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Geometrical representation of subshifts for primitive substitutions

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Abstract. For any primitive substitution whose Perron eigenvalue is a Pisot unit, we construct a domain exchange that is measurably conjugate to the subshift. Additionally, we give a condition for the subshift to be a finite extension of a torus translation. For the particular case of weakly irreducible Pisot substitutions, we show that the subshift is either a finite extension of a torus translation or its eigenvalues are roots of unity. Furthermore, we provide an algorithm to compute eigenvalues of the subshift associated with any primitive pseudo-unimodular substitution.

Key words: substitution subshifts, Rauzy fractals, finite extension of torus translation, eigenvalues of subshifts
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Contents

1. *Introduction and main results*

In the seminal paper [**Rauzy**], Rauzy constructed a geometrical representation of the subshift associated with some particular substitution. He constructed a compact subset of $\mathbb{R}^2$ which is called now a Rauzy fractal, and that tiles the plane and gives a measurable conjugacy between the subshift and a translation on the torus $\mathbb{T}^2$. It was generalized later by many people, such as Arnoux and Ito, see [**AI**].

For irreducible Pisot unit substitutions, it is conjectured that Rauzy fractals give a measurable conjugacy between the subshift and a translation on a torus. What is known is that it gives a finite extension of a torus translation.

THEOREM 1.1. (Host, unpublished) *Let σ be an irreducible Pisot unimodular substitution over an alphabet of $d+1$ letters. Then the uniquely ergodic subshift (Ω_σ, S) is a finite extension of a translation on the torus $\mathbb{T}^d$.*

Recently, Durand and Petite gave a very interesting proof of this result in [**DP**]. Their starting point is to construct a proper substitution whose subshift is conjugate to the subshift of the first substitution (see Theorem 2.4). However, this construction does not preserve irreducibility. So they have to deal with reducible substitutions. Moreover, primitive reducible substitutions naturally arise from some dynamical systems (see for example the 9-letter substitution in [**ABB**], coming from an interval exchange transformation).

We use the strategy for the proof of Durand and Petite to extend Theorem 1.1 to a large class of pseudo-unimodular substitutions, that is, substitutions whose product of all non-zero eigenvalues of the incidence matrix equals ± 1.

THEOREM 1.2. *Let σ be a proper primitive pseudo-unimodular substitution. Assume that the Perron eigenvalue of the incidence matrix is a Pisot number β of degree $d+1$. Additionally, assume that every generalized eigenvector for every other eigenvalue of modulus ≥ 1 has sum zero. Then, the subshift (Ω_σ, S) is a finite extension of a minimal translation on the torus $\mathbb{T}^d$.*

Moreover, we show that the Pisot hypothesis is necessary (see Proposition 8.1). This theorem, together with Theorem 2.4, permits to check easily that many non-proper substitutions have a subshift which is a finite extension of a torus translation. However, we do not know if the reciprocal of Theorem 1.2 is true. Nevertheless, in the particular case of weakly irreducible Pisot substitutions (that is, the only eigenvalue of the primitive incidence matrix of modulus greater that 1 is a Pisot unit, and every eigenvalue of modulus one is a root of unity), we have the following alternative.

THEOREM 1.3. *Let σ be a weakly irreducible Pisot substitution. Then one of the following is true:*

- *eigenvalues of the subshift (Ω_σ, S) are roots of unity;*
- *the subshift (Ω_σ, S) is a finite extension of a minimal translation of the torus $\mathbb{T}^d$, where $d+1$ is the degree of the Pisot number.*

Moreover, there is an algorithm to decide in which case we are.

Notice that for unimodular substitutions, the first point implies that the subshift is weakly mixing by Lemma 6.7. We give a geometrical representation of the subshift for any primitive substitutions whose Perron eigenvalue is a Pisot unit.

THEOREM 1.4. *Let σ be a primitive substitution such that the Perron eigenvalue of its incidence matrix is a unit Pisot number of degree $d+1$. Then, the uniquely ergodic subshift (Ω_σ, S) is measurably isomorphic to a domain exchange (R, E, λ), with $R \subseteq \mathbb{R}^d$.*

In the particular case of irreducible substitutions, this result is due to Durand and Petite. Notice that this result is similar to [**BS**, Theorem 6], but in this paper, we give a proof, and Theorem 2.4 of Durand and Petite permits us to avoid the strong coincidence hypothesis. The set R is a Rauzy fractal, but potentially for another substitution. This result, together with Theorem 1.2, gives a generalization of the main theorem in [**DP**].

We also give a way to compute eigenvalues of the subshift associated with any primitive pseudo-unimodular substitution.

THEOREM 1.5. *Let σ be a primitive proper pseudo-unimodular substitution over an alphabet A. Then $e^{2i\pi\alpha}$ is an eigenvalue of (Ω_σ, S) if and only if there exists a row vector $w \in \mathbb{Z}^A$ such that for every generalized eigenvector v of M_σ for eigenvalues of modulus ≥ 1, we have:*

- *$wv = \alpha$ if v has sum 1;*
- *$wv = 0$ if v has sum 0.*

In this theorem, it is enough to check the condition for any choice of bases of generalized eigenspaces for eigenvalues of modulus ≥ 1. Additionally, thanks to the proprification algorithm of Durand and Petite (see §7.1), this theorem permits to completely describe and compute the set of eigenvalues of the subshift for any primitive pseudo-unimodular substitution. We provide an algorithm and an implementation doing this computation (see §7). Notice that a different way to compute eigenvalues is given in [**FMN**].

Note that Theorem 1.5 implies that eigenvalues of (Ω_σ, S) are in the form $e^{2i\pi\alpha}$ with α in a free $\mathbb{Z}$-module of finite rank in $\mathbb{Q}(\beta)$, where $\beta > 1$ is the Perron eigenvalue of M_σ.

The hypothesis that σ is proper and pseudo-unimodular in Theorem 1.5 are needed only for the direct implication. Moreover, this hypothesis can be lightened. All we need is the fact that if $e^{2i\pi\alpha}$ is an eigenvalue of the subshift, then $\alpha(1, \ldots, 1)M^n \xrightarrow[n\to\infty]{} 0 \bmod \mathbb{Z}^A$. It is the case if there is no non-trivial coboundary and if the initials period is 1 (see [**Host**] for more details and see [**Mossé**]). It is in particular the case if a power of the substitution is left-proper. Additionally, in Theorem 1.2, the hypothesis that σ is proper can also be

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replaced with the hypothesis that a power of σ is left-proper, since it also implies the strong coincidence property.

1.1. *Organization of the paper.* We start in §2 by definitions and notation. Then, in §3, we introduce the notion of a generalized Rauzy fractal. It permits to generalize the notion of a Rauzy fractal to reducible substitutions, with various possible choices of projection. It will allow to get translations on a torus as a factor of the subshift and also to get domain exchanges, but with different projections. Section 4 focuses on the particular choice of projection giving usual Rauzy fractals, for which we get many nice properties. In §5, we prove Theorem 1.4, by constructing usual Rauzy fractals that permits to get domain exchanges. In §6, we prove Theorem 1.5. Then in §7, we provide an algorithm to compute the eigenvalues. In §8, we prove Theorems 1.2 and 1.3. We finish with §9 by giving examples.

2. *Definitions and notation*

This section aims to give all the definitions and notation that will be used in the paper.

2.1. *Algebraic numbers.* An *algebraic number* β is a root of a polynomial with rational coefficients. The smallest unitary polynomial P with rational coefficients such that $P(\beta) = 0$ is called *minimal polynomial*. The *degree* of β is the degree of its minimal polynomial. Two different algebraic numbers are *conjugate* if they have the same minimal polynomial. An algebraic number β is an *algebraic integer* if coefficients of its minimal polynomial are in $\mathbb{Z}$. An algebraic number β is a *unit* if it is an algebraic integer such that the constant term of its minimal polynomial is ± 1. This is equivalent to saying that β and $1/\beta$ are algebraic integers. A *Pisot number* is an algebraic integer $\beta > 1$ whose conjugates γ satisfy $|\gamma| < 1$.

2.2. *Words and worms.* An *alphabet* is a finite set. If A is an alphabet, then we denote by A^* the set of *finite words* over A. We denote by $|u|$ the length of a word u. An *occurrence* of a word w in a word u is the length $|p|$ of a word p such that $u = pws$, where $s \in A^*$ is a word. We denote by $|u|_w$ the number of occurrences of w in u. The *abelianization* of a finite word $u \in A^*$ is the vector $\mathrm{ab}(u) = (|u|_a)_{a \in A}$. For every letter $a \in A$, we denote $e_a = \mathrm{ab}(a)$. The family $(e_a)_{a \in A}$ is the canonical basis of $\mathbb{R}^A$.

The set of *bi-infinite words* over A is $A^{\mathbb{Z}}$. *Infinite words* over A are elements of $A^{\mathbb{N}}$. For a (bi-)infinite word u and for every $n \in \mathbb{N}$, we use the standard notation $u_{[0,n)} = u_0 u_1 \ldots u_{n-1}$. For $n < 0$, we use the convention $\mathrm{ab}(u_{[0,n)}) = -\,\mathrm{ab}(u_{[-n,0)})$.

The usual metric on $A^{\mathbb{Z}}$ is defined for $u \neq v$ by

$$d(u, v) = 2^{-n} \quad \text{where } n = \max\{k \in \mathbb{N} \mid u_{[-k,k]} = v_{[-k,k]}\}.$$

For this metric, $A^{\mathbb{Z}}$ is compact. A *subshift* (Ω, S) is a compact subset $\Omega \subseteq A^{\mathbb{Z}}$ which is invariant under the *shift map*:

$$S : \begin{array}{ccc} A^{\mathbb{Z}} & \to & A^{\mathbb{Z}}, \\ (u_i)_{i \in \mathbb{Z}} & \mapsto & (u_{i+1})_{i \in \mathbb{Z}}. \end{array}$$

The *orbit* of a bi-infinite word $u \in A^{\mathbb{Z}}$ is $\mathcal{O}(u) = \{S^n u \mid n \in \mathbb{Z}\}$. A subshift (Ω, S) is said to be *minimal* if every orbit is dense in Ω, and *aperiodic* if every orbit is infinite.

We define the *worm* associated to a bi-infinite word $u \in A^{\mathbb{Z}}$ as

$$W(u) = \{\text{ab}(u_{[0,n)}) \mid n \in \mathbb{Z}\}.$$

We also define

$$W_a(u) = \{x \in W(u) \mid x + e_a \in W(u)\}.$$

The notion of a worm can also be defined for infinite words in an obvious way (see [**Pythéas**]).

Properties 2.1

- For every $u \in \mathcal{A}^{\mathbb{Z}}$, we have $W(Su) = W(u) - \text{ab}(u_0)$.
- We have

$$W(u) = \bigcup_{a \in A} W_a(u) = \left(\bigcup_{a \in A} W_a(u) + e_a \right) \cup \{0\},$$

  and these unions are disjoint.

2.3. *Matrices and subspaces.* We denote by $I_n \in M_n(\mathbb{N})$, or just I when there is no ambiguity, the identity matrix. A matrix is said to be *irreducible* if its characteristic polynomial is irreducible. Let $M \in M_n(\mathbb{N})$ be a matrix. We say that M is *primitive* if there exists $n \geq 1$ such that every coefficient of M^n is strictly positive. We say that M is *pseudo-unimodular* if the product of all its non-zero eigenvalues, counting multiplicities, equals ± 1. In particular, unimodular matrices are pseudo-unimodular.

We use the following well-known theorem.

