A CLASS OF SINGULAR FUNCTIONS

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Kakutani (**2**) has proved a very general theorem, giving necessary and sufficient conditions for two infinite product measures to be mutually absolutely continuous. To formulate Kakutani's result, let us first recall that a *measurable space* is a pair (E, B), where B denotes a Borel field (also called σ-ring) of subsets of E, and a measure m on this space is a countably additive set function on B (see Halmos (**1**)). m is a *probability measure* if it is non-negative and $m(E) = 1$. If m_1 and m_2 denote two probability measures on the same space which are mutually absolutely continuous and f denotes the Radon-Nikodym derivative of m_2 with respect to m_1 (so that $m_2(S) = \int_S f\, dm_1$ for $S \in B$), we define the Hellinger-Kakutani functional

$$D(m_1, m_2) = \int_E f^{\frac{1}{2}}\, dm_1.$$

It is easily shown that $D(m_1, m_2) = D(m_2, m_1)$ and $0 < D(m_1, m_2) \leqq 1$ with equality only if $m_1 = m_2$ (see **2**, p. 215). Kakutani's remarkable theorem is then the following (**2**, p. 218).

THEOREM (Kakutani). *For each $n = 1, 2, \ldots$, let (E_n, B_n) be a measurable space, and m_n, m_n' mutually absolutely continuous probability measures on it. Let (E, B) be the infinite product of these spaces, and M, M' the corresponding product measures. Then*

(a) *M and M' are either mutually absolutely continuous or mutually singular*;

(b) *The necessary and sufficient condition that M and M' be mutually absolutely continuous is that*

$$\prod_{n=1}^{\infty} D(m_n, m_n') > 0.$$

In the present paper we consider the special case when each E_n is a finite set, and m_n' is the uniform distribution on E_n. Thus, if E_n has k_n points, m_n' gives to each of these the measure k_n^{-1}, and m_n gives to the ith point a positive measure a_{ni},

$$\sum_{i=1}^{k_n} a_{ni} = 1.$$

Then, Kakutani's theorem gives, as the necessary and sufficient condition that M and M' be mutually absolutely continuous,

$$\prod_{n=1}^{\infty} \left(k_n^{-\frac{1}{2}} \sum_{i=1}^{k_n} a_{ni}^{\frac{1}{2}} \right) > 0. \tag{1}$$

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Now, (E, B) can be identified in a natural way (measure theoretically, not topologically, of course) with the unit interval I and M' with the Lebesgue measure on I. Therefore, if k_n and the a_{ni} are so chosen that the product on the left side of (1) is *zero*, the measure on I associated with M will be singular with respect to Lebesgue measure. In other words, Kakutani's theorem gives a method for constructing a large class of singular measures (or singular functions) on the unit interval.

Our aim in the present paper is to explore this situation. We shall not use Kakutani's theorem (which could, however, also have been used as the basis for our work) but shall give another method tailored to the special situation, and work with *functions* (the primitive functions) rather than measures. Our method is based on notions from probability theory. We shall not explicitly deal with product measures, nor make the identification just referred to of the interval with an infinite product of discrete spaces. As an application of our main theorem, we construct (Theorem 4) an increasing singular function whose modulus of continuity is $O(tp(t))$, where $p(t)$ is any function which increases without limit as t tends to zero.

Actually, I had completed this work without knowledge of Kakutani's paper. My attention was drawn to Kakutani's work by A. L. Shields, and independently by A. M. Vershik of the Leningrad State University, whom I wish to thank also for a most enlightening conversation on the subject.

1. A general method for constructing singular functions. Let there be given for each $n = 1, 2, \ldots$ a finite sequence a_{ni} $(i = 1, 2, \ldots, k_n)$ of non-negative numbers with $\sum_i a_{ni} = 1$. Define step functions $g_n(x)$ on $[0, 1)$ by

$$g_n(x) = a_{ni}k_n, \qquad \frac{i}{k_n} \leqq x < \frac{i+1}{k_n} \quad (i = 0, 1, \ldots, k_n - 1).$$

Extend $g_n(x)$ to $0 \leqq x < \infty$ by periodicity, and define

$$f_n(x) = \prod_{i=1}^{n} g_i(r_i x), \qquad F_n(x) = \int_0^x f_n(t)\, dt, \qquad 0 \leqq x \leqq 1,$$

where $r_1 = 1$, $r_i = \Pi_{j=1}^{i} k_{j-1}$ for $i \geqq 2$. Clearly, $F_n(x)$ is continuous and non-decreasing. Since

$$\int_0^1 g_n(x)\, dx = 1$$

and the $g_i(r_i x)$ are easily seen to be statistically independent, we have that $F_n(1) = 1$.

