equivalence classes of multiplications on X from the knowledge of certain quotient sets of $H^*(B \wedge B, \Sigma)$ and some homotopy equivalences of B, where B represents the spaces in the Postnikov decomposition of X, and Σ denotes abelian groups corresponding to the homotopy groups of X. The results of this paper can be used to obtain Proposition A and B in [**6**], which in turn will give a counterexample to Problem 34 in [**5**], c.f. [**6**].

In § 1 we shall state some definitions, notations and some theorems from [**1**], [**4**] and [**6**]. In § 2 we shall define an equivalence relation among multipliers. We shall show in Theorem 2.4 that the H-equivalence class of multiplications is a disjoint union of $M_m(f)/R_f$, where R_f is a relation in $M_m(f)$. In § 3 we provide some more information about R_f and establish the main result Theorem 3.3. In § 4 we present a simple example to show how to use Theorem 2.4, 3.1 and 3.3 in a rather novel computation of the set of H-equivalence classes in certain situations.

We restrict ourselves to the CW-category.

1. Preliminary.

Definition 1.1. An H-space is a triple $(X, *, m)$ where $(X, *)$ is a space with base point $*$ and $m : X \times X \to X$ is a mapping which satisfies $m(x, *) = m(*, x) = x$ for any $x \in X$. Such a map is called a multiplication on X.

Let P, L and Ω be the free path functor, the path functor with fixed initial point and the loop functor respectively.

Definition 1.2. An H-map from $(X, *, m)$ to $(Y, *, n)$ is a map $f : (X, *) \to (Y, *)$ such that there exists a map $F : X \times X \to PY$ such that $e_0F = n \circ (f \times f)$, $e_1F = f \circ m$ and $e_tF(x, *) = e_tF(*, x) = f(x)$ where e_t is the evaluation at t. F is called a multiplier of the H-map f. If $e_tF = e_0F$ for all t, we call f a multiplicative map.

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Definition 1.3. Two multiplications m_1 and m_2 on X are called H-equivalent provided there exists an H-map $f : (X, m_1) \to (X, m_2)$ that is a homotopy equivalence. Let us denote this by $m_1 \simeq m_2$ (via f). In particular, if $f = \text{id}$, we denote it by $m_1 \cong m_2$.

It is well known that both $\simeq$ and $\cong$ are equivalence relations.

From now on let n be a multiplication on $K(\Sigma, l + 1)$ where Σ is an abelian group. (Note that up to homotopy, $K(\Sigma, l + 1)$ admits only one multiplication.)

Definition 1.4. Two multipliers $F_i (i = 1, 2)$ of an H-map $f : (X, m_i) \to K(\Sigma, l + 1) = K$ are called H-equivalent, denoted by $F_1 \sim F_2$, provided:

(i) there exist homotopy equivalence H-maps $g : (X, m_1) \to (X, m_2)$ and $g' : K \to K$ with Q and Q' as multipliers respectively,

(ii) there exists a homotopy G from $g' \circ f$ to $f \circ g$

(iii) there exists a secondary homotopy $D : X^2 \to P(PK)$ such that it preserves the boundary conditions

$$e_0 D = Q' \circ (f \times f), \quad e_1 D = f \circ Q$$
$$Pe_0 D(x, y) = F_2(g(x), g(y)) + P_n(G(x), G(y))$$
$$Pe_1 D(x, y) = G(m_1(x, y)) + g' F_1(x, y)$$

where e_0 and e_1 are the evaluation of each path at initial and terminal points (see Diagram 1).

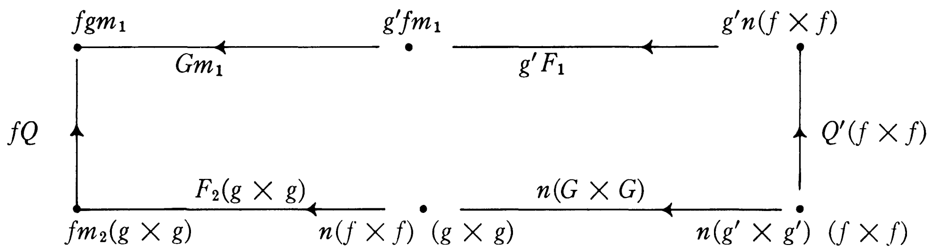

Diagram 1

Definition 1.5. In the above definition, if we let $m_1 = m_2$, $g = \text{id}$, $g' = \text{id}$, $Q(x, y)(t) = m_1(x, y)$, $Q'(u, v)(t) = n(u, v)$ and $G(t, x) = f(x)$, then F_1 and F_2 are called equivalent multipliers, denoted by $F_1 \approx F_2$.

(Note that, equivalent multipliers were called H-homotopic multipliers in [**1**]).

It is easy to verify that both "$\sim$" and "$\approx$" are equivalence relations.