THEOREM 2.2. (Perron–Frobenius) *If $M \in M_n(\mathbb{N})$ is primitive, then M has a simple real eigenvalue equal to the spectral radius of M. Moreover, the corresponding eigenvector can be chosen with strictly positive components.*

We call this maximal eigenvalue the *Perron eigenvalue* of M and we call the associated eigenvector a *Perron eigenvector*.

We say that $v \in \mathbb{C}^n$ is a *generalized eigenvector* of M for an eigenvalue β if v is a non-zero vector in the *generalized eigenspace* $\ker((M - \beta I)^k)$ where $k \geq 1$ is the algebraic multiplicity of β.

We extend the notion of projector to linear maps that are not endomorphisms. We say that a linear map $V : \mathbb{R}^n \to \mathbb{R}^d$ is a *projection* along a subspace F of $\mathbb{R}^n$ if $\ker(V) = F$ and $d + \dim(F) = n$. Such a map is onto.

We have the following lemma.

LEMMA 2.3. *Let $V : \mathbb{R}^n \to \mathbb{R}^d$ be a projection along F and let $M : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map such that $M(F) = F$. Then there exists a unique linear map $N : \mathbb{R}^d \to \mathbb{R}^d$ such that $NV = VM$ and we have* $\det(N) = \det(M')$, *where $M' : \mathbb{R}^n/F \to \mathbb{R}^n/F$ is the quotient map.*

Proof. The map V gives an isomorphism $V' : \mathbb{R}^n/F \to \mathbb{R}^d$ and the map M gives a map $M' : \mathbb{R}^n/F \to \mathbb{R}^n/F$. Then, we can define N by $N = V'M'(V')^{-1}$, and it satisfies $NV = VM$ and $\det(N) = \det(M')$. The unicity comes from the fact that V is onto: if N and N' are two such maps, then $(N - N')V = 0$, so $N = N'$. □

We say that a subspace is *rational* if it admits a basis with coefficients in $\mathbb{Q}$. We say that a vector v has a *totally irrational direction* if the coefficients of v are linearly independent over $\mathbb{Q}$. A projection is *totally irrational* if it is a projection along a vector with a totally irrational direction.

2.4. *Substitutions.* We say that a morphism $\sigma : A^* \to A^*$ is *non-erasing* if for every $a \in A$, $|\sigma(a)| \geq 1$. A *substitution* over an alphabet A is a non-erasing morphism of A^*. The *incidence matrix* of a substitution σ is the matrix $M_\sigma = (|\sigma(a)|_b)_{(b,a)\in A^2}$. For any finite word $u \in A^*$, we have the relation $\mathrm{ab}(\sigma(u)) = M_\sigma \, \mathrm{ab}(u)$. We say that a substitution is *primitive*, *irreducible*, or any property that has a meaning for a matrix, if its incidence matrix has the corresponding property.

The *subshift* of a primitive substitution σ is the dynamical system (Ω_σ, S), where Ω_σ is the smallest non-empty compact subset of $A^{\mathbb{Z}}$ invariant under the substitution and by the shift map. We denote it by Ω when there is no ambiguity. It can be shown that for every primitive substitution σ, the subshift (Ω_σ, S) is minimal and uniquely ergodic (see [**Queffélec**, §V.2 and Theorem V.13]). We say that the substitution σ is *aperiodic* if every orbit in the subshift is infinite. Notice that if a substitution is primitive and pseudo-unimodular, then it is aperiodic since the Perron eigenvalue is irrational.

For every finite word $v, w \in A^*$, we denote by $[v \cdot w]$ the *cylinder* of Ω:

$$[v \cdot w] = \{u \in \Omega \mid u_{[0,|w|)} = w \text{ and } u_{[-|v|,0)} = v\}.$$

Additionally, we denote $[w] = \{u \in \Omega \mid u_{[0,|w|)} = w\}$.

A *fixed point* of a substitution σ is a bi-infinite word $u \in A^{\mathbb{Z}}$ such that $\sigma(u) = u$. A *periodic point* of σ is a bi-infinite word $u \in A^{\mathbb{Z}}$ such that there exists $n \geq 1$ such that u is a fixed point of σ^n. We say that a fixed point or a periodic point is *admissible* if it is an element of the subshift Ω_σ. Every primitive substitution has an admissible periodic point.

We say that a substitution σ is *left-proper* (respectively *right-proper*) if there exists a letter $a_0 \in A$ such that for every $a \in A$, $\sigma(a)$ starts (respectively ends) with letter a_0. The substitution is *proper* if it is left-proper and right-proper.

The following theorem is due to F. Durand and S. Petite (see [**DP**, Corollary 9]).

THEOREM 2.4. (Durand–Petite) *Let σ be a primitive substitution. Then there exists a proper primitive substitution ξ such that:*

- *(Ω_σ, S) is conjugate to (Ω_ξ, S);*
- *there exists $l \geq 1$ such that the substitution matrices M_σ^l and M_ξ have the same eigenvalues, except perhaps 0 and 1.*

Moreover, the proof is effective.

We call a *proprification algorithm* an algorithm that inputs a primitive substitution σ and that outputs a proper substitution ξ as in this theorem. We say that we *proprify* a substitution σ if we apply to it such an algorithm, and the output substitution is called a *proprification* of σ. See §7.1 for more details about the proprification algorithm of Durand and Petite.

We say that a substitution is *weakly irreducible Pisot* if it is primitive, the Perron eigenvalue β is a unit Pisot number, and every other eigenvalue of its incidence matrix is either a conjugate of β, a root of unity, or zero. Note that the class of primitive pseudo-unimodular substitutions is strictly larger than this (e.g. Example 9.1 and irreducible Salem substitutions).

2.5. *Prefix–suffix automaton and Dumont–Thomas numeration.* Let σ be a substitution over an alphabet A. The *prefix–suffix automaton* of σ is an automaton whose states are the set A, and whose transitions are $a \xrightarrow{p,s} b$ for every letter $a, b \in A$ and word $p, s \in A^*$ such that $\sigma(a) = pbs$. In all of this article, we denote by $a \xrightarrow{p,s} b$ if and only if $\sigma(a) = pbs$, if there is no ambiguity on what is the substitution σ.

The *abelianized prefix automaton* is the same automaton where we replace transitions $a \xrightarrow{p,s} b$ by $a \xrightarrow{t} b$, where $t = \mathrm{ab}(p)$.

For the subshift Ω_σ and for every letter $a \in A$, we have the relation

$$[a] = \bigcup_{b \xrightarrow{p,s} a} S^{|p|}\sigma([b]).$$

Additionally, we have a similar relation for worms: for every $u \in \Omega$, we have

$$W_a(\sigma(u)) = \bigcup_{b \xrightarrow{t} a} M_\sigma W_b(u) + t.$$

Any word $u \in \Omega$ can be written uniquely in the form

$$u = \sigma^n(v_n)\sigma^{n-1}(p_{n-1}) \ldots \sigma(p_1)p_0 \cdot bs_0\sigma(s_1) \ldots \sigma^{n-1}(s_{n-1})\sigma^n(w_n),$$

for v_n a left-infinite word, w_n a right-infinite word, and such that we have a path $\xrightarrow{p_{n-1},s_{n-1}} \ldots \xrightarrow{p_0,s_0} b$ in the prefix–suffix automaton. We call a *sequence of prefixes* the sequence $(p_n)_{n\in\mathbb{N}} = (p_n(u))_{n\in\mathbb{N}}$ associated to u. The *sequence of abelianized prefixes* of $u \in \Omega$ is defined by $t_n(u) = \mathrm{ab}(p_n(u))$.

Hence, to any word $u \in \Omega$, we associate a unique left-infinite path in the prefix–suffix automaton or in the abelianized prefix automaton (see [**CS**, Proposition 3.2]). Notice that such a path can be considered as a path in a Bratelli diagram of Ω.

2.6. *Eigenvalues of a subshift.* We denote by $\mathbb{S}^1$ the set of complex numbers of modulus one. We say that $\eta \in \mathbb{S}^1$ is an *eigenvalue* of a subshift (Ω, S) if there exists a continuous function $f : \Omega \to \mathbb{S}^1$ called an *eigenfunction* such that $f \circ S = \eta f$. Notice that for primitive substitutions, if we allow the eigenfunctions to be only measurable rather than continuous and the image to be $\mathbb{C}$ rather than $\mathbb{S}^1$, it does not give more eigenvalues (see [**Host**, Theorem 1.4]).

We say that a subshift (Ω, S) is an *extension* of a translation on a torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, if there exists a continuous map $f : \Omega \to \mathbb{T}^d$ and $\alpha \in \mathbb{T}^d$ such that $f \circ S = f + \alpha$. We say moreover that this extension is *finite* if the cardinality of $f^{-1}(x)$ is finite for almost every $x \in \mathbb{T}^d$, for the *Lebesgue measure* that we denote as λ.

Notice that if $1, \alpha_1, \ldots, \alpha_d \in \mathbb{R}$ are linearly independent over $\mathbb{Q}$, then the translation by $\alpha = (\alpha_1, \ldots, \alpha_d)$ on the torus $\mathbb{T}^d$ is minimal and uniquely ergodic. It implies that an eigenfunction f is necessarily almost everywhere constant-to-one, but with a constant that can be infinite if the extension is not finite.

We say that a subshift (Ω, S) is *weakly mixing* if its only eigenvalue is 1 and if this eigenvalue 1 is simple. Notice that if σ is a primitive substitution, then eigenvalues of (Ω_σ, S) are simple since it is uniquely ergodic.

2.7. *Domain exchange.* We call a *domain exchange* a subset $R \subseteq \mathbb{R}^d$, with a map $E : R \to R$ almost everywhere defined for the Lebesgue measure λ such that there exists a finite number of subsets R_a, $a \in A$, such that:

- $R = \bigcup_{a \in A} R_a$ and the union is Lebesgue disjoint;
- each R_a is the closure of its interior;
- the boundary of each R_a has zero Lebesgue measure;
- for every $a \in A$, $E|_{\mathrm{int}(R_a)}$ is a translation;
- $\lambda(R) = \lambda(E(R))$.

We say that a map $f : R \to \mathbb{R}^d$, where $R \subset \mathbb{R}^d$, is a *translation by pieces* if there exists a finite measurable partition $R = \bigcup_{i \in I} R_i$ such that for every $i \in I$, the restriction $f|_{E_i}$ is a translation.

Notice that the map E associated with a domain exchange is a translation by pieces. And note that a translation by pieces is finite-to-one.

3. *Generalized Rauzy fractals*

In this section, we generalize the usual notion of Rauzy fractal. As we will see, the construction depends on the choice of a projection map. For subshifts associated to irreducible substitutions, the choice of the projection is obvious, but for primitive substitutions several choices can be made. One choice gives a domain exchange, and another one permits to get a translation on a torus as a factor.

PROPOSITION 3.1. *Let (Ω, S) be a minimal aperiodic subshift over an alphabet A and let $V : \mathbb{R}^A \to \mathbb{R}^d$ be a linear map. Assume that there exists $u \in \Omega$ such that $VW(u)$ is bounded. Then, the map*

$$\phi : \begin{array}{ccc} \mathcal{O}(u) & \to & \mathbb{R}^d \\ S^n u & \mapsto & V\,\mathrm{ab}(u_{[0,n)}) \end{array}$$

can be extended by continuity to the whole subshift Ω.

