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SANDWICH THEOREMS FOR SEMICONTINUOUS OPERATORS

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ABSTRACT. We provide vector analogues of the classical interpolation theorems for lower semicontinuous functions due to Dowker and to Hahn and Katetov-Tong.

RÉSUMÉ. Le but de cet article est de montrer que sous certaines conditions, les théorèmes d'interposition de Dowker, Hahn et Katetov-Tong ont des analogues pour des applications à valeurs vectorielles et semi-continues inférieurement.

0. **Introduction.** One of the most flexible versions of the Tietze extension principle is Katetov-Tong's theorem [E], [Ja]. This asserts that when X is normal, $f: X \longrightarrow \mathbb{R}$ and $g: X \longrightarrow \mathbb{R}$ are respectively lower and upper semicontinuous while $f(x) \geq g(x)$ for all x, then there is a continuous mapping $h: X \longrightarrow \mathbb{R}$ with $f(x) \geq h(x) \geq g(x)$. There are two relevant refinements of Katetov-Tong's theorem for X paracompact (and Hausdorff). First, if f is allowed to take the value $+\infty$ and g the value $-\infty$ the result still holds and is due in the metric setting to Hahn [Str]. Second, if actually $f(x) > g(x)$ for all x, there is a continuous mapping $h: X \longrightarrow \mathbb{R}$ with $f(x) > h(x) > g(x)$. This is due to Dowker [Str], [Du]. In this paper we allow f and g to take extended values in a partially ordered vector space (Y, S) where S is an ordering convex cone, and give versions of Hahn's theorem and of Dowker's theorem in this setting. To do this we make appropriate definitions of semicontinuity for functions and for multifunctions. We are then able to apply Michael's selection theorem to the lower semicontinuous multifunction $H(x) := [f(x) - S] \cap [g(x) + S]$ to obtain Hahn-type results. We provide a similar selection result for *strongly lower semicontinuous* multifunctions which we apply to $K(x) := [f(x) - \operatorname{Int} S] \cap [g(x) + \operatorname{Int} S]$ to obtain Dowker-type results. In each case we place restrictions on S to insure that the selection theorem applies.

1. **Preliminaries.** Throughout this paper X and Z denote Hausdorff topological spaces and Y denotes a real Hausdorff topological vector space partially ordered by a convex cone S. $\operatorname{Int} A$ (respectively $c\ell A$) will denote the topological interior (respectively the closure) of A. We write $a \geq_S b$ or $a \geq b$ if $a - b$ is in S and adjoin signed infinities to Y with $\infty \geq y \geq -\infty$ for all y in Y. $(Y \cup \{\infty\}, S \cup \{\infty\})$ and $(Y \cup \{-\infty\}, S \cup \{-\infty\})$

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are denoted by $(Y^\bullet, S^\bullet)$ and $(Y_\bullet, S_\bullet)$ respectively. We write $a >_S b$ or $a > b$ if $a - b$ is in $\operatorname{Int} S$. See [Bo-Pe-Th] for details.

Then $[a,b]_S$ or $[a,b]$ will denote the *order interval* $(b-S)\cap(a+S)$, *i.e.*, $\{x \in X : a \leq x \leq b\}$.

Let $\Omega\colon X \rightrightarrows Z$ be a *set-valued* (or multivalued) mapping, *i.e.*, a mapping which assigns to every $x \in X$ a (possibly empty) subset $\Omega(x)$ of Z. We denote by $\operatorname{Dom}\Omega := \{x \in X : \Omega(x) \neq \emptyset\}$. Given $\Omega\colon X \rightrightarrows Z$, $\operatorname{Int}\Omega$ and $c\ell\,\Omega$ are defined for each $x \in X$ by $(\operatorname{Int}\Omega)(x) := \operatorname{Int}\big(\Omega(x)\big)$ and $(c\ell\,\Omega)(x) := c\ell\big(\Omega(x)\big)$, respectively. Given Ω_1 and $\Omega_2\colon X \rightrightarrows Z$, $\Omega_1 \cap \Omega_2$ is defined for each $x \in X$ by $(\Omega_1 \cap \Omega_2)(x) := \Omega_1(x) \cap \Omega_2(x)$.