From now on let l be a positive integer and B be a space such that $\Pi_k(B) = 0$ if $k \geqq l$. Let $K = K(\Sigma, l + 1)$ where Σ is an abelian group. Let the fibering

$$\Omega K \to E \xrightarrow{\pi} B$$

be induced from $\Omega K \to LK \to K$ by a map $f : B \to K$. E can be represented by $\{(b, \lambda) | b \in B, \lambda \in LK \text{ and } e_1 \lambda = f(b)\}$. The principal results of [**4**] are:

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THEOREM 1.6. *If E is an H-space, then B can be made into an H-space so that π and f are H-maps.*

THEOREM 1.7. *If $f : (B, m) \to K$ is an H-map, then for each multiplication s on E that makes π an H-map, there exists a multiplier F of f such that s is equivalent to $s(F) : E \times E \to E$ defined by*

$$s(F)\ ((b_1, \lambda_1), (b_2, \lambda_2)) = (m(b_1, b_2), Pn(\lambda_1, \lambda_2) + F(b_1, b_2))$$

where "+" means the usual path joining. We will call $s(F)$ the multiplication on E obtained from the multiplier F.

Let $H(E, m)$ be the family of all $\cong$ equivalence classes of multiplications on E such that π is multiplicative with respect to at least one multiplication on E in the $\cong$ equivalence class and the multiplication m on B. And let $M_m(f)$ be $\approx$ equivalence classes of multipliers of $f : (B, m) \to K$.

Remark. Theorem 1.7 implies $H(E, m) = \{\{s(F)\} | \{F\} \in M_m(f)\}$.

THEOREM 1.8. *If $m \cong m'$ on B, then there exists a bijection $\Phi : M_m(f) \to M_{m'}(f)$ such that for any $\{F\} \in M_m(f)$, $\{s(F)\} = \{s(G)\}$ where $G \in \Phi\{F\}$.*

THEOREM 1.9. *$H(E, m) = H(E, m')$ provided $m \cong m'$. $H(E, m) \cap H(E, m') = \emptyset$ if $m \ncong m'$.*

Let $H(E)$ be the family of all $\cong$ classes on E.

THEOREM 1.10. *$H(E) = \bigcup_{m \in \Gamma} H(E, m)$ where Γ is an arbitrary representation of the set $\{\alpha \in H(B) | f$ is an H-map with respect to $\alpha\}$. Moreover the union is disjoint union.*

The proofs of Theorem 1.6 and 1.7 can be found in [**1**; **4**; **6**]. The proofs of Theorem 1.8, 1.9 and 1.10 can be found in [**1**].

2. $\simeq$ relations.

THEOREM 2.1. *If $g'' : E \to E$ is a homotopy equivalence then there exist homotopy equivalences $g : B \to B$ and $g' : K \to K$ such that $\pi \circ g''$ is homotopic to $g \circ \pi$ and $f \circ g$ is homotopic to $g' \circ f$. Conversely if $g : B \to B$ and $g' : K \to K$ are homotopy equivalences such that $f \circ g$ is homotopic to $g' \circ f$, then each homotopy equivalence $g'' : E \to E$, such that $\pi \circ g$ is homotopic to $g \circ \pi$, is homotopic to one of the form*

$$g''(b, \lambda) = (g(b), Pg'(\lambda) + G(b))$$

where $G : B \to PK$ is a choice of the homotopy from $g' \circ f$ to $f \circ g$.

Proof. The proof of the first part of the theorem is contained in the material on pp. 438-441 of [**3**], or Proposition 2 in [**6**]. The converse can be easily proved using the exactness of $\to [E, \Omega B] \to [E, \Omega K] \to [E, E] \to [E, B]$ and dimensional considerations.

Let m_1 and m_2 be two multiplications on B. Let $\{F_1\} \in M_{m_1}(f)$, $\{F_2\} \in M_{m_2}(f)$, $s(F_1) = s_1$ and $s(F_2) = s_2$. In the following diagram

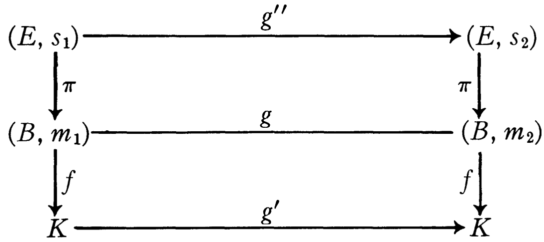

Diagram 2

g'', g and g' are homotopy equivalences, all squares are commutative up to homotopy and π is multiplicative. In the light of Theorem 2.1 we assume $g''(b, \lambda) = (g(b), Pg'(\lambda) + G(b))$, where $G : B \to PK$ is a homotopy from $g' \circ f$ to $f \circ g$.

THEOREM 2.2. *$g'' : (E, s_1) \to (E, s_2)$ is an H-map if and only if*

(i) *$g : (B, m_1) \to (B, m_2)$ and g' are H-maps, and*

(ii) *there exists multipliers Q and Q' of g and g' respectively and there exists a secondary homotopy $D : B \times B \to P(PK)$ such that the $Q, Q', G, F_1, F_2, D, f, g, g', m_1, m_2$, satisfy Diagram 1. (i.e. $F_1 \sim F_2$). Moreover $Q''((b_1, \lambda_1)(b_2, \lambda_2)) = (Q(b_1, b_2), PQ'(\lambda_1, \lambda_2) + D(b_1, b_2))$ is a multiplier of g''.*

Proof. It is easy to show that (i) and (ii) imply Q'' to be a multiplier of g''.