Proof. This proposition is a generalization of [**AM**, Lemma 8.2.5], with a more general projection map V and with bi-infinite words rather than right-infinite words. However, the same proof works. □

We call the image $\phi(\Omega)$ a *Rauzy fractal* of Ω. We denote this map by $\phi_{u,V,\Omega}$ and we will omit u, V, or Ω when there is no ambiguity.

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Remark 3.2. With this definition, a Rauzy fractal is always compact. It is possible to give a more general definition that allows unbounded Rauzy fractal as in [**Andrieu**].

The following proposition gives properties of the map ϕ.

PROPOSITION 3.3. *Under the hypothesis of Proposition 3.1, we have the following.*

- *The Rauzy fractal $R = \phi(\Omega)$ is the closure of $VW(u)$.*
- *For every $v \in \Omega$, $\phi(Sv) = \phi(v) + V\ \mathrm{ab}(v_0)$.*
- *For every $v \in \Omega$, ϕ_v is well defined and $\phi_u - \phi_v$ is constant.*
- *If v and w are two bi-infinite words of Ω with the same left-infinite or right-infinite part, then $\phi(v) = \phi(w)$.*

Proof. By continuity of ϕ, R is the closure of $\phi(\mathcal{O}(u))$. Additionally, by construction, $\phi(\mathcal{O}(u)) = VW(u)$. Thus, R is the closure of $VW(u)$.

By construction, we have for every $n \in \mathbb{Z}$, $\phi(S^{n+1}u) - \phi(S^n u) = V\ \mathrm{ab}(u_n)$. Since Ω is minimal, the orbit of u is dense in Ω, and since ϕ is continuous, we get that for every $v \in \Omega$, $\phi(Sv) = \phi(v) + V\ \mathrm{ab}(v_0)$.

Let $v \in \Omega$. Then, the set

$$VW(v) = \{V\ \mathrm{ab}(v_{[0,n)}) \mid n \in \mathbb{Z}\} = \{\phi_u(S^n v) - \phi_u(v) \mid n \in \mathbb{Z}\}$$

is bounded, so ϕ_v is well defined. Additionally, for every $n \in \mathbb{Z}$, we have $\phi_u(S^n v) - \phi_v(S^n v) = \phi_u(v) + V\ \mathrm{ab}(v_{[0,n)}) - V\ \mathrm{ab}(v_{[0,n)}) = \phi_u(v)$. By density of the orbit of v and by continuity, we get that $\phi_u - \phi_v$ is constant to $\phi_u(v)$.

If u and v are two elements of Ω having their right-infinite parts in common, then the proof of Proposition 3.1 shows that $\phi(u) = \phi(v)$. If it is the left-infinite parts that u and v have in common, then we come back to the previous case by symmetry, looking at the mirror of the words. □

The following proposition permits us to show that the Rauzy fractal is well defined for the subshift of a substitution, as soon as the projection and the incidence matrix are compatible.

LEMMA 3.4. *Let σ be a primitive and aperiodic substitution over an alphabet A, and let u be an admissible fixed point of σ. If $V : \mathbb{R}^A \to \mathbb{R}^d$ is a linear map such that $\sum_{n\in\mathbb{N}} \|VM_\sigma^n\|$ converges, then the hypothesis of Proposition 3.1 is satisfied and $\phi_{u,V,\Omega_\sigma} : \Omega_\sigma \to \mathbb{R}^d$ is well defined. Moreover, for every $v \in \Omega$, we have the equality*

$$\phi_{u,V,\Omega}(v) = \sum_{n=0}^{\infty} VM^n t_n(v),$$

where $t_n(v) = \mathrm{ab}(p_n(v))$ is defined in §2.5.

Proof. The subshift (Ω_σ, S) is minimal since σ is primitive. We have $\phi_V(\mathcal{O}(u)) = \{V\ \mathrm{ab}(u_{[0,n)}) \mid n \in \mathbb{Z}\}$. The positive part is described by

$$\{V\ \mathrm{ab}(u_{[0,n)}) \mid n \in \mathbb{N}\} = \left\{ \sum_{n=0}^{N} VM^n t_n \mid u_0 \xrightarrow{t_N} a_N \ldots a_1 \xrightarrow{t_0} a_0,\ N \in \mathbb{N} \right\}.$$

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Since t_n are in a finite set (abelianizations of prefixes of $\sigma(a)$, $a \in A$) and since $\sum_{n\in\mathbb{N}} \|VM^n\|$ converges, we get that the set is bounded. The negative part can be described in the same way and is also bounded. Thus, $VW(v)$ is bounded, so $\phi_V : \Omega_\sigma \to \mathbb{R}^d$ is a well-defined continuous map.

To prove the last equality, note that the sum $f = \sum_{n=0}^{\infty} VM^n t_n$ defines a continuous map $f : \Omega \to \mathbb{R}^d$ since every $t_n : \Omega \to \mathbb{R}^A$ is continuous and since the series is normally convergent. Hence, it is enough to check the equality on the dense subset $\{S^k u \mid k \in \mathbb{N}\}$. Let $k \in \mathbb{N}$. There exists $N \in \mathbb{N}$ such that $t_n(S^k u) = 0$ for every $n \geq N$. Then, we have

$$\begin{aligned} f(S^k u) &= \sum_{n=0}^{N-1} VM^n t_n(S^k u) \\ &= V\ \mathrm{ab}(\sigma^{N-1}(p_{N-1}(S^k u)) \ldots \sigma(p_1(S^k u)) p_0(S^k u)) \\ &= V\ \mathrm{ab}(u_{[0,k)}) \\ &= \phi(S^k u). \end{aligned}$$

□

4. *Usual Rauzy fractal*

The previous section defined a Rauzy fractal for general subshifts and for various choices of projections. In this section, we focus on subshifts associated with primitive substitutions whose Perron eigenvalue of the incidence matrix is a Pisot unit, and we consider a particular choice of projection that permits to have many nice properties. More precisely, we assume the following.

HYPOTHESIS 4.1

- *σ is a primitive substitution over an alphabet A such that the Perron eigenvalue of M_σ is a unit Pisot number β of degree $d+1$.*
- *$u \in \Omega_\sigma$ is an admissible fixed point of σ.*
- *$V : \mathbb{R}^A \to \mathbb{R}^d$ is a projection along* $\ker((M - \beta I)P(M))$, *where $P \in \mathbb{Z}[X]$ is such that the characteristic polynomial of M_σ has the form $\pi_\beta P$, where π_β is the minimal polynomial of β. In other words, V is a projection along every generalized eigenspace except for the conjugates of modulus less than* 1 *of the Perron eigenvalue β.*
- *$\phi = \phi_{u,V,\sigma}$, $R = \phi(\Omega_\sigma)$ and for every $a \in A$, $R_a = \phi([a])$.*

Definition 4.2. Under Hypothesis 4.1, we say that R is a *usual Rauzy fractal* of σ.

Such usual Rauzy fractals have a lot of nice properties.

Properties 4.3. Under Hypothesis 4.1, we have the following properties:

- there exists a unique invertible endomorphism N of $\mathbb{R}^d$ such that $NV = VM_\sigma$ and $|\det(N)| = 1/\beta$;
- the union $R_a = \bigcup_{b \xrightarrow{t} a} NR_b + Vt$ is disjoint in Lebesgue measure;
- V restricted to $W(v)$ is one-to-one, for every $v \in A^{\mathbb{Z}}$;
- each R_a is the closure of its interior;
- each R_a has a boundary of zero Lebesgue measure.

In the remainder of this section, we prove these properties.

The map N is given by Lemma 2.3. The determinant of N is equal to the determinant of the quotient map $M : \mathbb{R}^A/F \to \mathbb{R}^A/F$, where $F = \ker((M - \beta I)P(M))$. The eigenvalues of this quotient map are all the roots of π_β but β. The hypothesis that the Perron eigenvalue is a Pisot unit gives us that the product of the roots of π_β is ± 1, thus we get $\det(N) = \pm 1/\beta$.

Now, we give several lemmas that permits to prove the other properties.

LEMMA 4.4. *We assume Hypothesis 4.1. Then, the pieces R_a, $a \in A$, of the Rauzy fractal are the smallest non-empty compact solutions of the equations*

$$N^{-1}R_a = \bigcup R_b + \mathcal{D}_{a,b}, \quad a \in A,$$

where $\mathcal{D}_{a,b} = \{N^{-1}Vt \mid b \xrightarrow{t} a\}$.

Proof. Since u is a fixed point, we have the union

$$W_a(u) = \bigcup_{b \xrightarrow{t} a} MW_b(u) + t.$$

Then, applying V to both sides and using $NV = VM$, we get

$$R_a = \bigcup_{b \xrightarrow{t} a} NR_b + Vt.$$

Now, assume that R'_a, $a \in A$, are non-empty compact sets satisfying such equations. Since we have $\|N\| < 1$, iterating such equations gives for every $a \in A$,

$$\left\{ \sum_{n=0}^{\infty} N^n V t_n \mid \ldots \xrightarrow{t_n} \cdots \xrightarrow{t_0} a \right\} \subset R'_a.$$

Thus, by Lemma 3.4, we have $R_a \subseteq R'_a$. So R_a, $a \in A$, are indeed the smallest non-empty compact subsets satisfying the equations. □

If we iterate the equations of this lemma, we get

$$N^{-n}R_a = \bigcup R_b + \mathcal{D}^n_{a,b}, \quad a \in A,$$

where $\mathcal{D}^n_{a,b} = \{\sum_{i=0}^{n-1} N^{i-n}Vt_i \mid b \xrightarrow{t_{n-1}} \cdots \xrightarrow{t_0} a\}$.

LEMMA 4.5. *There exists $\epsilon > 0$ such that for every $a, b \in A$ and every $n \in \mathbb{N}$, the set $\mathcal{D}^n_{a,b}$ is ϵ-separated: for all $x \neq y \in \mathcal{D}^n_{a,b}$, $\|x - y\| > \epsilon$.*

In the following, we need some notation. The projection map $V : \mathbb{R}^A \to \mathbb{R}^d$ can be factorized: $V = V_\beta V_P$, where $V_P : \mathbb{R}^A \to \ker(\pi_\beta(M))$ is the projection onto $\ker(\pi_\beta(M))$ along $\ker(P(M))$, and where $V_\beta : \ker(\pi_\beta(M)) \to \mathbb{R}^d$ is a projection along the Perron eigenspace of M. The projection V_β is totally irrational, since it is a projection along the Perron eigenspace of the endomorphism $M|_{\ker(\pi_\beta(M))}$ whose characteristic polynomial π_β is irreducible.

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Proof. As V_P is a projection along a rational subspace, $\Lambda = V_P \mathbb{Z}^A$ is a lattice of $\ker(\pi_\beta(M))$. Additionally, we have $M\Lambda \subseteq \Lambda$ since $\ker(\pi_\beta(M))$ is invariant under M. We have $\det(M|_{\ker(\pi_\beta(M))}) = \pm 1$ since β is assumed to be an algebraic unit. Thus, we have $M^{-1}\Lambda = \Lambda$.