According to [Ku] or [Ber], $\Omega\colon X \rightrightarrows Z$ is said to be LSC (lower semicontinuous) at $x_0 \in \operatorname{Dom}\Omega$ if for each open set U in Z, such that $\Omega(x_0) \cap U \neq \emptyset$ then $\Omega(x) \cap U \neq \emptyset$ for x near x_0. Ω is called LSC if it is LSC at every point of its domain.

We recall the following standard result:

LEMMA 1.1. *If Ω is* LSC *and if Λ satisfies $\Omega \subset \Lambda \subset c\ell\,\Omega$, then Λ is* LSC.

PROOF. Let x_0 be fixed and let O be open; let y_0 such that $y_0 \in O \cap \Lambda(x_0)$. Since $y_0 \in c\ell\,\Omega(x_0)$, $O \cap \Omega(x_0) \neq \emptyset$. Since Ω is LSC, let U be a neighbourhood of x_0 such that $O \cap \Omega(x) \neq \emptyset$ for each $x \in U$. In particular, $O \cap \Lambda(x) \neq \emptyset$ near x_0 and Λ is LSC as required. ∎

In the setting of vector-valued mappings, there exist several non-equivalent forms of lower semicontinuity. For a mapping $f\colon X \longrightarrow (Y^\bullet, S^\bullet)$, one natural way of saying that it is lower semicontinuous (*l.s.c.*) is that the set-valued mapping $H_f\colon X \rightrightarrows Y$ given by $H_f(x) := f(x) - S = \{z \in Z : z \leq_S f(x)\}$ is LSC. A mapping g is said to be *u.s.c.* if $-g$ is l.s.c. Note that g may take the value $-\infty$. This notion of lower semicontinuity coincides with the standard ones when the range space is the reals. The reader interested in these notions may for instance consult [Bo-Pe-Th] and the references therein.

Let us notice that every l.s.c. mapping has a closed *epigraph*

$$\operatorname{epi} f := \{(x,y) \in X \times Y : f(x) \leq_S y\}.$$

The converse is false as the following simple example in $\mathbb{R}^2$ (with the usual topology and order) shows [Th]:

$$f(x) := (0,0) \text{ if } x = 0 \text{ and } f(x) := \left(\frac{1}{|x|}, -1\right) \text{ otherwise.}$$

Then f has clearly a closed epigraph while $f^{-1}\left(\left\{(x,y) : y > \tfrac{1}{2}\right\}\right)$ is not open.

LEMMA 1.2 [BO-PE-TH]. *(a) If Y is a topological lattice then the supremum or infimum of a finite family of l.s.c. mappings is l.s.c.;*

(b) If Y is a Dini lattice (every increasing net with a supremum converges to this supremum), then the supremum of an arbitrary family of l.s.c. mappings is l.s.c.

A set-valued mapping $\Omega\colon X \rightrightarrows Z$ will be said to be SLSC (*strongly lower semicontinuous*) at $x_0 \in \operatorname{Dom}\Omega$, if its values locally have nonempty interior at x_0 and if for each $z \in \operatorname{Int}\Omega(x_0)$, there exists some neighbourhood N of x_0 such that $z \in \operatorname{Int}\Omega(x)$, for each $x \in N$.

As usual, we say that Ω is SLSC if it is SLSC at every point of its domain.

LEMMA 1.3. *(a) The intersection of a finite family of* SLSC *multifunctions is* SLSC*;*

(b) If Ω is SLSC*, then* $\operatorname{Int}\Omega$ *is* LSC *but the converse fails to be true;*

(c) If Ω is LSC *with an open graph, then Ω is* SLSC*;*

(d) If Ω is SLSC *with convex values and which have nonempty interior, then Ω is* LSC.

PROOF. (a) is immediate.