Assume g'' is an H-map. Let $J : E \times E \to PE$ be a multiplier of g''. The composite function

$$E \times E \xrightarrow{J} PE \xrightarrow{\text{Proj}} P(PK)$$

provides a map $\theta_1 : I \times I \times E \times E \to K$, such that $\theta_1 | \dot{I}^2 \times E \times E$ is the function indicated in the following diagram (functions in the diagram are evaluated at (b_1, b_2) unless otherwise specified),

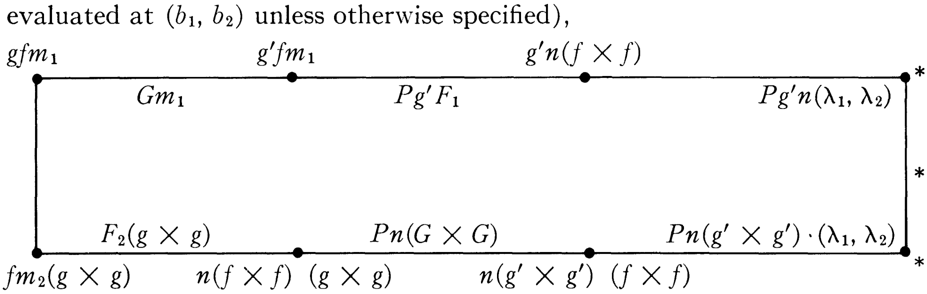

Diagram 3

where the top line is the second coordinate of $g'' \circ s_1(b_1, \lambda_1), (b_2, \lambda_2))$ and the bottom line is the second coordinate of $s_2 \circ (g'' \times g'')((b_1, \lambda_1), (b_2, \lambda_2))$.

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There exists a cross section $\xi : B^l \to E$. Let $\theta_2 = \theta_1 \circ (\mathrm{id} \times \xi \times \xi)$: $I^2 \times B^l \times B^l \to K$. By using Diagram 3, we can represent θ_2 in Diagram 4:

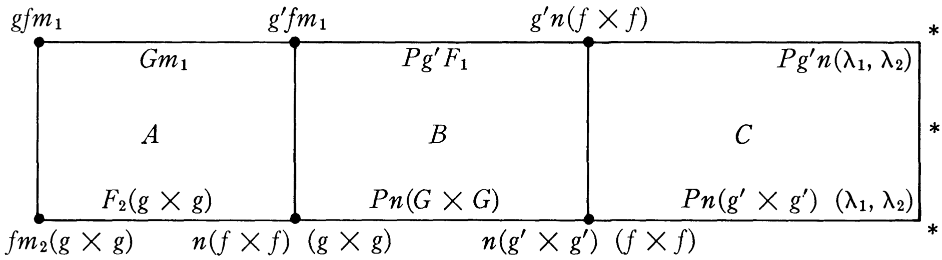


Diagram 4

where $\xi(b_i) = (b_i, \lambda_i)$, $i = 1, 2$, and A, B and C indicate the restriction of θ_2 on appropriate parts of I^2.

Because g' is a homotopy equivalence and K is an Eilenberg-Maclane space, g' is an H-map. Therefore there exists a multiplier Q' of g'. (Note that, Q' is unique up to homotopy.) Define the map $C_1 : I \times I \times B^l \times B^l \to K$ as follows:

$$C_1(t, s, b_1, b_2) = Q'(\lambda_1(s), \lambda_2(s))(t).$$

Therefore on $\dot{I}^2 \times B^l \times B^l$ the function C_1 is the function indicated in the following diagram.

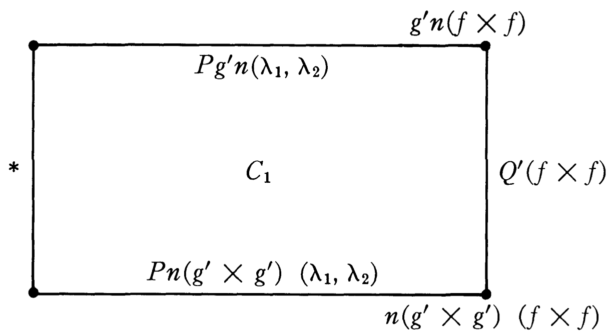
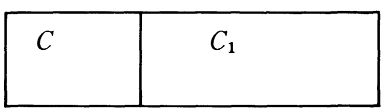


Diagram 5

If we glue C_1 to C from the right, the top and bottom line are the paths $-Pg'n(\lambda_1, \lambda_2) + Pg'n(\lambda_1, \lambda_2)$ and $-Pn(g' \times g')(\lambda_1, \lambda_2) + Pn(g' \times g')(\lambda_1, \lambda_2)$, $ng'(f \times f)(b_1, b_2)$ respectively. Therefore, by the homotopy extension property, we can deform



to a map $C_2 : I^2 \times B^l \times B^l \to K$ such that $C_2|\dot{I}^2 \times B^l \times B^l$ preserves the boundary conditions and satisfies the conditions indicated in the following

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square

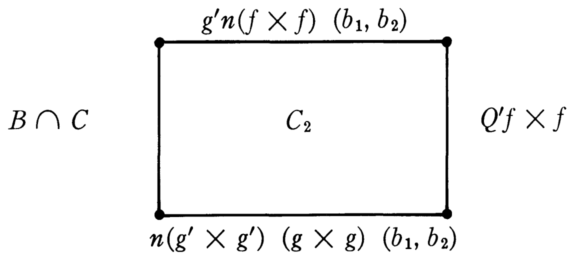

Diagram 6

where $B \cap C$ is the restriction of C to its left vertical side. Therefore if we glue C_1 to θ_2 from the right, we obtain a boundary condition preserving map $D_1 : I \times I \times B^l \times B^l \to K$, such that $D_1|\dot{I}^2 \times B^l \times B^l$ satisfies the properties indicated in the following diagram