Now let us consider the set

$$\mathcal{T}^n_{a,b} = \bigg\{ \sum_{i=0}^{n-1} M^{i-n} V_P t_i \mid b \xrightarrow{t_{n-1}} \cdots \xrightarrow{t_0} a \bigg\}.$$

It is a subset of Λ since every t_i is in $\mathbb{Z}^A$. Moreover, since the t_i are in a finite set, it stays at a bounded distance $D > 0$ of a hyperplane $\mathcal{P}$ of $\ker(\pi_\beta(M))$ which is the orthogonal complement of a left Perron eigenvector of $M|_{\ker(\pi_\beta(M))}$. Let

$$\mathcal{T}_D = \{x \in \Lambda \mid d(x, \mathcal{P}) \leq D\}.$$

For every $\alpha > 0$, the set $\mathcal{T}_{2D} \cap V_\beta^{-1} B(0, \alpha)$ is finite, since $\ker(V_\beta) \oplus \mathcal{P} = \operatorname{Im}(V_P)$. Thus, as V_β is totally irrational, there exists $\epsilon > 0$ such that $B(0, \epsilon) \cap V_\beta \mathcal{T}_{2D}$ has cardinality one. Then, for every $x, y \in V_\beta \mathcal{T}_D$ such that $\|x - y\| \leq \epsilon$, we have $x - y \in V_\beta \mathcal{T}_{2D} \cap B(0, \epsilon)$ by triangular inequality, so $x = y$. In other words, the set $V_\beta \mathcal{T}_D$ is ϵ-separated.

Since we have for every $a, b \in A$ and every $n \in \mathbb{N}$, $\mathcal{D}^n_{a,b} = V_\beta \mathcal{T}^n_{a,b} \subset V_\beta \mathcal{T}_D$, we get the result. □

Such subsets $V_\beta \mathcal{T}_D$ of $\mathbb{R}^d$ are sometimes called cut-and-project sets.

LEMMA 4.6. *Under Hypothesis 4.1, the projection V is one-to-one on $W(v)$ for any bi-infinite word $v \in A^{\mathbb{Z}}$.*

Proof. Let $V_{\hat\beta} : \mathbb{R}^A \to \ker(M - \beta I)$ be the projection on $\ker(M - \beta I)$ along $\ker(Q(M))$ where $(X - \beta)Q(X)$ is the characteristic polynomial of M. Let us show that $V_{\hat\beta}$ is one-to-one on $W(v)$. We have $V_{\hat\beta} M = \beta V_{\hat\beta}$, so the matrix of $V_{\hat\beta}$ is a left Perron eigenvector of M for any choice of basis of $\ker(M - \beta I)$. Thus, we can choose a basis of $\ker(M - \beta I)$ such that it has strictly positive coordinates. Now, if we take two distinct elements of $W(v)$, their difference is the abelianization of a non-empty word, so it is a non-negative and non-zero vector of $\mathbb{Z}^A$. Thus, its image by $V_{\hat\beta}$ is strictly positive and $V_{\hat\beta}$ is one-to-one on $W(v)$. As, we have $V_{\hat\beta} = V_{\hat\beta} V_P$, it proves that V_P is also one-to-one on $W(v)$. Then, the total irrationality of V_β and the fact that $V_P W(v)$ is rational give the result. □

LEMMA 4.7. *Under Hypothesis 4.1, the Lebesgue measure of R_a is non-zero for every $a \in A$.*

Proof. The proof is similar to the proof of [**SW**, Proposition 2.8]. Thanks to Lemma 4.5, one can choose $\epsilon > 0$ such that for every $n \in \mathbb{N}$, the set $\mathcal{D}^n_a = \bigcup_{b \in A} \mathcal{D}^n_{a,b}$ is 2ϵ-separated. Thanks to Lemma 4.6, the cardinality of the set $\mathcal{D}^n_{a,u_0}$ is $|\sigma^n(u_0)|_a = e_a^t M^n e_{u_0}$, where u_0 is the first letter of the fixed point u. Hence,

$$\lambda\Big(\bigcup_{x \in \mathcal{D}^n_a} B(x, \epsilon)\Big) \geq \sum_{x \in D^n_{a,u_0}} \lambda(B(x, \epsilon)) = e_a^t M^n e_{u_0} \lambda(B(0, \epsilon)).$$

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Let D be large enough such that for every $b \in A$, $N \bigcup_{x \in \mathcal{D}_b^n} B(x, D) \subseteq B(0, D)$. Then the sequence of sets $N^n \bigcup_{x \in \mathcal{D}_a^n} B(x, D)$ decreases and its intersection is R_a, thus

$$\lambda(R_a) = \lim_{n \to \infty} \lambda\bigg(N^n \bigcup_{x \in \mathcal{D}_a^n} B(x, D)\bigg) \geq \liminf_{n \to \infty} \lambda(N^n \bigcup_{x \in \mathcal{D}_{a,u_0}^n} B(x, \epsilon)).$$

This limit is strictly greater than zero since $1/\beta^n M^n$ converges to the matrix in the canonical basis of the projector $V_{\hat{\beta}}$ defined in the proof of Lemma 4.6 and since we have $e_a^t V_{\hat{\beta}} e_{u_0} > 0$. □

LEMMA 4.8. *Under Hypothesis 4.1, for every $a \in A$, R_a has a non-empty interior and is the closure of its interior.*

Note that this lemma has similarities with [**AM**, Lemma 8.3.4] but it is not equivalent: it has different hypotheses, different conclusion, and the tools used are not the same even if in both cases the idea is to use the self-similarity of the objects. To show this lemma, we use the following theorem due to Sirvent and Wang, see [**SW**, Theorem 3.1].

THEOREM 4.9. (Sirvent–Wang) *Let $(X_1, \ldots, X_J)$ be the attractor of a strongly connected graph-directed IFS*

$$A(X_i) = \bigcup_{j=1}^{J} (X_j + \mathcal{D}_{ij}), \quad i = 1, \ldots, J.$$

Assume that there exists $\epsilon > 0$ such that the sets $\mathcal{D}_{i,j}^m$ are ϵ-separated for all i, j, m, and assume that X_1 has a positive Lebesgue measure. Then every X_i has a non-empty interior and is the closure of its interior.

Proof of Lemma 4.8. Let us show that the hypotheses of Theorem 4.9 are fulfilled. The sets R_a, $a \in A$, are the attractor of an equation of this form, with $A = N^{-1}$ and $\mathcal{D}_{ij} = \{N^{-1} V t \mid j \xrightarrow{t} i\}$ thanks to Lemma 4.4. The sets $\mathcal{D}_{a,b}^n$ are ϵ-separated by Lemma 4.5. Additionally, the sets R_a have non-zero Lebesgue measure by Lemma 4.7. Therefore, we can apply the theorem and it gives the result. □

It remains to show that the union

$$R_a = \bigcup_{b \xrightarrow{t} a} N R_b + V t$$

is disjoint in measure. We follow a classical argument due to Host (see [**AI**]). We have the inequality

$$\lambda(R_a) \leq \sum_{b \xrightarrow{t} a} \lambda(N R_b) = \frac{1}{\beta} \sum_{b \xrightarrow{t} a} \lambda(R_b).$$

Let $X = (\lambda(R_a))_{a \in A}$. We get the inequality $X \leq (1/\beta) M X$ since the matrix of the prefix–suffix automaton is M. Now we use the following lemma.

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LEMMA 4.10. (Perron–Frobenius) *Let M be a primitive positive matrix, with maximal eigenvalue λ. Suppose that v is a positive vector such that $Mv \geq \lambda v$. Then the inequality is an equality and v is an eigenvector with respect to λ.*

Proof. See [**AI**, Lemma 11]. □

We deduce from this lemma that the inequality $X \leq (1/\beta)MX$ is an equality, thus the union $R_a = \bigcup_{b \xrightarrow{t} a} NR_b + Vt$ is disjoint in Lebesgue measure.

Now, to prove that each R_a has a boundary of zero Lebesgue measure, it suffices to use that some R_{a_0} has non-empty interior and to iterate

$$R_{a_0} = \bigcup_{b \xrightarrow{t_n} \cdots \xrightarrow{t_0} a_0} N^{n+1} R_b + \sum_{k=0}^{n} VM^k t_k,$$

up to having a term of the union of the form $N^{n+1}R_a + t$ completely included in the interior of R_{a_0}. As the union is disjoint in Lebesgue measure, it gives that the boundary of R_a has zero Lebesgue measure.

It finishes the proof of Properties 4.3.

5. *Conjugacy with a domain exchange*

In this section, we prove Theorem 1.4. The domain exchange is obtained as a usual Rauzy fractal for a proper substitution thanks to the following. The following proposition will also be useful to construct finite extensions of torus translations. It is a generalization of [**AM**, Lemma 8.2.7].

PROPOSITION 5.1. *Assume Hypothesis 4.1 and assume that σ is proper. Then we have the following:*

- *the unions $R = \bigcup_{a \in A} R_a = \bigcup_{a \in A} R_a + Ve_a$ are disjoint in Lebesgue measure;*
- *we can define a domain exchange almost everywhere by*

$$E : \begin{array}{rcl} R & \to & R \\ x & \mapsto & x + Ve_a \quad \text{if } x \in R_a, \end{array}$$

  and it is invertible;
- *ϕ is a measurable conjugacy between the uniquely ergodic subshift (Ω_σ, S) and the domain exchange (R, E, λ).*

In particular, we have the following theorem.

THEOREM 5.2. *Let σ be a primitive proper substitution such that the Perron eigenvalue of the incidence matrix is a unit Pisot number of degree $d+1$. Then, the uniquely ergodic subshift (Ω_σ, S) is measurably isomorphic to a domain exchange (R, E, λ), with $R \subset \mathbb{R}^d$.*

In these results, the hypothesis that σ is proper can be replaced with the strong coincidence hypothesis (see [**AI**]). Notice that this result is already stated in [**BS**, Theorem 6], but without proof, and it is proven but not stated in [**SW**] (they assume additional hypotheses that are not really used in their proof). The result could be generalized by

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avoiding the hypothesis that the Pisot number is a unit by considering p-adic spaces, but it would complicate the proof.

Proof of Proposition 5.1. Let $a_0 \in A$ be the letter such that for every $b \in A$, $\sigma(b)$ starts with the letter a_0. Hence for every letter $b \in A$, $b \xrightarrow{0} a_0$ is a transition in the abelianized prefix automaton. Thus, the union $\bigcup_{b\in A} NR_b$ appears in the union $R_{a_0} = \bigcup_{b \xrightarrow{t} a_0} NR_b + Vt$, so it is Lebesgue disjoint by Properties 4.3. Then,

$$\lambda(R) = \lambda\bigg(\bigcup_{b\in A} R_b + Ve_b\bigg) \leq \sum_{b\in A} \lambda(R_b) = \lambda\bigg(\bigcup_{b\in A} R_b\bigg) = \lambda(R),$$

thus the union $\bigcup_{b\in A} R_b + Ve_b$ is also Lebesgue disjoint.

Then, the domain exchange E can be defined almost everywhere and is invertible. Then, let $F_0 = (\bigcup_{a\in A} \mathrm{int}(R_a)) \cap (\bigcup_{a\in A} \mathrm{int}(R_a) + Ve_a)$. The maps E and E^{-1} are everywhere defined in F_0. Then, for every $n \in \mathbb{N}$, we define by induction the open subsets $F_{n+1} = E(F_n) \cap E^{-1}(F_n) \cap F_0$. The intersection $F = \bigcap_{n\in\mathbb{N}} F_n$ is a subset of R of full Lebesgue measure being invariant under E and E^{-1}.

Now we define the natural coding. Since V is one-to-one on $W(u)$, ϕ is one-to-one on $\mathcal{O}(u)$, and we can define the map $\chi : G \to A$, where $G = F \cup \phi(\mathcal{O}(u))$, by $\chi(x) = a$ if $x \in F \cap R_a$ and $\chi(x) = u_n$ if $x = \phi(S^n u)$. Additionally, we can also define E on $\phi(\mathcal{O}(u))$ by $E(\phi(S^n u)) = \phi(S^{n+1}u) = \phi(S^n u) + V\ \mathrm{ab}(u_n)$ and E is well defined on G. Then, we define the coding map, well defined on G by

$$\mathrm{cod} : \begin{array}{ccc} G & \to & A^{\mathbb{Z}}, \\ x & \mapsto & (\chi(E^n x))_{n\in\mathbb{Z}}. \end{array}$$

We have $\mathrm{cod} \circ E = S \circ \mathrm{cod}$ and the restriction of $\mathrm{cod} \circ \phi$ to $\mathcal{O}(u)$ is the identity.