Let us now show that for $m > n$, $F_m(x) = F_n(x)$, whenever x is an integral multiple of $(r_{n+1})^{-1}$. Indeed, writing $h = (r_{n+1})^{-1}$ we have, for integral q, that

$$F_m(gh) = \sum_{i=0}^{q-1} \int_{ih}^{(i+1)h} f_m(x)\, dx$$

and

$$\int_{ih}^{(i+1)h} f_m(x)\,dx = \int_{ih}^{(i+1)h} f_n(x) g_{n+1}(r_{n+1}x) \ldots g_m(r_m x)\,dx.$$

Since for $j \leqq n$, $g_j(r_j x)$ is constant for $ih < x < (i+1)h$, the same is true for $f_n(x)$ and therefore the last integral equals

$$\left(\frac{1}{h}\int_{ih}^{(i+1)h} f_n(x)\,dx\right)\left(\int_{ih}^{(i+1)h} g_{n+1}(r_{n+1}x) \ldots g_m(r_m x)\,dx\right) = \left(\int_{ih}^{(i+1)h} f_n(x)\,dx\right)\left(\frac{1}{h}\int_0^h g_{n+1}(r_{n+1}x) \ldots g_m(r_m x)\,dx\right),$$

and the last integral is just

$$\int_0^1 g_{n+1}(t) g_{n+2}(k_{n+1}t) \ldots g_m(k_{n+1} \ldots k_{m-1}t)\,dt,$$

which equals one (because of the statistical independence of the factors in the integrand), and thus finally,

$$F_m(qh) = \sum_{i=0}^{q-1} \int_{ih}^{(i+1)h} f_n(x)\,dx = F_n(qh),$$

as asserted.

Now, let R_n denote the set of numbers $\{i/r_n\}$, $i = 0, 1, \ldots, r_n$, and $R = \bigcup R_n$. Clearly, R is dense in $[0, 1]$. We define, for $x \in R$,

$$F(x) = \lim F_n(x).$$

This definition is meaningful since, as we just showed, if $x \in R$, the sequence $\{F_n(x)\}$ is constant from some point on. Finally, let us extend $F(x)$ to all of $[0, 1]$ by

$$F(x) = \sup_{y \leqq x; y \in R} F(y).$$

We have thus defined a function $F(x)$, which is non-decreasing on $[0, 1]$, such that $F(0) = 0$, $F(1) = 1$.

THEOREM 1. *The necessary and sufficient condition that $F(x)$ be continuous is that*

$$\lim_{n\to\infty} \prod_{i=1}^{n} A_i = 0, \tag{1}$$

where $A_n = \max_i a_{ni}$.

Proof. Suppose first that (1) holds. We shall prove continuity at each point $t \notin R$; when $t \in R$, a trivial modification of the argument is necessary, which we leave to the reader. Since F is non-decreasing, it is sufficient to show, for every $\epsilon > 0$, that there exist points $x_1 < t$ and $x_2 > t$ with

$$F(x_2) - F(x_1) < \epsilon.$$

Now, for each n we can find i such that

$$ih < t < (i+1)h \qquad (h = (r_{n+1})^{-1}).$$

Hence,

$$F((i+1)h) - F(ih) = F_n((i+1)h) - F_n(ih) = \int_{ih}^{(i+1)h} f_n(x)\,dx \leqq \prod_{i=1}^{n} A_i,$$

and since this is less than ϵ for large n, the result follows. Suppose now, to prove the necessity of (1), that F is continuous. Then it is uniformly continuous, and therefore

$$\operatorname*{Max}_{i} \int_{ih}^{(i+1)h} f_n(x)\,dx$$

tends to zero as $n \to \infty$. This quantity is, however, precisely $\Pi_{i=1}^{n} A_i$, and the theorem is proved.

It may be well to interpret the construction of $F(x)$ in terms of measures. Since $F(x)$ is already defined precisely, we may permit ourselves a somewhat intuitive description. We start with Lebesgue measure on $[0, 1]$. At the first stage we break up $[0, 1]$ into k_1 equal subintervals and redistribute the original unit mass so that these subintervals receive (permanently) the masses $a_{11}, \ldots, a_{1k_1}$, uniformly distributed within the respective subintervals. At the second stage we break up each of these intervals into k_2 equal subintervals, and redistribute the mass within each interval of the first subdivision in proportion to the numbers $a_{21}, \ldots, a_{2k_2}$. Proceeding in this manner, each interval with end points in R is eventually assigned a fixed measure. The limiting measure is that which corresponds to the function $F(x)$. Condition (1) assures that intervals of small lengths are assigned measures which tend to zero; thus there are no "mass points" in the limit measure.