(b) Let O be open and suppose $y_0 \in \operatorname{Int}\Omega(x_0) \cap O$. Since Ω is SLSC we may suppose that $y_0 \in \operatorname{Int}\Omega(x)$ for x near x_0. Hence, $O \cap \operatorname{Int}\Omega(x) \neq \emptyset$ for x near x_0 and $\operatorname{Int}\Omega$ is LSC, as desired.

It should be noticed that the converse of (b) fails to be true: if we take $\Omega\colon \mathbb{R} \rightrightarrows \mathbb{R}$ defined by $\Omega(0) := \mathbb{R}$ and $\Omega(x) := \mathbb{R} \setminus \{1\}$ for $x \neq 0$. Then $\operatorname{Int}\Omega$ is clearly LSC at 0 while Ω fails to be SLSC at 0.

(c) If $y_0 \in \operatorname{Int}\Omega(x_0)$, since the graph of Ω is open, we may find a neighbourhood N of x_0 and a neighbourhood V of y_0 such that $V \subset \Omega(x)$ for each $x \in N$. Hence, Ω is SLSC as desired.

(d) By virture of (b), $\operatorname{Int}\Omega$ is LSC. Since Ω has convex images we have

$$\operatorname{Int}\Omega \subset \Omega \subset c\ell(\Omega) = c\ell(\operatorname{Int}\Omega),$$

and by Lemma 1.1 we obtain that Ω is LSC. ∎

Let us remark that without any convexity assumption (d) is no longer true: take $\Omega(0) :=]0,1[\cup \{2\}$ and $\Omega(x) :=]0,1[$ if $x \neq 0$.

The following lemma provides an example of SLSC set-valued mapping:

LEMMA 1.4. *Suppose* $\operatorname{Int} S \neq \emptyset$. *Then, $f\colon X \longrightarrow (Y^\bullet, S^\bullet)$ is l.s.c., if and only if, the set-valued mapping $x \rightrightarrows H(x) := f(x) - \operatorname{Int} S$ is* SLSC *(and therefore* LSC *on its domain (by Lemma 1.3 (d)).*

PROOF. Let $y_0 \in H(x_0)$; since H has open images, we have to prove that $y_0 \in H(x)$ for x near x_0. From the assumption we get $f(x_0) - y_0 + w \in S$ for w near 0; therefore if we pick $e \in \operatorname{Int} S$ we may suppose that $f(x_0) - y_0 - 3\varepsilon e \in S$ for some $\varepsilon > 0$. Hence, for $W := \varepsilon[-e, e]$ (which is a neighbourhood of 0) we get

$$(y_0 + 2\varepsilon e + W) \cap \big(f(x_0) - S\big) \neq \emptyset.$$

Since f is l.s.c. we derive that

$$(y_0 + 2\varepsilon e + W) \cap \big(f(x) - S\big) \neq \emptyset \text{ for } x \text{ near } x_0.$$

Hence, if we take z in the last intersection we obtain:

$$y_0 <_S y_0 + \varepsilon e <_S y_0 + 2\varepsilon e + w \leq_S z \leq_S f(x)$$

and $y_0 \in H(x)$ for x near x_0.

Conversely, if H is SLSC then according to Lemma 1.3 (d) H is LSC as is $c\ell(H) = H_f$ and therefore f is l.s.c. as claimed. ∎

COROLLARY 1.5. *Suppose* Int S *is nonempty,* $f: X \longrightarrow (Y^\bullet, S^\bullet)$ *is l.s.c. and* $g: X \longrightarrow (Y_\bullet, S_\bullet)$ *is u.s.c. Then* $\Omega(x) := [g(x), f(x)]_S$ *defines a* LSC *multifunction at each* $x \in X$ *such that* $g(x) <_S f(x)$.

PROOF. By virtue of Lemmas 1.4 and 1.3 (a), the multifunction

$$x \rightrightarrows \operatorname{Int}\Omega(x) = [f(x) - \operatorname{Int} S] \cap [g(x) + \operatorname{Int} S]$$

is SLSC on its domain. Therefore by Lemma 1.3 (d), Ω is LSC at each $x \in X$ such that $g(x) < f(x)$. ∎

A set-valued mapping $\Omega: X \rightrightarrows Y$ is said to be *locally convex-valued* at $a \in X$, if $\Omega(x)$ is convex for x near a.