Let us show that $\mathrm{Im}(\mathrm{cod}) \subseteq \Omega$. For $x \in \phi(\mathcal{O}(u))$, we have $\mathrm{cod}(x) \in \mathcal{O}(u) \subseteq \Omega$. Let $x \in F$. For every $N \in \mathbb{N}$, since the set F_N is open, there exists a neighborhood U of x such that for every $n \in [-N, N]$, $\chi \circ E^n|_U$ is a constant. Additionally, since $\phi(\mathcal{O}(u))$ is dense in R, U contains an element of $\phi(\mathcal{O}(u))$. Thus, $\mathrm{cod}(x)$ is arbitrarily close to an element of $\mathcal{O}(u)$ so it is in Ω.

Moreover, the map ϕ is continuous. Thus for every $x \in F$ and every $\epsilon > 0$, there exists a neighborhood U of x whose image by $\phi \circ \mathrm{cod}$ has a diameter at most ϵ. Additionally, for $y \in U \cap \phi(\mathcal{O}(u)) \cap B(x, \epsilon)$, we have $\phi \circ \mathrm{cod}(y) = y$, thus $|\phi \circ \mathrm{cod}(x) - x| \leq |\phi \circ \mathrm{cod}(x) - \phi \circ \mathrm{cod}(y)| + |y - x| \leq 2\epsilon$. We deduce that $\phi \circ \mathrm{cod}$ is the identity map of G.

Now, let μ be the push-forward measure of the Lebesgue measure λ by the continuous map $\mathrm{cod}\,|_F$. Then μ is an invariant measure and we get that (Ω, S, μ) is isomorphic to (R, E, λ). □

Thank to Theorem 2.4, we can proprify σ. Thus, Theorem 1.4 is a consequence of Theorem 5.2.

6. *Eigenvalues of the dynamical system*

This section aims to prove Theorem 1.5. Note that for one implication, we do not need properness or unimodularity.

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PROPOSITION 6.1. *Let σ be a primitive aperiodic substitution. Assume that there exists a row vector $w \in \mathbb{Z}^A$ such that for every generalized eigenvector v for an eigenvalue of modulus ≥ 1, $wv = \alpha$ if v has sum 1 and $wv = 0$ if v has sum 0, then $e^{2i\pi\alpha}$ is an eigenvalue of (Ω_σ, S).*

For the reciprocal, we need the following proposition (see [DP, Proposition 13]).

PROPOSITION 6.2. *Let σ be a primitive proper substitution. If $e^{2i\pi\alpha}$ is an eigenvalue of (Ω_σ, S), then $\alpha(1, \ldots, 1)M_\sigma^n \xrightarrow[n\to\infty]{} 0$ mod $\mathbb{Z}$.*

Additionally, we give a characterization of this condition.

LEMMA 6.3. *Let $\alpha \in \mathbb{R}$ and let $M \in M_d(\mathbb{Z})$ be a pseudo-unimodular matrix. We have the equivalence:*

$$\alpha(1, \ldots, 1)M^n \xrightarrow[n\to\infty]{} 0 \textit{ mod } \mathbb{Z}^d,$$

$$\Updownarrow$$

$$\textit{there exists } w \in \mathbb{Z}^d,\ (\alpha(1, \ldots, 1) - w)M^n \xrightarrow[n\to\infty]{} 0.$$

To prove this equivalence, we need the following.

LEMMA 6.4. *Let $M \in M_d(\mathbb{Z})$ be a pseudo-unimodular matrix. Then there exists $m \in \mathbb{N}_{\geq 1}$ such that for all $y \in \mathbb{Z}^d \cap \mathrm{Im}(M^m)$, there exists $x \in \mathbb{Z}^d \cap \mathrm{Im}(M^m)$ such that $M^m x = y$.*

Proof. Let $m \geq 1$ such that $\ker(M^m)$ and $\mathrm{Im}(M^m)$ are supplementary subspaces. Since $\mathrm{Im}(M^m)$ is a rational subspace, the intersection $\Lambda = \mathrm{Im}(M^m) \cap \mathbb{Z}^d$ is a lattice of $\mathrm{Im}(M^m)$. Let $f : \mathrm{Im}(M^m) \to \mathrm{Im}(M^m)$ be the restriction of M^m to $\mathrm{Im}(M^m)$. We have $f(\Lambda) \subseteq \Lambda$ and the pseudo-unimodular hypothesis gives $\det(f) = \pm 1$. The matrix of f in a basis of the lattice is in $GL_r(\mathbb{Z})$, where r is the rank of M^m, thus $f^{-1}(\Lambda) \subseteq \Lambda$. □

Proof of Lemma 6.3. Assume that

$$\alpha(1, \ldots, 1)M^n \xrightarrow[n\to\infty]{} 0 \bmod \mathbb{Z}^d.$$

Let $a_n \in \mathbb{Z}^d$ be the row vector such that $\alpha(1, \ldots, 1)M^n - a_n \in (-1/2, 1/2]^d$. We have $a_n M - a_{n+1} \xrightarrow[n\to\infty]{} 0$, so there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $a_{n+1} = a_n M$. Let $m \geq 1$ be given by Lemma 6.4 for the matrix M^t. Then, there exists a row vector $w \in \mathbb{Z}^d$ such that $a_n = wM^n$ for every $n \geq (n_0 + 1)m$. Hence, we have $(\alpha(1, \ldots, 1) - w)M^n \xrightarrow[n\to\infty]{} 0$. The reciprocal is obvious. □

Now we give another characterization of the condition.

LEMMA 6.5. *Let $\alpha \in \mathbb{R}$, let M be a matrix of size d, and let $w \in \mathbb{Z}^d$ be a row vector. We have $(\alpha(1, \ldots, 1) - w)M^n \xrightarrow[n\to\infty]{} 0$ if and only if for every generalized eigenvector v for an eigenvalue of modulus ≥ 1,*

$$\begin{cases} wv = \alpha & \textit{if } v \textit{ has sum } 1, \\ wv = 0 & \textit{if } v \textit{ has sum } 0. \end{cases}$$

Moreover, the convergence is exponential.

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Proof. [$\Longrightarrow$] Let us show that for every generalized eigenvector v for an eigenvalue β with $|\beta| \geq 1$, we have $(\alpha(1, \ldots, 1) - w)v = 0$. We show it by induction on $k \geq 1$ such that $(M - \beta I)^k v = 0$ and $(M - \beta I)^{k-1} v \neq 0$.

- If $k = 1$, then $(\alpha(1, \ldots, 1) - w)M^n v = \beta^n(\alpha(1, \ldots, 1) - w)v \xrightarrow[n\to\infty]{} 0$. Thus, it implies that $(\alpha(1, \ldots, 1) - w)v = 0$.
- If $k > 1$, for every $n \in \mathbb{N}$, we have

$$M^n v - \beta^n v = \sum_{i=1}^{k-1} \binom{n}{i} \beta^{n-i}(M - \beta I)^i v \in \ker((M - \beta I)^{k-1}),$$

so by the induction hypothesis, we have $\beta^n(\alpha(1, \ldots, 1) - w)v \xrightarrow[n\to\infty]{} 0$ and we conclude.

Now, if v has sum 1, the equality $(\alpha(1, \ldots, 1) - w)v = 0$ implies $\alpha = wv$. If v has sum 0, it implies $wv = 0$.

[$\Longleftarrow$] Let $x \in \mathbb{R}^d$. Then, there exists coefficients $c_1, \ldots, c_d \in \mathbb{C}$ and generalized eigenvectors $v_1, \ldots, v_d$ such that $x = c_1 v_1 + \cdots + c_d v_d$. We can assume that each vector v_i as a sum 0 or 1. Then, we have $(\alpha(1, \ldots, 1) - w)M^n x = \sum_{i=1}^{d} c_i(\alpha(1, \ldots, 1) - w)M^n v_i$. If v_i is associated to an eigenvalue β_i with $|\beta_i| < 1$, then $M^n v_i \xrightarrow[n\to\infty]{} 0$. Otherwise, the hypothesis gives $(\alpha(1, \ldots, 1) - w)v_i = 0$. Thus, for all $x \in \mathbb{R}^d$, $(\alpha(1, \ldots, 1) - w)M^n x \xrightarrow[n\to\infty]{} 0$. We conclude that $(\alpha(1, \ldots, 1) - w)M^n \xrightarrow[n\to\infty]{} 0$. Moreover, the convergence is exponential. □

LEMMA 6.6. *Let σ be a primitive aperiodic substitution and let $\alpha \in \mathbb{R}$. Suppose there exists a row vector $w \in \mathbb{Z}^A$ such that* $(\alpha(1, \ldots, 1) - w)M_\sigma^n \xrightarrow[n\to\infty]{} 0$.

Then, $e^{2i\pi\alpha}$ is an eigenvalue of the dynamical system (Ω_σ, S).

Proof. Let $v = \alpha(1, \ldots, 1) - w$. Thanks to Lemma 6.5, the convergence of vM^n is exponential. Now by Lemma 3.4, the map $\phi_v = \phi_{u,v,\Omega_\sigma}$ is well defined, for an admissible fixed point u of σ (we can assume that σ has an admissible fixed point up to replace σ by a power of itself). For every $a \in A$, we have $ve_a = \alpha \bmod \mathbb{Z}$, so we have $\phi_v \circ S = \phi_v + \alpha \bmod \mathbb{Z}$, by Proposition 3.3.

Thus, we get that $\pi \circ \phi_v : \Omega_\sigma \to \mathbb{R}/\mathbb{Z}$ is well defined, where $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is the canonical projection, and we have $\pi \circ \phi_v \circ S = \pi \circ \phi_v + \alpha$. We conclude that $e^{2i\pi\alpha}$ is an eigenvalue of (Ω_σ, S) for the continuous eigenfunction $e^{2i\pi(\pi\circ\phi_v)}$. □

Now, Proposition 6.1 and Theorem 1.5 are obvious consequences of these lemmas.

In the particular case of unimodular substitutions, Proposition 6.2 gives the following.

LEMMA 6.7. *Let σ be a primitive proper unimodular substitution. Then, the only eigenvalue of the subshift (Ω_σ, S) being a root of unity is* 1.

Proof. Let $\alpha \in \mathbb{Q}$ such that $e^{i2\pi\alpha}$ is an eigenvalue of the subshift (Ω_σ, S), and let $p \geq 1$ be an integer such that $\alpha p \in \mathbb{Z}$. Since M_σ is unimodular, it is in the finite group $GL(\mathbb{Z}/p\mathbb{Z})$ modulo p, thus there exists $k \geq 1$ such that M_σ^k is the identity matrix modulo p. By Proposition 6.2, we have $\alpha(1, \ldots, 1)M_\sigma^{kn} \xrightarrow[n\to\infty]{} 0 \bmod \mathbb{Z}^A$. Thus, $\alpha(1, \ldots, 1) \xrightarrow[n\to\infty]{} 0 \bmod \mathbb{Z}^A$, so $\alpha \in \mathbb{Z}$. □

7. *Explicit computation of eigenvalues*

Thanks to Theorems 2.4 and 1.5, we can compute the eigenvalues of the subshift for any primitive pseudo-unimodular substitution. The aim of this section is to provide an explicit computation algorithm. From an input substitution, we compute a proper substitution in §7.1 and then we compute eigenvalues of the subshift from the proper substitution in §7.2.