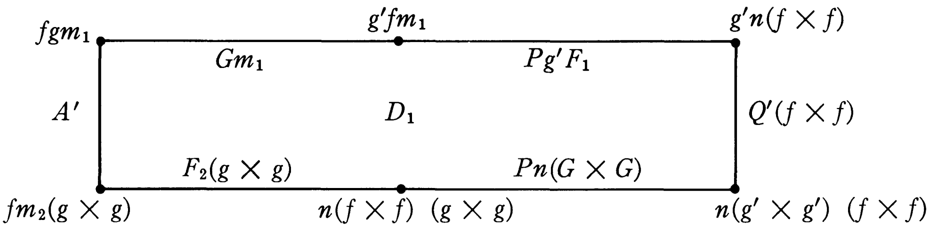

Diagram 7

where A' is the restriction of A to its left vertical side.

Using obstruction theory we can extend the composite map

$$I \times B^l \times B^l \xrightarrow{\text{id} \times \xi \times \xi} I \times E \times E \xrightarrow{\quad J \quad} E \xrightarrow{\quad \pi \quad} B$$

to a multiplier of $g : (B, m_1) \to (B, m_2)$, and let us call this multiplier Q. It is obvious that $f \circ Q|B^l \times B^l = A'$.

By adding boundary conditions to D_1, we define a map D_2 roughly as indicated in the following diagram, where the dotted lines have the direction pointing out of the paper.

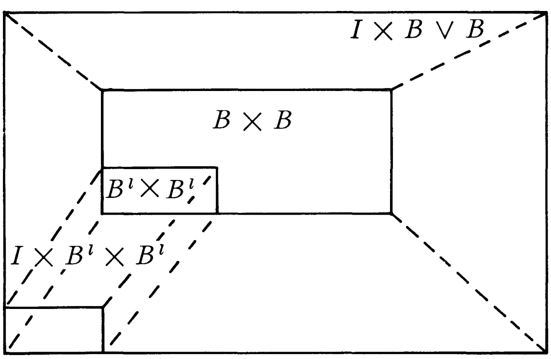

Diagram 8

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The precise definition of $D_2 : I \times I \times B^l \times B^l \cup I \times I \times B \vee B \cup \dot{I}^2 \times B \times B \to K$ is as follows:

$$D_2(t, x, b, *) = D_2(t, s, *, b) = G(b)(s + 2t - 1).$$

(Note that we use the convention that $G(b)(r) =$ initial point if $r < 0$ and $G(b)(r) =$ terminal point if $r > 1$.)

$$\begin{aligned} D_2(t, 1, b_1, b_2) &= g'F_1(b_1, b_2)(2t - 1) && \text{if } t \geqq 1/2 \\ &= Gm(b_1, b_2)(2t) && \text{if } t \leqq 1/2 \\ D_2(t, 0, b_1, b_2) &= n(G(b_1, b_2))(2t - 1) && \text{if } t \geqq 1/2 \\ &= F_2(g(b_1), g(b_2))(2t) && \text{if } t \leqq 1/2 \end{aligned}$$

$$\begin{aligned} D_2(0, s, b_1, b_2) &= fQ(b_1, b_2)(s) \\ D_2(1, s, b_1, b_2) &= Q'(f(b_1), f(b_2))(s) \\ D_2|I \times I \times B^l \times B^l &= D_1. \end{aligned}$$

It is routine to verify that D_2 is a well defined function. By obstruction theory, D_2 can be extended to a map $D_3 : I \times I \times B \times B \to K$, which will provide us a secondary homotopy D. And Q, Q' and D which we found above will satisfy (ii) of Theorem 2.2. Hence the theorem is proved.

We shall use $F_1 \sim F_2$ (via g and g') to mean that we use $g : (B, m_1) \to (B, m_2)$ and $g' : K \to K$ as the functions in (i) of Definition 1.4.

COROLLARY. $s(F_1) \simeq s(F_2)$ (via g'') *if and only if that* $F_1 \sim F_2$ (*via* g *and* g') *where* g'', g *and* g' *are related as in Diagram* 2.

Let $M(E)$ denote the H-equivalence classes of multiplications on E. Since every multiplication on E is $\simeq$ equivalent to some $s(F)$ for some multiplier F, in the light of Theorem 2.2 and the corollary, to study $M(E)$ we need only to study the "$\sim$" equivalence classes of multipliers.

Let $s : M_m(f) \to H(E)$ be defined by $s\{F\} = \{s(F)\}$ and let $\Psi : H(E) \to M(E)$ be the quotient map which is defined by the fact that $\simeq$ is finer than $\cong$.