The next proposition provides another example of SLSC set-valued mapping:

PROPOSITION 1.6. *Let X be first countable and let* $\Omega: X \rightrightarrows \mathbb{R}^n$ *be a set-valued mapping which is* LSC *and locally convex at* $a \in X$ *and such that* $\Omega(a)$ *has nonempty interior. Then* Ω *is* SLSC *at* a.

PROOF. If Ω fails to be SLSC at a, then there exist $b \in \operatorname{Int}\Omega(a)$, a sequence $\{x_n; n \in \mathbb{N}\}$ with limit a and a sequence $\{b_n; n \in \mathbb{N}\}$ converging to b such that $b_n \notin \Omega(x_n)$ for each $n \in \mathbb{N}$. Since the range space is finite dimensional, there exists a continuous linear form ϕ_n with norm 1 such that

$$\sup_{y \in \Omega(x_n)} \phi_n(y) \le \phi_n(b_n).$$

On selecting a subsequence, we may suppose that $\{\phi_n; n \in \mathbb{N}\}$ tends to $\phi_0 \neq 0$ whence

$$(*) \qquad \lim_{x_n \to a} \sup_{y \in \Omega(x_n)} \phi_n(y) \le \phi_0(b).$$

If there exists y such that $\phi_0(y) > \phi_0(b)$, then by lower semicontinuity of Ω at a, for each sequence $\{x_n; n \in \mathbb{N}\}$ converging to a we may find a sequence $\{y_n; n \in \mathbb{N}\}$ converging to y such that $y_n \in \Omega(x_n)$ for each $n \in \mathbb{N}$. Using $(*)$ and the fact that $\phi_n(y_n)$ tends to $\phi_0(y)$, this yields $\phi_0(y) \le \phi_0(b)$ and a contradiction. Hence,

$$\sup_{y \in \Omega(a)} \phi_0(y) \le \phi_0(b)$$

which yields $b \notin \operatorname{Int}\Omega(a)$ and a contradiction. ∎

It should be noticed that when the assumption of local convexity fails to be satisfied, the proposition is no longer true:

Take $\Omega(0) := \mathbb{R}$ and $\Omega(x) := \mathbb{R} \setminus \{1\}$ for $x \neq 0$. Then Ω is LSC at 0 but is not SLSC at 0.

The following corollary deals with the question of when the intersection of multifunctions is lower semicontinuous. We present this result here as an illustration of the concept of SLSC although it could also easily deduced from the proof Theorem 3.1 of [L-P].

COROLLARY 1.7 [LEC-SPA]. *Let X be first countable and let* $\Omega_1, \Omega_2 \colon \rightrightarrows \mathbb{R}^n$ *be* LSC *and locally convex at a. Suppose also that* $\mathrm{Int}(\Omega_1 \cap \Omega_2)$ *is nonempty at a, then* $\Omega_1 \cap \Omega_2$ *is* LSC *at a.*

PROOF. By virtue of Proposition 1.6, Ω_1 and Ω_2 are SLSC at a, as is also $\Omega := \Omega_1 \cap \Omega_2$. It follows from Lemma 1.3 (b) that $\mathrm{Int}(\Omega_1 \cap \Omega_2)$ is LSC at a and therefore by Lemma 1.1 that $\Omega_1 \cap \Omega_2$ is LSC at a as desired. ∎

REMARK 1.8. Robert [Ro] has given a counter-example which shows that Corollary 1.7 does not hold in infinite dimensions.

A partially ordered space is said to be *lattice-like* if whenever x_1, x_2, y_1, y_2 with $x_i \geq_S y_j$ $i, j = 1, 2$, there exists z in X such that $x_i \geq_S z \geq_S y_j$.