As a simple example we may consider the classical Lebesgue function constructed in terms of the Cantor ternary decomposition. Here, $k_n = 3$ for all n and $a_{n1} = a_{n3} = \frac{1}{2}$, $a_{n2} = 0$. Thus, the functions we construct may be thought of as generalizing Lebesgue's construction. Note, however, that if all a_{ni} are *positive*, the function obtained is strictly increasing.

Before formulating our main result, it will be helpful to introduce some further notation. Let $\{X_n\}$ be a sequence of discrete independent random variables with the following distribution. The event $X_n = t$ has non-zero probability if and only if t is one of the numbers $\{k_n a_{ni}\}$, $i = 1, 2, \ldots, k_n$, and in that case it has probability m/k_n, where m is the number of occurrences of t in the sequence.

THEOREM 2. *$F(x)$ is singular if and only if the random variables*

$$Y_n = \prod_{i=1}^{n} X_i$$

tend in probability to zero.

Proof. Suppose first that Y_n tends to zero in probability, that is, for every positive ϵ and δ we have that

$$\Pr(Y_n > \delta) < \epsilon \quad \text{for } n \geqq N(\epsilon, \delta).$$

Consider now the set E_δ, where $F'(x)$ exists and $F'(x) > \delta$. We shall show that E_δ has (Lebesgue) measure zero for every $\delta > 0$. Let us define $E_{n,\delta}$ to be the union of those intervals of the form $[i/r_{n+1}, (i+1)/r_{n+1}]$ with respect to which $F(x)$, or, which is the same thing, $F_n(x)$, has difference quotient greater than δ. From the definition of Y_n, it is readily seen that the measure of $E_{n,\delta}$ is just $\Pr(Y_n > \delta)$, and hence less than ϵ for n large enough.

Now, each $x \in E_\delta$ lies in every $E_{n,\delta}$ from some value of n (depending on x) onwards. Thus, $E_\delta = \bigcup_{n \geqq 1} G_{n,\delta}$, where $G_{n,\delta} = \bigcap_{j \geqq n} E_{j,\delta}$. Since $G_{n,\delta}$ is an increasing family of sets, each having measure less than ϵ, E_δ has measure not exceeding ϵ. Since ϵ is arbitrary, E_δ has measure zero. Since δ is arbitrary, it follows that $F'(x) = 0$ almost everywhere. Suppose, on the other hand, that the Y_n do not tend in probability to zero. Then there exist positive numbers ϵ_0, δ_0 such that

$$\Pr(Y_n > \delta_0) > \epsilon_0$$

for an infinite sequence of n. That is, E_{n,δ_0} has measure greater than or equal to ϵ_0 for an infinite sequence of n. Hence, there is a set H of measure greater than or equal to ϵ_0, each point of which belongs to infinitely many of the E_{n,δ_0}. For almost all points $x \in H$, we have that $F'(x) \geqq \delta_0$, and therefore $F(x)$ is not singular. Theorem 2 is proved.

Remark. It follows from Kakutani's theorem that F is absolutely continuous whenever the hypothesis of Theorem 2 is not satisfied, once one identifies the present set-up with that of infinite product measures as discussed in the introduction.

Theorem 2 enables us to compute effectively whether $F(x)$ is singular, in particular cases. We shall carry out a detailed analysis only in the following case, fully sufficient for the application we have in mind. Let $k_n = 2$ for all n, and $a_{n1} = \frac{1}{2}(1 + \lambda_n)$, $a_{n2} = \frac{1}{2}(1 - \lambda_n)$. Here we assume that $0 \leqq \lambda_n \leqq \frac{1}{4}$, so that (1) is satisfied and $F(x)$ is continuous.

THEOREM 3. *The necessary and sufficient condition that $F(x)$ be singular is that*

$$\sum_{1}^{\infty} \lambda_n^2 = \infty. \tag{2}$$

Proof. Sufficiency. In the present case, X_n takes the values $1 + \lambda_n$ and $1 - \lambda_n$ with equal probability $\frac{1}{2}$. The random variable $\log X_n$ has mean

$$M_n = \tfrac{1}{2}\log(1 - \lambda_n^2)$$

and dispersion

$$D_n = \frac{1}{2}\left(\log \frac{1 + \lambda_n}{1 - \lambda_n}\right)^2.$$

Thus, recalling that $\lambda_n \leqq \frac{1}{4}$, we readily verify that $M_n \leqq -\frac{1}{2}\lambda_n^2$ and $D_n \leqq 8\lambda_n^2$.