THEOREM 2.3. *Suppose* $g : (B, m_1) \to (B, m_2)$ *is a homotopy equivalence and* H*-map. If there exists a map* $g' : K \to K$ *such that* $f \circ g$ *is homotopic to* $g' \circ f$, *then*

$$\operatorname{Im}\{\Psi \circ s : M_{m_1}(f) \to M(E)\} = \operatorname{Im}\{\Psi \circ s : M_{m_2}(f \to M(E)\}.$$

Proof. For any $\{F\} \in M_{m_2}(f)$, by using the homotopy extension property, the map $g'^{-1} \circ (-Gm_1 + fQ + F(g \times g) + n(G \times G) - Q'(f \times f))$ (see Diagram 1) can be deformed to a multiplier F^* of $f : (B, m_1) \to K$, where "+" and "−" are the usual joining of paths with direction and G, Q, and Q' are as given in Picture 1. It is obvious that $F \sim F^*$ (via g and g').

Note on Theorem 2.3. The F^* defined in the proof is unique up to homotopy relative to boundary conditions, and we shall use F^* frequently in Sections 3 and 4.

Define a relation "\$" in $H(B)$ as follows: $\{m_1\}\$\{m_2\}$ if and only if there exist homotopy equivalences $g : (B, m_1) \to (B, m_2)$ and $g' : K \to K$ which are H-maps such that $f \circ g$ is homotopic to $g' \circ f$. It is easy to see "\$" is a well defined equivalence relation on $H(B)$. Let $Q(B)$ be a set of representatives, one from each "\$" equivalence classes in $H(B)$.

For any multiplication m on B, let us define a relation, $R_m(f)$ on $M_m(f)$ as follows: $(\{F_1\}, \{F_2\}) \in R_m(f)$ if and only if

(i) there exist homotopy equivalences $g : (B, m) \to (B, m)$ and $g' : K \to K$ which are H-maps such that $f \circ g$ is homotopic to g' of, and

(ii) $F_1 \sim F_2$ (via g and g').

It is easy to verify that $R_m(f)$ is an equivalence relation.

From Theorem 1.10, 2.3 and corollary, we get:

THEOREM 2.4. *$M(E) = \bigcup_{m \in Q(B)} M_m(f)/R_m(f)$. Moreover it is a disjoint union.*

3. Some information about $R_m(f)$.

Note. From now on we deliberately make $F \in M_m(f)$ ambiguous; $F \in M_m(f)$ means a class or a representative of the class of multipliers.

THEOREM 3.1. (i) $M_m(*) = H^l(B \wedge B, \Sigma)$.

(ii) *Let $J_1, J_2 \in M_m(*)$. Then $J_1 \sim J_2$ if and only if there exist homotopy equivalences $g : (B, m) \to (B, m)$ and $g' : K \to K$ which are H-maps such that*

$$g'_*(J_1) - (g \times g)^*(J_2) \in \{\mathrm{Im}\ (\tilde{m}^* : H^l(B, \Sigma) \to H^l(B \wedge B, \Sigma))\}$$

where $\tilde{m}^*(x) = m^*(x) - 1 \otimes x - x \otimes 1$.

Proof. (i) is Lemma 2.1 in [**1**].

(ii) From the definition of $J_1 \sim J_2$ (Definition 1.4), there exist homotopy equivalences $g : (B, m) \to (B, m)$ and $g' : K \to K$ which are H-maps. There also exist $G : B \to \Omega K$ and a secondary homotopy D as in Diagram 1. The edges of the rectangle in Diagram 1 are loops in K when $f = *$. In fact on the two vertical sides, they are trivial loops. Therefore the existence of the secondary homotopy D is equivalent to

$$g'_*(J_1) + m^*(G) = 1 \otimes G + G \otimes 1 + (g \times g)^*(J_2).$$

It is equivalent to say

$$g'_*(J_1) - (g \times g)^*(J_2) \in \mathrm{Im}\ (\tilde{m}^* : H^l(B, \Sigma) \to H^l(B \wedge B, \Sigma)).$$

From Theorem 1.1 in [**2**], there exists a map $q : K \times K \to K$ such that the following composite

$$K \to \begin{array}{c} K \xrightarrow{\mathrm{id}} K \\ \times \qquad \times \\ K \xrightarrow[q]{} K \end{array} \xrightarrow{n} K$$

is homotopic to the trivial map. In fact, K can be taken as an abelian group: thus q as a strict inverse.

Let $F \in M_m(f)$. For any $J \in M_m(*)$, we define $F \oplus J$ to be the class containing or the multiplier equal to (we allow this ambiguity) the composite.

$$I \times B \times B \xrightarrow{\text{diagonal}} \begin{array}{c} I \times B \times B \xrightarrow{F} K \\ \times \qquad\qquad \times \\ I \times B \times B \xrightarrow[J]{} K \end{array} \xrightarrow{n} K$$

Let $F' \in M_m(f)$. By the property of q, the restriction of the composite

$$I \times B \times B \xrightarrow{\text{diagonal}} \begin{array}{c} I \times B \times B \xrightarrow{F} K \xrightarrow{q} K \\ \times \qquad\qquad \times \qquad \times \\ I \times B \times B \xrightarrow[F]{} K \xrightarrow[\text{id}]{} K \end{array} \xrightarrow{n} K$$

on $I \times B \vee B$ is homotopic to the trivial map. Therefore by using the homotopy extension property, we can deform uniquely up to the homotopy relative to boundary conditions to a multiplier $J' \in M_m(*)$. We define $F \ominus F' = J'$.