7.1. *Proprification algorithm.*

In this subsection, we compute a proprification, as defined in §2.4. We do it by following the proprification algorithm of Durand and Petite (see [**Durand**] and see [**DP**, Corollary 9] for more details).

We start with an input primitive substitution σ. The first step is to compute the *return substitution*. To do it, perform the following.

- Replace σ by a power of itself to ensure that it has an infinite fixed point.
- Let $a \in A$ be the first letter of a left-infinite fixed point $u \in A^{\mathbb{N}}$. We call a *return word* on letter a, a word w such that wa is a subword of u, and such that w has a unique occurrence of the letter a, at the first position.Let $w_0 \in A^*$ be the unique return word such that u starts by w_0. Start with $\mathcal{S} = \{w_0\}$.
- Take out an element w from $\mathcal{S}$. Decompose $\sigma(w)$ as a product of return words (such decomposition is unique). Add to $\mathcal{S}$ every return word w not already seen. Continue until $\mathcal{S}$ is empty.

The number of return words being finite, this terminates and gives a return substitution τ, whose alphabet is the set $\mathcal{R}$ of return words.

Then, we define a substitution ξ over the alphabet $B = \{(r, p) \mid r \in \mathcal{R}, 1 \leq p \leq |r|\}$ by

$$\xi(r, p) = \begin{cases} \psi((\tau(r))_p) & \text{if } 1 \leq p < |r|, \\ \psi((\tau(r))_{[|r|,|\tau(r)|]}) & \text{if } p = |r|, \end{cases}$$

where $\psi : \mathcal{R}^* \to B^*$ is the morphism defined by $\psi(r) = (r, 1)(r, 2) \ldots (r, |r|)$. We can show that a power of ξ is left-proper. Additionally, we easily get a proper substitution from this. However, the fact that a power of ξ is left-proper is enough to apply our results.

Example 7.1. Let $\sigma : 1 \mapsto 213, 2 \mapsto 4, 3 \mapsto 5, 4 \mapsto 1, 5 \mapsto 21$.

The cube of σ has a left-infinite fixed point

$$1421352142135213142135214213142135213142 \ldots$$

Return words on 1 are $a = 142$, $b = 1352$, and $c = 13$, and we get the return substitution $\tau : a \mapsto ababc, b \mapsto abacabc, c \mapsto abac$ which is left-proper.

Then, the substitution ξ is $0 \mapsto 012$, $1 \mapsto 3456$, $2 \mapsto 012345678$, $3 \mapsto 012$, $4 \mapsto 3456$, $5 \mapsto 012$, $6 \mapsto 78012345678$, $7 \mapsto 012$, $8 \mapsto 345601278$, with the identifications $0 = (a, 1)$, $1 = (a, 2)$, $2 = (a, 3)$, $3 = (b, 1)$, $4 = (b, 2)$, $5 = (b, 3)$, $6 = (b, 4)$, $7 = (c, 1)$, $8 = (c, 2)$. The square of ξ is left-proper and its subshift is conjugate to the subshift of σ.

7.2. *Computation of eigenvalues for a proper substitution.* Now, a power of σ is assumed to be left-proper. The computation of eigenvalues of the subshift is as follows.

Let $\mathbb{K}$ be the splitting field of the characteristic polynomial of the incidence matrix M. Let $v_0, \ldots, v_k$ be a family of vectors of $\mathbb{K}^A$ formed as a concatenation of bases of generalized eigenspaces for eigenvalues of modulus ≥ 1, and with v_0 the Perron eigenvector of sum 1. Then, we compute a set $\mathcal{S}$ as follows. Start with $\mathcal{S} = \emptyset$. Then, for every $i \in \{1, \ldots, k\}$:

- if v_i has sum zero, then add it to $\mathcal{S}$;
- otherwise normalize v_i such that it has sum 1, then add $v_i - v_0$ to $\mathcal{S}$.

Now, the possible row vectors $w \in \mathbb{Z}^A$ of Theorem 1.5 are exactly those that are orthogonal to every vector of $\mathcal{S}$. We can describe this set as the kernel of an integer matrix by the following.

Choose a basis of the number field $\mathbb{K}$ seen as a $\mathbb{Q}$-vector space. Decompose each vector of $\mathcal{S}$ in this basis. Form a matrix $M_{\mathcal{S}}$ whose columns are these components. Multiply in place the matrix $M_{\mathcal{S}}$ by an integer to have $M_{\mathcal{S}}$ with coefficients in $\mathbb{Z}$.

Then, we obtain eigenvalues as the set

$$\{e^{2i\pi w v_0} \mid w \in \mathbb{Z}^A \text{ and } w M_{\mathcal{S}} = 0\}.$$

Additionally, we can completely describe this set by computing a basis of the $\mathbb{Z}$-module of such w, and then computing a basis of the $\mathbb{Z}$-module of possible wv_0, using for example the Schmidt normal form.

Example 7.2. [**FMN**, Example 2] Let $\sigma : a \mapsto abdd, b \mapsto bc, c \mapsto d, d \mapsto a$. Its incidence matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix}$$

is irreducible and has two eigenvalues of modulus ≥ 1, associated to eigenvectors of non-zero sum. Hence, the set $\mathcal{S}$ is reduced to one element which is the difference of the two different eigenvectors of sum 1. Here, both eigenvectors live in the field $\mathbb{Q}(\beta)$, with β the Perron eigenvalue, so we can do the computation in this field rather than the splitting field. The computation in this field gives $\mathcal{S} = \{1/41(-6\beta^3 + 9\beta^2 - 17\beta + 7,\ 18\beta^3 - 27\beta^2 - 31\beta + 20,\ -14\beta^3 + 21\beta^2 + 15\beta - 11,\ 2\beta^3 - 3\beta^2 + 33\beta - 16)^t\}$. Thus, in the basis $(1, \beta, \beta^2, \beta^3)$, the matrix $M_{\mathcal{S}}$ is

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$$M_{\mathcal{S}} = \begin{pmatrix} 7 & -17 & 9 & -6 \\ 20 & -31 & -27 & 18 \\ -11 & 15 & 21 & -14 \\ -16 & 33 & -3 & 2 \end{pmatrix}.$$

Then, the set of row vectors $w \in \mathbb{Z}^4$ such that $wM_{\mathcal{S}} = 0$ is the $\mathbb{Z}$-module generated by (1, 1, 1, 1) and (0, 3, 4, 1). Then, we get

$$\{wv_0 \mid w \in \mathbb{Z}^A \text{ and } wM_{\mathcal{S}} = 0\} = \mathbb{Z} + (-\beta^2 + \beta + 2)\mathbb{Z} = \mathbb{Z}[\sqrt{2}],$$

where v_0 is the Perron eigenvector of sum 1.

We can moreover check that σ has no non-trivial coboundary and it has an initial period of 1 (see [**Host**] for more details), so the eigenvalues of the subshift are indeed $e^{2i\pi n\sqrt{2}}$, $n \in \mathbb{Z}$. The computation from a proprification leads to the same result. Notice that there is a mistake in [**FMN**, Example 2]. They claim that eigenvalues are $e^{i\pi n\sqrt{2}}$, $n \in \mathbb{Z}$, but their computation leads to the same result as us, they just forgot a 2 in their conclusion.

We provide an implementation of this algorithm in the Sage computing system (see https://www.sagemath.org/) in the Supplementary Material.

8. *Finite extension of a torus translation*

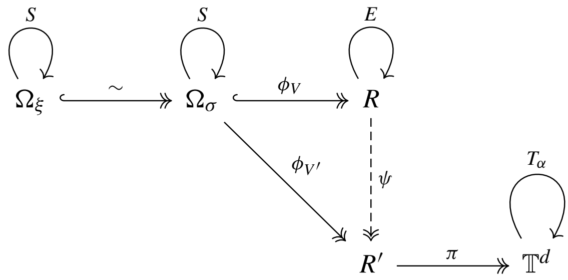
This section mainly aims to prove Theorem 1.2. However, before we show that, the hypothesis that the Perron eigenvalue is Pisot is necessary.

PROPOSITION 8.1. *Let σ be a primitive pseudo-unimodular substitution. If the Perron eigenvalue β of the incidence matrix is not Pisot, then the subshift has strictly less rationally independent eigenvalues than the degree of β.*

Proof. Thanks to Theorem 2.4, we can assume that σ is proper. Let $\beta > 1$ be the Perron eigenvalue of the incidence matrix. Let γ be a conjugate of β with $|\gamma| \geq 1$. Let $\varphi : \mathbb{Q}(\beta) \to \mathbb{Q}(\gamma)$ be the morphism of fields such that $\varphi(\beta) = \gamma$. Let $e^{2i\pi\alpha}$ be an eigenvalue of the subshift. Let v_β be the Perron eigenvector of sum 1. Then, $v_\gamma = \varphi(v_\beta)$ is an eigenvector of sum 1 for the eigenvalue γ. By Theorem 1.5, there exists a row vector $w \in \mathbb{Z}^A$ such that $\alpha = wv_\beta = wv_\gamma$. Thus, we have $\varphi(\alpha) = \alpha$. Hence, α lives in the $\mathbb{Q}$-vector space $\{x \in \mathbb{Q}(\beta) \mid \varphi(x) = x\}$ whose dimension is strictly less than $\deg(\beta)$ since it does not contain β. □

8.1. *Proof of Theorem 1.2.*

Our proof of Theorem 1.2 is similar to the proof of the main theorem in [**DP**]. The idea is the following. We start by showing that there exists rationally independent eigenvalues. Then, we get a minimal translation on the torus $\mathbb{T}^d$ as a factor, with a Rauzy fractal R'. Then, we use the fact that the substitution is proper to construct a domain exchange on another Rauzy fractal R, which is conjugate to the subshift. Additionally, we define a map $\psi : R \to R'$ being a translation by pieces, thus finite-to-one, so the extension is finite.



8.1.1. *Existence of rationally independent eigenvalues.* Let us show that there exists rationally independent numbers $1, \alpha_1, \ldots, \alpha_d$ such that $e^{2i\pi\alpha_1}, \ldots, e^{2i\pi\alpha_d}$ are eigenvalues of the subshift. We decompose the minimal polynomial of $M = M_\sigma$ in the form $\pi_\beta X^k Q$, where π_β is the minimal polynomial of the Perron eigenvalue β, and with $Q(0) \neq 0$. We can assume that $k = 1$ up to replacing σ by σ^k. Then, the row vector $(1, \ldots, 1)$ is orthogonal to $\ker(Q(M))$ thanks to the following lemma.

LEMMA 8.2. *Let M be a matrix with integer coefficients. If γ is an eigenvalue of M such that every associated generalized eigenvector has sum zero, then every generalized eigenvector associated with a conjugate of γ also has sum zero.*

Proof. Let β be a conjugate of γ and let $\varphi : \mathbb{Q}(\beta) \to \mathbb{Q}(\gamma)$ be the morphism of fields sending β to γ. Let v be a generalized eigenvector for the eigenvalue β. Let $k \geq 1$ such that $(M - \beta I)^k v = 0$. Then it gives $(M - \gamma I)^k \varphi(v) = 0$. Thus, $\varphi(v)$ is a generalized eigenvector for the eigenvalue γ. Additionally, we have $\varphi((1, \ldots, 1)v) = (1, \ldots, 1)\varphi(v) = 0$, so $(1, \ldots, 1)v = 0$. □

Let $w_0 = (1, \ldots, 1)$, $w_1, \ldots, w_d$ be row vectors in the orthogonal complement of $\ker(Q(M))$ that are linearly independent modulo the orthogonal complement of $\ker((\pi_\beta Q)(M))$. As we have rational subspaces, we can assume that every vector w_i is in $\mathbb{Z}^A$.