The *Riesz interpolation property* is said to hold if the relation

$$[0, x + y]_S = [0, x]_S + [0, y]_S$$

is satisfied for each $x, y \geq 0$. (The addition sign on the right denotes the algebraic sum, *i.e.*, the set of all $x_1 + x_2$ with $x_1 \in [0, x]_S$ and $x_2 \in [0, y]_S$).

PROPOSITION 1.9. *(a) Every vector lattice is lattice-like;*

(b) Every generating inductively ordered lattice-like space is a lattice;

(c) A partially ordered space is lattice-like, if and only if S has the Riesz interpolation property;

(d) If X is a reflexive Banach space with a normal generating closed convex ordering cone, then X is a lattice if it is lattice-like.

PROOF. (a) $x_1 \wedge x_2 \geq y_1 \vee y_2$. Hence, any z which belongs to the order interval $[y_1 \vee y_2, x_1 \wedge x_2]_S$ is as desired.

(b) Let w, x_1, x_2 in X be given with $x_1 \geq_S w, x_2 \geq_S w$. Let $W := \{x \in X : w \leq_S x \leq_S x_1, w \leq_S x \leq_S x_2\}$. Then $W \neq \emptyset$ and every chain in W has a supremum by hypothesis, which must be in W. Thus by Zorn's lemma W has a maximal element y_1. Let y_2 lie in W; as X is lattice-like, we can find $z \geq_S y_1, y_2$ in W. Thus by maximality $z = y_1 \geq_S y_2$ and $y_1 = x_1 \wedge x_2$. This suffices to show X is a lattice.

(c) may be found in [Lux-Za].

(d) If X is normal generating and lattice-like, it is well-known [Sch$_2$] that (X^*, S^*) is a Banach lattice as is (X^{**}, S^{**}). Since X is reflexive the proof is finished. ∎

EXAMPLES 1.10. (1) When X is not reflexive assertion (d) may fail: let X be the space of all continuous real functions x on $[-1, 1]$ such that $x(1) + x(-1) = x(0)$ with the usual ordering. Then X is lattice-like but is not a lattice [Sch$_1$], and the order is normal and generating. Note we may similarly prescribe n distinct points:

$$X := C[-1, 1] \cap \left\{x : \sum_{i=1}^{n} \lambda_i x(t_i) = x(t_0), \lambda_i > 0\right\}.$$

(2) [Lux-Za, p. 77]: Let a and b be finite real numbers, and X be the ordered vector space of all functions of the form $f(x) = \frac{p(x)}{q(x)}$ defined on $[a,b]$, where p, q are real polynomials such that $q(x) > 0$ on $[a,b]$ and where the algebraic operations and the partial ordering are pointwise. E is lattice-like but is not a lattice, and moreover, the ordering is generating. ■

COROLLARY 1.11. *(a) In $\mathbb{R}^n$, S is lattice-like, if and only if, S is a lattice order for $S - S$;*

(b) If S has a bounded complete base, S is lattice-like, if and only if S is latticial i.e., S is a lattice order for $S - S$.

2. **Hahn-Katetov-Tong-type sandwich theorems.** We will write $f \geq g$ if $f(x) \geq g(x)$ for all x in X. Recall that S (or Y) is *normal* if there is a base of neighbourhoods U of the origin with $U = (U - S) \cap (U + S)$. For instance, any Banach lattice is normal as is every convex pointed locally compact cone. We say (Y, S) is *locally decomposable* if given a neighbourhood W of zero we may find a neighbourhood V of zero such that $V \subset W \cap S - W \cap S$. In particular, S is locally decomposable whenever S has nonempty interior or if S is latticial. It follows from the open mapping theorem [Bo,Ja] that when Y is a Fréchet space and S is closed, S is locally decomposable, if and only if, S is generating, *i.e.*, $S - S = Y$.