As in § 2 in [**1**], we can show $F\oplus$ and $F\ominus$ are well defined bijections and are inverse to each other. ($F\ominus$ defined here differs from $F\ominus$ defined in [**1**], but it is easy to show there is a homotopy between them which preserves the boundary condition.)

Now we want to show that $F\oplus$ preserves the "$\sim$" relation. As a matter of fact, we shall see, in Theorem 3.2, that by using $F\oplus$, the "$\sim$" relation on $M_m(*)$ will determine the "$\sim$" relation on $M_m(f)$.

Before the statement of Theorem 3.2, we will first set up some notation.

Let $g : (B, m) \to (B, m)$ and $g' : K \to K$ be fixed homotopy equivalences and H-maps such that $g' \circ f$ is homotopic to $f \circ g$. Let Q and Q' be fixed multipliers of g and g' respectively. For any $F_1, F_2 \in M_m(f)$ and a homotopy G from $g' \circ f$ to $f \circ g$, let $D(G, F_1, F_2)$ be a secondary homotopy in Definition 1.4 which relates to G, F_1, F_2 and those fixed g, g', Q and Q' according to Diagram 1.

Let g, g', Q, Q' and G be fixed. Fix an $F \in M_m(f)$ and as in the note and proof of Theorem 2.3 we let F^* be the multiplier of f derived from deforming

$$f'^{-1} \circ (-Gm_1 + fQ + F(g \times g) + n(G \times G) - Q'(f \times f)).$$

Let $D_0 = D(G, F^*, F)$ be the obvious secondary homotopy of F and F^*. From Theorem 2.2 in [**1**], for each $H_1, H_2 \in M_m(f)$, there exist $J_1, J_2 \in M_m(*)$ such that $F \oplus J_1 = H_1$ and $F^* \oplus J_2 = H_2$ in $M_m(f)$. Then we have the following theorem.

THEOREM 3.2. *There exists a secondary homotopy $D(G', H_1, H_2)$ for $H_1 \sim H_2$ if and only if there exists a map $H : B \to \Omega K$ and a secondary homotopy $D(H, J_1, J_2)$ for $J_1 \sim J_2$.*

Proof. For any homotopy G, G' from $g' \circ f$ to $f \circ g$, there exists a map $H : B \to \Omega K$ such that the composite

$$B \to \begin{matrix} B \xrightarrow{G} PK \\ \times \qquad \times \\ B \xrightarrow[H]{} \Omega K \end{matrix} \to PK$$

is homotopic to G' relative to the initial and terminal points.

For any $D(H, J_1, J_2)$, define $D_0 + D(H, J_1, J_2)$ to be the composite of

$$I \times I \times B \times B \longrightarrow \begin{matrix} I \times I \times B \times B \xrightarrow{D_0} K \\ \times \qquad\qquad \times \\ I \times I \times B \times B \xrightarrow[D(H, J_1, J_2)]{} K \end{matrix} \xrightarrow{n} K .$$

It is a straightforward argument to show $D_0 + D(H, J_1, J_2)$ is a secondary homotopy for $H_1 = F^* \oplus J_1 \sim F \oplus J_2 = H_2$. Define a function $D' : I \times I \times B \vee B \to K$ by

$$D'(t, s, x, *) = D'(t, s, *, x) = \begin{cases} H(s + 2t - 1), & \text{when } 0 \leqq s + 2t - 1 \leqq 1 \\ *, & \text{otherwise.} \end{cases}$$

By the property of q, for any $D(G', H_1, H_2)$, where $H_1, H_2 \in M_m(f)$, the restriction to $I \times I \times BvB$ of the composite

$$I \times I \times B \times B \xrightarrow{\text{diagonal}} \begin{matrix} I \times I \times B \times B \xrightarrow{D(G', H_1, H_2)} \\ \times \\ I \times I \times B \times B \xrightarrow[D_0]{} \end{matrix}$$

$$\begin{matrix} K \xrightarrow{\text{id}} K \\ \times \qquad \times \\ K \xrightarrow[q]{} K \end{matrix} \xrightarrow{n} K$$

is homotopic to D'. Therefore this composite can be deformed to a secondary homotopy D'' for $F^* \ominus H_1 \sim F \ominus H_2$ in $M_m(*)$, which satisfies the properties described in the following diagram.

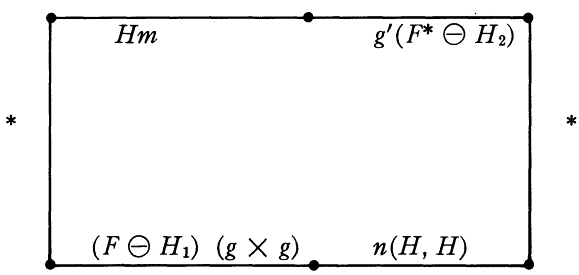

Diagram 9

Since $J_1 \sim F^* \ominus H_1$ and $J_2 \sim F \ominus H_2$, the theorem is proved.

Since K is an Eilenberg-Maclane space, we can use obstruction theory to show that Q' is unique up to homotopy.