Let v_0 be the Perron eigenvector of sum 1. Let us show that $1, w_1 v_0, \ldots, w_d v_0$ are rationally independent. Let $c_0, \ldots, c_d \in \mathbb{Q}$ such that $c_0 w_0 v_0 + \cdots + c_d w_d v_0 = 0 = (c_0 w_0 + \cdots + c_d w_d) v_0$. Then, as $c_0 w_0 + \cdots + c_d w_d$ is rational, it is orthogonal to every conjugate of v_0, thus to $\ker(\pi_\beta(M))$. As $c_0 w_0 + \cdots + c_d w_d$ is orthogonal to $\ker((\pi_\beta Q)(M))$, every c_i is zero. Additionally, by Proposition 6.1, $e^{2i\pi\alpha_i}$ are eigenvalues of the dynamical system, where $\alpha_i = w_i v_0$, since every generalized eigenvector of zero sum, for an eigenvalue of modulus at least 1, is in $\ker(Q(M))$.

8.1.2. *Minimal torus translation as a factor.* Now, we show that the subshift is an extension of a minimal translation of the torus $\mathbb{T}^d$. Let V' be the matrix whose rows are $\alpha_i(1, \ldots, 1) - w_i$, $i = 1, \ldots, d$. By Lemma 6.5, the convergence $V' M^n \xrightarrow[n\to\infty]{} 0$ is exponential. Thus, by Lemma 3.4, the map $\phi_{V'} = \phi_{u,V',\Omega_\sigma}$ of Proposition 3.1 is well defined. It permits to define a Rauzy fractal $R' = \phi_{V'}(\Omega_\sigma)$. Additionally, for every $x \in \Omega_\sigma$, we have $\phi_{V'}(Sx) = \phi_{V'}(x) + V' \operatorname{ab}(x_0) = \phi_{V'}(x) + \alpha \mod \mathbb{Z}^A$, where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$. Thus, by the map $\pi \circ \phi_{V'} : \Omega \to \mathbb{T}^d$, the subshift is a

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measurable extension of the translation by α on the torus $\mathbb{T}^d$. This translation is minimal since $1, \alpha_1, \ldots, \alpha_d$ are linearly independent over $\mathbb{Q}$.

To end the proof of Theorem 1.2, it remains to show that this extension is finite.

8.1.3. *Construction of a domain exchange.* Now, we show that the subshift is conjugate to a domain exchange. Let Π be the projection on $\mathrm{Im}(M) = \ker((\pi_\beta Q)(M))$ along $\ker(M)$. We have $\Pi M = M$ and we can use the following lemma.

LEMMA 8.3. *Let Π be a rational matrix such that $\Pi M = M$. Then, the row vectors $(1, \ldots, 1)\Pi$, $w_1\Pi, \ldots, w_d\Pi$ are linearly independent.*

Proof. Let $c_0, c_1, \ldots, c_d$ be such that $c_0(1, \ldots, 1)\Pi + c_1 w_1 \Pi + \cdots + c_d w_d \Pi = 0$. We can assume that every c_i is in $\mathbb{Q}$ since Π and w_i have rational coordinates. Then, $w_i M^n = \alpha_i(1, \ldots, 1)M^n + o_{n\to\infty}(1)$, so

$$(c_0 + c_1\alpha_1 + \cdots + c_d\alpha_d)(1, \ldots, 1)M^n = o_{n\to\infty}(1).$$

As, $(1, \ldots, 1)M^n$ diverges when n tends to infinity, we get $c_0 + c_1\alpha_1 + \cdots + c_d\alpha_d = 0$. Then, the linear independence over $\mathbb{Q}$ gives $c_i = 0$ for every i. □

This lemma tells us that $V = V'\Pi$ is of rank d. Now, let us show that V satisfies Hypothesis 4.1 to have a usual Rauzy fractal.

We want to show that $\ker((M - \beta I)MQ(M)) \subseteq \ker(V)$. As $\mathrm{rank}(V) = d = \mathrm{codim}\ker((M - \beta I)MQ(M))$, it will give also the other inclusion. Since polynomials $X - \beta$, X and Q are pairwise coprime, we have

$$\ker((M - \beta I)MQ(M)) = \ker(M - \beta I) \oplus \ker(M) \oplus \ker(Q(M)).$$

By definition of Π, we have $\ker(M) = \ker(\Pi) \subset \ker(V)$. By construction, each w_i is orthogonal to $\ker(Q(M))$ and the row vector $(1, \ldots, 1)$ is also ortho- gonal to $\ker(Q(M))$, thus we have $\ker(Q(M)) \subseteq \ker(V')$. We also have $\ker(M - \beta I) \subseteq \ker(V')$ by Lemma 6.5. Since, $\Pi M = M\Pi$, we obtain $\ker(M - \beta I) \subset \ker \Pi(M - \beta I) = \ker(M - \beta I)\Pi \subseteq \ker(V)$ and similarly $\ker Q(M) \subset \ker(V)$. Then, we get the equality $\ker(V) = \ker((M - \beta I)MQ(M))$.

Hence, Hypothesis 4.1 is satisfied, and ϕ_{u,V,Ω_σ} is well defined and defines a usual Rauzy fractal, for an admissible fixed point u (which exists up to replacing σ by a power of itself).

Moreover, since the substitution σ is proper, Proposition 3.3 gives us a domain exchange on $R = \phi_V(\Omega_\sigma)$ measurably conjugated to the subshift.

8.1.4. *Translation by pieces.* Now, we show that the almost everywhere defined map $\psi = \phi_{V'} \circ \phi_V^{-1} : R \to R'$ is a translation by pieces. For almost every $x \in \Omega$, we have

$$\psi\left(\sum_{n\in\mathbb{N}} VM^n t_n(x)\right) = \sum_{n\in\mathbb{N}} V'M^n t_n(x).$$

However, for every $n \geq 1$, we have $VM^n = V\Pi M^n = V'M^n$. Thus, ψ is a translation by $(V' - V)t$ on each piece $\phi_V(t_0^{-1}(t)) = \bigcup_{b \xrightarrow{t} a} NR_b + Vt$ for every label t of the abelianized prefix automaton, where $t_0 = \mathrm{ab} \circ p_0 : \Omega_\sigma \to \mathbb{Z}^d$ is the map defined in §2.5.

8.1.5. *End of the proof of Theorem 1.2.* Since ψ is a translation by pieces, it is finite-to-one. The map $\pi : \mathbb{R}^d \to \mathbb{T}^d$ restricted to R' is also finite-to-one since R' is bounded. Hence, $\pi \circ \psi : (R, E, \lambda) \to (\mathbb{T}^d, T_\alpha, \lambda)$ is finite-to-one, where T_α is the translation by $\alpha = (\alpha_1, \ldots, \alpha_d)$. Thus, the continuous map $\pi \circ \phi_{V'} : (\Omega, S) \to (\mathbb{T}^d, T_\alpha)$ is almost everywhere finite-to-one. It ends the proof of Theorem 1.2.

Remark 8.4. The map $\psi = \phi_{V'} \circ \phi_V^{-1}$ can always be defined almost everywhere, as in this proof, as soon as V and V' are well defined with ϕ_V almost everywhere invertible. Additionally, we have

$$\psi : \begin{array}{ccc} R & \to & R' \\ \sum_{n=0}^{\infty} VM^n t_n & \mapsto & \sum_{n=0}^{\infty} V'M^n t_n, \end{array}$$

where t_n are labels of left-infinite paths in the abelianized prefix automaton. However, in general, this map does not seem to be always finite-to-one (see Example 9.8).

In the following subsection, we prove the last remaining theorem to prove.

8.2. *Proof of Theorem 1.3.* By Theorem 2.4, we can assume that σ is proper. Additionally, up to replacing σ by a power of itself, we can also assume that the only eigenvalue of the matrix being a root of unity is 1.

If there is no generalized eigenvector v of sum 1 for the eigenvalue 1, then the hypothesis of Theorem 1.2 is satisfied and the subshift is a finite extension of a translation on the torus $\mathbb{T}^d$.

Otherwise, we use Theorem 1.5. Let v be a generalized eigenvector of sum 1 for the eigenvalue 1. As the generalized eigenspace for the eigenvalue 1 is rational, we can assume that v has rational coordinates. Then the eigenvalues of the subshift are of the form $e^{2i\pi\alpha}$ with $\alpha = wv \in \mathbb{Q}$, for some row vectors $w \in \mathbb{Z}^A$. Hence, eigenvalues of the subshift are roots of unity, and it ends the proof of Theorem 1.3.

9. *Examples*

Example 9.1. (Non-Pisot, [**FMN**]) For the primitive unimodular substitution

$$a \mapsto abbbccccccccccdddddddd, b \mapsto bccc, c \mapsto d, d \mapsto a,$$

the subshift has eigenvalues $e^{2in\pi\sqrt{2}}$, $n \in \mathbb{Z}$. The characteristic polynomial $x^4 - 2x^3 - 7x^2 - 2x + 1$ of M_σ is irreducible and the Perron eigenvalue θ_1 is not Pisot since there is another root θ_4 of modulus > 1. The subshift is not weakly mixing, in contrast to what is said in [**FMN**, Example 1]. They made a miscalculation. They say that $\alpha = \theta_1^3/((1+\sqrt{10})\theta_1 + 11 - \sqrt{10})Q(1/\theta_1) = \theta_4^3/((1-\sqrt{10})\theta_4 + 11 + \sqrt{10})Q(1/\theta_4)$ can take only integer values, where Q is a polynomial over $\mathbb{Z}$, but it is false since, for example, $Q(X) = 6X - 11X^2 + 19X^3$ is a solution that gives $\alpha = (7 + \sqrt{2})/3$.

Example 9.2. (Conjugate to a torus translation) The subshift of the weakly irreducible Pisot substitution $1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 14, 4 \mapsto 5, 5 \mapsto 1425$ is measurably conjugate to a translation on $\mathbb{T}^2$. The incidence matrix of a left-proprification has only one eigenvalue of

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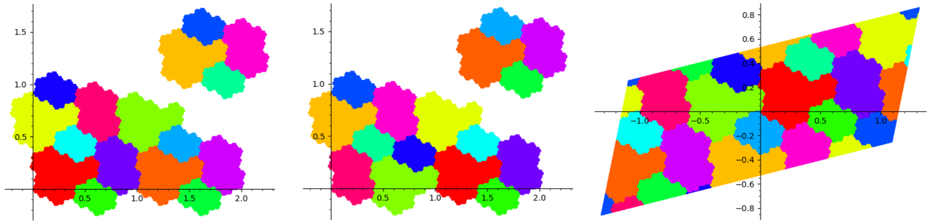

FIGURE 1. Domain exchange and fundamental domain of $\mathbb{T}^2$ for a subshift being measurably conjugate to a translation of $\mathbb{T}^2$.

modulus ≥ 1, thus Theorem 1.2 applies. The domain exchange and fundamental domain are depicted in Figure 1. The eigenvalues of the dynamical system are $e^{2i\pi\alpha}$, with $\alpha \in ((2\beta^2 + \beta - 6)/11)\mathbb{Z} + (\beta^2 + \beta)\mathbb{Z}$, where β is the Pisot number root of $X^3 - X^2 - X - 1$.