Thus, $\log Y_n$ has mean not exceeding $-\frac{1}{2}B_n^2$ and standard deviation not exceeding $3B_n$, where $B_n^2 = \sum_1^n \lambda_i^2$. By the Čebyšev inequality, the probability that $\log Y_n$ exceeds $-N$, where N is any positive number, does not exceed $(3B_n/(\frac{1}{2}B_n^2 - N))^2$ which tends to 0 as $n \to \infty$ if (2) holds.

Necessity. The characteristic function of $\log X_n$ is

$$\phi_n(x) = \tfrac{1}{2}[\exp(i \log(1 + \lambda_n)x) + \exp(i \log(1 - \lambda_n)x)].$$

It is easy to verify that $\Pi_{n=1}^{\infty}\phi_n(x)$ converges uniformly on every finite x-interval, if $\sum \lambda_n^2 < \infty$. Thus, if (2) does not hold, the distribution functions of the variables $\log Y_n$ converge to a limiting distribution, and Y_n cannot tend in probability to zero.

Remark. Theorem 3 is a special case of Kakutani's theorem, once the necessary identifications are made. On the left side of equation (1) of the introduction, take $k_n = 2$, $a_{n1} = \frac{1}{2}(1 + \lambda_n)$, $a_{n2} = \frac{1}{2}(1 - \lambda_n)$ and we obtain as the necessary and sufficient condition for *singularity* the divergence of

$$\prod_{n=1}^{\infty} \tfrac{1}{2}(\sqrt{(1 + \lambda_n)} + \sqrt{(1 - \lambda_n)})$$

which is equivalent to (2).

As an application of Theorem 3, we prove the following theorem.

THEOREM 4.* *Let $p(t)$ be any function defined and decreasing for $0 < t \leqq 1$ such that $\lim_{t\to 0}p(t) = \infty$. Then there exists an increasing singular function $F(x)$ on $[0, 1]$ such that, for $0 \leqq x_1 < x_2 \leqq 1$,*

$$F(x_2) - F(x_1) \leqq hp(h), \quad \text{where } h = x_2 - x_1. \tag{3}$$

Proof. We shall show that λ_n can be found satisfying (2) such that (3) holds. We may obviously assume, without loss of generality, that $p(t) > 3$. Now, consider a *fixed* choice of x_1 and x_2. Let n be the unique positive integer such that

$$\frac{1}{2^n} \leqq h < \frac{1}{2^{n-1}}, \quad h = x_2 - x_1,$$

and let i be the largest integer not exceeding $2^n x_1$. Then

$$x_2 = x_1 + h < \frac{i+1}{2^n} + \frac{1}{2^{n-1}} = \frac{i+3}{2^n},$$

$$F(x_2) - F(x_1) < F\left(\frac{i+3}{2^n}\right) - F\left(\frac{i}{2^n}\right) = \tag{4}$$

$$F_n\left(\frac{i+3}{2^n}\right) - F_n\left(\frac{i}{2^n}\right) \leqq 3\cdot 2^{-n} \prod_{i=1}^{n} (1 + \lambda_i) \leqq 3he^{S_n},$$

where $S_n = \sum_{i=1}^n \lambda_i$.

**Added in proof.* This theorem was proved, using other methods, by P. Hartman and R. Kershner (*The structure of monotone functions*, Amer. J. Math. *59* (1937), 809–822). We also take this opportunity to refer the reader to forthcoming papers by J.-P. Kahane (in Enseignement Math.) and the author (in Michigan Math. J.) where the analogous problem for the second-order modulus of smoothness is solved.

We wish to make the last quantity in (4) not greater than $hp(h)$, and for this it suffices to show that permissible $\{\lambda_n\}$ can be chosen such that for all $n \geqq 1$, $3e^{S_n} \leqq p(1/2^{n-1})$, since the last quantity does not exceed $p(h)$. Now, writing

$$T_n = \log\left[\frac{1}{3}\, p\left(\frac{1}{2^{n-1}}\right)\right],$$

we see that $\{T_n\}$ is an increasing sequence of positive numbers and $T_n \to \infty$, and to complete the proof we have only to arrange $S_n \leqq T_n$ for all n. Define now λ_n to be zero if there is no integer m such that $T_n \leqq m < T_{n+1}$, and $\lambda_n = \frac{1}{4}$, otherwise. It is readily seen that $\sum_{i=1}^{n} 4\lambda_i \leqq T_n$. Moreover, since $T_n \to \infty$, $\lambda_n = \frac{1}{4}$ infinitely often; therefore, (2) holds and the theorem is proved.

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