For any $T \in M_m(* : B \to B)$ and $Q \in M_m(g : B \to B)$, let $Q \oplus T$ be the composite

$$B \times B \to \begin{array}{c} B \times B \xrightarrow{Q} PB \\ \times \qquad\quad \times \\ B \times B \xrightarrow[T]{} \Omega B \end{array} \xrightarrow{m} B$$

By a similar argument as to show $F\oplus$ is a bijection or as in pp. 1057-1059 in [**1**], $Q \oplus : M_m(* : B \to B) \to M_m(g : B \to B)$ is a bijection.

Let $L^l(B \wedge B, \Sigma)$ be the subgroup of $H^l(B \wedge B, \Sigma)$ generated by $\operatorname{Im}(\tilde{m}^* : H^l(B, \Sigma) \to H^l(B \wedge B, \Sigma))$ and $\operatorname{Im}(\Omega f)^* : [B \wedge B, \Omega B] \to H^l(B \wedge B, \Sigma))$. Let F and F^* be as before. Then we have:

THEOREM 3.3. *For any $F^* \oplus J_1$, $F \oplus J_2 \in M_m(f)$, where $J_1, J_2 \in M_m(*)$, $F^* \oplus J_1 \sim F \oplus J_2$ (via g and g') if and only if*

$$g_*'(J_1) - (g \times g)^*(J_2) \in L^l(B \wedge B, \Sigma).$$

Proof. $F^* \oplus J_1 \sim F \oplus J_2$ (via g and g') if and only if there exists a multiplier $Q \oplus T$ of $g : (B, m) \to (B, m)$, where Q is the fixed multiplier of g and $T \in M_m(* : B \to B)$, and a homotopy G' from $g' \circ f$ to $f \circ g$ such that there exists a secondary homotopy D_1 which satisfies the properties indicated in the following diagram.

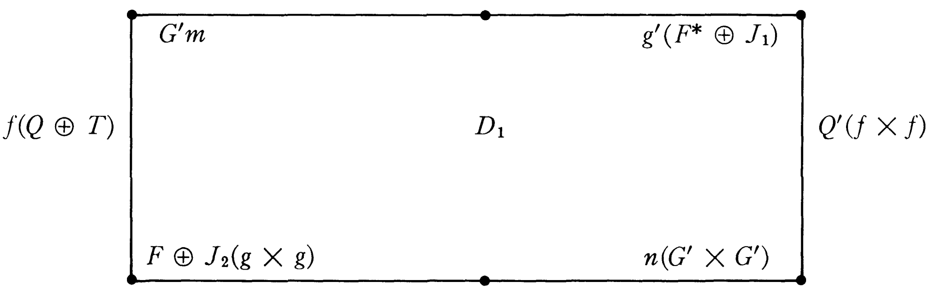

Diagram 10

Because $Q \oplus T$ and $Q + m_2(T, m_2(g \times g))$ are homotopic relative to end points, and so are $(F \oplus J_2)(g \times g) + fm_2(T, m_2(g \times g))$ and $F \oplus (J_2 \oplus fT(g^{-1} \times g^{-1}))(g \times g)$, where "+" means the joining of two paths, therefore D_1 can be deformed to a secondary homotopy D_2 which satisfies the properties indicated in the following diagram.

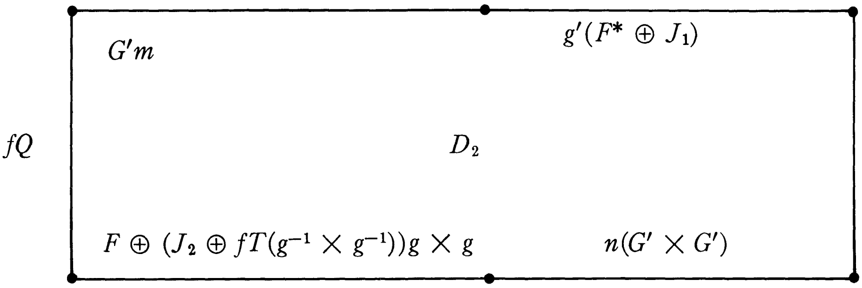

Diagram 11

Therefore from Theorem 3.2 we know there exists a secondary homotopy $D(H, J_1, J_2 \oplus fT(g^{-1} \times g^{-1}))$ for J_1 and $J_2 \oplus fT(g^{-1} \times g^{-1})$. By Theorem 3.1, $g_*'(J_1) - (g \times g)^*(J_2 \oplus fT(g^{-1} \times g^{-1})) \in \operatorname{Im} \tilde{m}^*$. Therefore $g_*'(J_1) - (g \times g)^*(J_2) \in L^l(B \wedge B, \Sigma)$. We omit the proof for the converse part, which is a straight forward argument.

Evidently Theorem 3.3 provides a way to determine $R_m(f)$.

4. Example. In general it is hard to find $R_m(f)$, because of the lack of information about g's and what is $F \ominus F^* \in M_m(*)$. In the case when $B = K(Z_{p^n}, t)$, we can give a method to compute $M(E)$. We shall demonstrate this method in the following example.

Let $B = K(Z_5, 3)$, $K = K(Z_5, 12)$ and $f = p^1\beta a$ where a is the fundamental class of $H^3(B, Z_5)$. Let $b = \beta a$.