Example 9.3. (2-to-1 extension of a torus translation) It can be shown that the subshift of the weakly irreducible Pisot substitution $\sigma : a \mapsto Ab, b \mapsto A, A \mapsto aB, B \mapsto a$ is a 2-to-1 extension of a translation on $\mathbb{T}^1$. Note that it is a 2-to-1 cover of the subshift of the original Fibonacci substitution $a \mapsto ab, b \mapsto a$. A left-proprification of σ has an incidence matrix with eigenvalues zero, roots of unity associated to generalized eigenvectors of sum zero, and golden number and conjugate. Thus, Theorem 1.2 applies. An approximation of the graph of ψ is plotted in Figure 4(right). The eigenvalues of the subshift are $e^{2i\pi n\varphi}$, $n \in \mathbb{Z}$, where φ is the golden number.

Example 9.4. (Presumably infinite extension of a torus translation) The subshift of the primitive substitution $1 \mapsto 11116, 2 \mapsto 1, 3 \mapsto 1111112, 4 \mapsto 1111113, 5 \mapsto 466, 6 \mapsto 566$ is a (presumably infinite) extension of a minimal circle translation. Its incidence matrix is not diagonalizable. Its characteristic polynomial is $(x^2 - 4x - 1)(x^2 - x - 1)^2$. The eigenvalues of the dynamical system are $e^{2i\pi\alpha}$ where $\alpha \in (45/2)\varphi\mathbb{Z}$, where φ is the golden number.

Example 9.5. (Two Pisot, intermediate) The subshift of the primitive substitution

$$1 \mapsto 1116, 2 \mapsto 1, 3 \mapsto 2, 4 \mapsto 3, 5 \mapsto 1146, 6 \mapsto 566$$

is neither a finite extension of a minimal translation of $\mathbb{T}^3$ nor weakly mixing. Indeed, the eigenvalues of the subshift are $e^{2i\pi\alpha}$, where $\alpha \in 57\varphi\mathbb{Z}$, where φ is the golden number, but the degree of the Perron eigenvalue is 4. The characteristic polynomial of this matrix is

$$(x^2 - x - 1) \cdot (x^4 - 4x^3 + 2x^2 - x + 1).$$

Example 9.6. (Two Pisot, weakly mixing) The primitive substitution

$$1 \mapsto 15, 2 \mapsto 2122, 3 \mapsto 122, 4 \mapsto 13, 5 \mapsto 14122$$

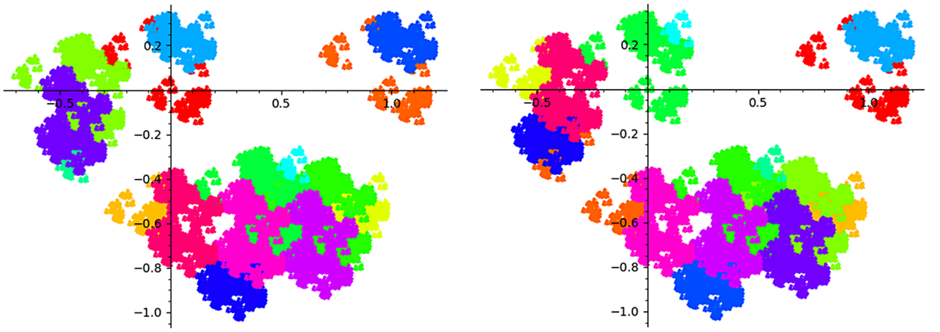

FIGURE 2. Weakly mixing domain exchange associated to a substitution whose incidence matrix has two Pisot eigenvalues.

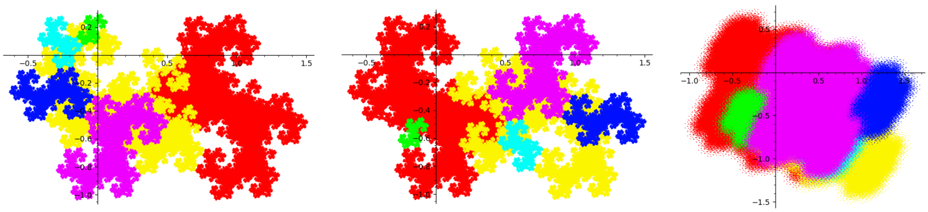

FIGURE 3. Domain exchange and multi-tiling associated to a substitution whose incidence matrix has two Pisot eigenvalues, and whose subshift is an (presumably infinite) extension of a translation on $\mathbb{T}^2$.

gives a weakly mixing subshift. Its incidence matrix has two Pisot eigenvalues of degrees 2 and 3. However, we can describe it geometrically with a domain exchange, see Figure 2.

Example 9.7. (Two Pisot, presumably infinite extension of $(\mathbb{T}^2, T)$) The subshift of the substitution $1 \mapsto 16$, $2 \mapsto 122$, $3 \mapsto 12$, $4 \mapsto 3$, $5 \mapsto 124$, $6 \mapsto 15$ is an (presumably infinite) extension of a translation on the torus $\mathbb{T}^2$. Note that the square of this substitution is left-proper. The incidence matrix has two Pisot eigenvalues of degrees 3. The eigenvalues of the dynamical system are $e^{2i\pi\alpha}$, where $\alpha \in 3\beta\mathbb{Z} + 3\beta^3\mathbb{Z}$, where β is the Perron eigenvalue of the incidence matrix. We show the domain exchange and the image by ψ in Figure 3.

Example 9.8. (Two Pisot, presumably infinite extension of $(\mathbb{T}^1, T)$) The subshift of the substitution $1 \mapsto 114$, $2 \mapsto 122$, $3 \mapsto 2$, $4 \mapsto 13$ is an (presumably infinite) extension of a translation on $\mathbb{R}/\mathbb{Z}$. The incidence matrix has two Pisot eigenvalues of degrees 2. We plot an approximation of the graph of the function ψ in Figure 4(left). We can see that the almost everywhere defined function ψ seems to be infinite-to-one. The eigenvalues of the subshift are $e^{2i\pi\sqrt{5}n}$ where $n \in \mathbb{Z}$.

Example 9.9. (Due to Timo Jolivet) The primitive substitution

$$\sigma : 1 \mapsto 213,\ 2 \mapsto 4,\ 3 \mapsto 5,\ 4 \mapsto 1,\ 5 \mapsto 21$$

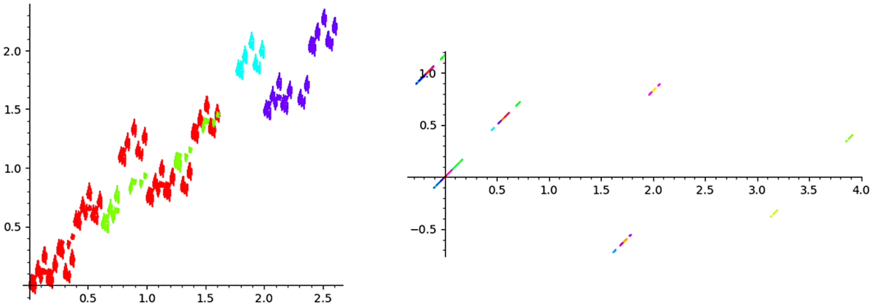

FIGURE 4. Approximation of the graph of the function ψ, for an example with two Pisot eigenvalues (left) and for an example weakly irreducible Pisot (right).

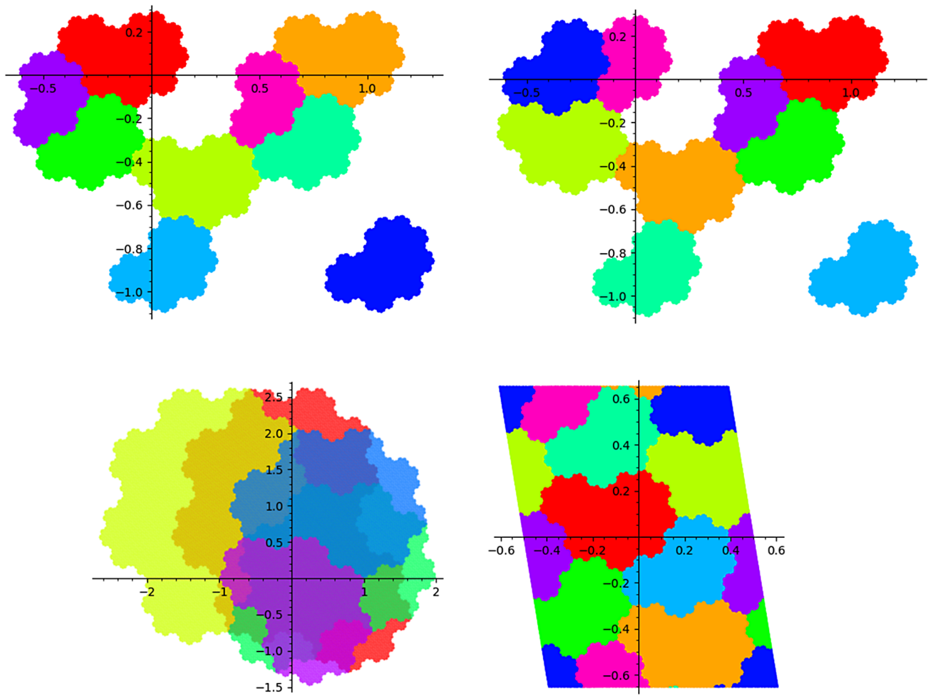

FIGURE 5. Domain exchange, Rauzy fractal with overlaps, and fundamental domain of $\mathbb{T}^2$ for Example 9.9.

is an example for which the usual Rauzy fractal overlaps (see Figure 5). However, we can proprify it and it permits to obtain a domain exchange that is measurably conjugate to the subshift of σ. Moreover, the proprification of σ satisfies Theorem 1.2, thus it is also a finite extension of a torus translation.

Notice that the return substitution of σ^3 on any letter has only three letters. The eigenvalues of the subshift are $e^{2i\pi\alpha}$, where $\alpha \in \mathbb{Z}[\beta]$, where β is the real root of $X^3 - 2X^2 + X - 1$.

Example 9.10. (Two eigenvalues) The square of the substitution

$$\sigma : \begin{cases} 0 \mapsto 1203, \\ 1 \mapsto 12, \\ 2 \mapsto 13, \\ 3 \mapsto 03, \end{cases}$$

is proper, primitive, and pseudo-unimodular. The associated subshift has eigenvalues $\{-1, 1\}$, thus the eigenvalue 1 of the square (Ω_σ, S^2) is not simple, so the square is not ergodic. By [**Dekking**, Theorem IV.1], it implies that (Ω_σ, S^2) is not minimal and this can be indeed easily checked directly.

Example 9.11. (Family of weakly mixing subshifts) For every $n \geq 1$, the substitution

$$\begin{cases} a \mapsto ab, \\ b \mapsto ac^{2n-1}, \\ c \mapsto ac^{2n}, \end{cases}$$

is primitive, unimodular, and left-proper. Its incidence matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 2n-1 & 2n \end{pmatrix}$ has eigenvalues $\{1, n \pm \sqrt{n^2+1}\}$, and the eigenvalue 1 is associated with the eigenvector $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ of sum 1 with integer coefficients. Thus, by Theorem 1.5, the unique eigenvalue of the subshift is 1, thus it is weakly mixing.

Supplementary material. The Supplementary Material is available online at https://doi.org/10.1017/etds.2023.101.

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