We know:

$$H^{11}(B \wedge B, Z_5) = Z_5 \langle 2b \otimes ab, 2ab \otimes b, b^2 \otimes a, a \otimes b^2 \rangle$$

$$\tilde{m}^*(H^{11}(B, Z_5)) = Z_5 \langle b^2 \otimes a + a \otimes b^2 + 2b \otimes ab + 2ab \otimes b \rangle = \operatorname{Im} \tilde{m}^*$$

and $\operatorname{Im} (f^* : [B \wedge B, \Omega B] \to [B \wedge B, \Omega K]) = 0$, where $Z_5\langle x,y \rangle$ denotes the Z_5 module generated by x and y.

The homotopy equivalences and H-maps $g : B \to B$ and $g' : K \to K$ such that $g' \circ f$ is homotopic to $f \circ g$ are maps induced from Iso $(Z_5, Z_5) = \{\text{units of } Z_5\} = \{1, 2, 3, 4\}$. We will use 1, 2, 3, 4 to indicate the corresponding homotopy equivalences.

Fix $F \in M_m(f)$, let F_i^* be the multiplier in $M_m(f)$ defined in the Note to Theorem 2.3 with $g = g' = i$, $1 \leqq i \leqq 4$ and any choice of G, Q and Q'. From Theorem 2.2 in [**1**], there exists $A_i \in M_m(*)$ such that $F \oplus A_i = F_i^*$. We want to know more about A_i. By the definition of F_i^* we know the difference between $F_i^* = F \oplus A_i$ and the composite of

$$B \times B \xrightarrow{i \times i} B \times B \xrightarrow{F} K^I \xrightarrow{P(i^{-1})} K^I$$

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is in the Im $\tilde{m}^*$. Therefore the difference of the following two composites is in Im $\tilde{m}^*$:

$$(i)\ B \times B \xrightarrow{2\times 2} B \times B \xrightarrow{2\times 2} B \times B \xrightarrow{F} K^I \xrightarrow{P(3)} K^I \xrightarrow{P(3)} K^I$$

$$(ii)\ B \times B \rightarrow \begin{matrix} B \times B \xrightarrow{2\times 2} B \times B \xrightarrow{F} K^I \xrightarrow{P(3)} K^I \\ \times \qquad\qquad \times \qquad\qquad \times \qquad\qquad \times \\ B \times B \xrightarrow{2\times 2} B \times B \xrightarrow{A_2} K^I \xrightarrow{P(3)} K^I \end{matrix} \xrightarrow{P(n)} K^I$$

where n is the multiplication on K. Therefore $A_4 - 3A_2 \in \operatorname{Im} \tilde{m}^*$. Similarly we can show that $A_3 - 2A_2 \in \operatorname{Im} \tilde{m}^*$. Since 1 is the identity map, it is obvious that $A_1 \in \operatorname{Im} \tilde{m}^*$. Let $H = F \oplus 4A_2 \in M_m(f)$, and let H_i^* be the multiplier in $M_m(f)$ defined in the Note to Theorem 2.3 with $g = g' = i$ $1 \leqq i \leqq 4$. From Theorem 2.2 in [**1**], there exists $B_i \in M_m(*)$ such that $H \oplus B_i = H_i^*$. A similar technique and the fact that $H = F \oplus Z_2$ shows that $B_2 \in \operatorname{Im} \tilde{m}^*$. Therefore by suitable choice of F we can assume $A_2 \in \operatorname{Im} \tilde{m}^*$.

Using Theorem 3.1, it can be shown that for any $x, y \in M_m(*) = H^{11}(B \wedge B, Z_5)$ the following four statements hold:

(i) $x \sim y$ (via 1 and 1) if and only if $x - y \in \operatorname{Im} \tilde{m}^*$

(ii) $x \sim y$ (via 2 and 2) if and only if $2x - 4y \in \operatorname{Im} \tilde{m}^*$

(iii) $x \sim y$ (via 3 and 3) if and only if $3x - 4y \in \operatorname{Im} \tilde{m}^*$

(iv) $x \sim y$ (via 4 and 4) if and only if $4x - y \in \operatorname{Im} \tilde{m}^*$.

Using Theorem 3.3, it can be shown that for any $w, z \in M_m(f)$ the following statement holds for $i = 1, 2, 3, 4$:

(*) $w \sim z$ (via i and i) if and only if

$$(w, z) \in \{(F \oplus A_i \oplus x,\ F \oplus y) | x \sim y \text{ (via } i \text{ and } i)\}.$$

Since $A_2 \in \operatorname{Im} \tilde{m}^*$, A_3 and A_4 are both in $\operatorname{Im} \tilde{m}^*$. Hence $M(E) = \{F \oplus V | V \in M_m(*) = H^{11}(B \wedge B, Z_5)\}/R_m(f)$ where

$$R_m(f) = \{(F \oplus V, F \oplus V') | V - V' \in \operatorname{Im} \tilde{m}^*, 2V - 4V' \in \operatorname{Im} \tilde{m}^*,$$
$$3V - 4V' \in \operatorname{Im} \tilde{m}^* \text{ or } 4V - V' \in \operatorname{Im} \tilde{m}^*\}.$$

In the above example the computations for the relations among A_2, A_3 and A_4 rely on the fact that Iso (Z_5, Z_5) is a cyclic group. Since Iso (Z_{p^n}, Z_{p_n}) is a cyclic group if p is odd or $p = 2$ but $n = 1$, and Iso (Z, Z) is a cyclic group, therefore we can apply the same argument to those cases.

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