THEOREM 2.1. *Let $f: X \longrightarrow (Y^\bullet, S^\bullet)$ be l.s.c. and let $g: X \longrightarrow (Y_\bullet, S_\bullet)$ be u.s.c. such that $g \leq f$. Suppose further that (a) (Y, S) is locally decomposable while (b) S is normal and (c) S is lattice-like. Then the set-valued mapping $\Omega: X \rightrightarrows Y$ given by $\Omega(x) := [g(x), f(x)]$ is* LSC.

PROOF. Let W be a full neighbourhood of the origin in X such that by (b),

$$W = (W + S) \cap (W - S).$$

By virtue of (a) we may find a neighbourhood V of zero such that

$$V \subset W \cap S - W \cap S.$$

Select $y_0 \in \Omega(x_0) \neq \emptyset$ and choose a neighbourhood U of x_0 such that

$$\big(g(x) + S\big) \cap (y_0 - V) \neq \emptyset$$
$$\big(f(x) - S\big) \cap (y_0 + V) \neq \emptyset$$

for each $x \in U$. Then,

$$g(x) \in y_0 - V - S \text{ and } f(x) \in y_0 + V + S$$

from which we get

$$g(x) \in y_0 + (W \cap S) - S \text{ and } f(x) \in y_0 - (W \cap S) + S.$$

This yields

$$g(x) \leq_S y_0 + x_1 \text{ and } f(x) \geq_S y_0 - x_2 \text{ for some } x_1, x_2 \text{ in } W \cap S,$$

i.e.,

$$g(x) - x_1 \leq_S y_0 \leq_S f(x) - x_2 \text{ for each } x \in U.$$

By (c) we may find z such that

$$y_0 = z - w \text{ and } g(x) \leq_S z \leq_S f(x), \quad -x_2 \leq_S w \leq_S x_1.$$

Since W is full, $w \in W$. Furthermore, since $z \in y_0 + W$ and $z \in \Omega(x)$ we derive that Ω is LSC at x_0, as desired. ∎

This result permits us to extend to the vector setting Hahn's interpolation theorem established for extended real-valued functions on a metric space, and most cases of Katetov-Tong's interpolation theorem for real-valued lower semicontinuous functions on a normal space. See [Ja, p. 121], and [Str, p. 133].

THEOREM 2.2 (KATETOV-TONG-HAHN TYPE RESULTS). *Under the assumptions of Theorem 2.1, if furthermore X is paracompact and Y is a Banach space, then for each $x_0 \in X$ and each $y_0 \in Y$ such that $g(x_0) \leq_S y_0 \leq_S f(x_0)$ there exists a continuous mapping $h: X \longrightarrow (Y, S)$ such that $h(x_0) = y_0$ and $g(x) \leq_S h(x) \leq_S f(x)$ for each $x \in X$.*

PROOF. Since Ω in Theorem 2.1 has closed convex nonempty images and is LSC, Michael's selection theorem [Ho] applies and any continuous selection h from Ω such that $h(x_0) = y_0$ is as desired. ∎

COROLLARY 2.3. *The preceding theorem applies whenever (i) Y is Banach, $S - S = Y$, and S is a closed lattice-like normal ordering cone; or in particular if (ii) Y is a Banach lattice.*

EXAMPLE 2.4. Let us consider the *mean-value transformation* $T_\lambda: L_1^+[-2, 2] \longrightarrow \mathcal{C}[-1, 1]$ defined for $\lambda \in (0, 1]$ by:

$$T_\lambda(f)(x) := \frac{1}{2\lambda} \int_{x-\lambda}^{x+\lambda} f(t)\, dt.$$

As noticed by [Be], such mappings are used in the standard proof of the Fréchet-Kolmogorov theorem [Yo] and have also been used by [Ce]. As observed by [Be], these mappings are l.s.c. from $L_1^+[-2, 2]$ to $\mathcal{C}[-1, 1]$ equipped with the metric of the convergence in measure and the norm of uniform convergence, respectively. Therefore by virtue of Theorem 2.2, T_λ admits a continuous minorant.

COROLLARY 2.5. *Under the assumptions of Theorem 2.2, suppose $f_i: X \longrightarrow (Y, S)$, $i \in \{1, \ldots, n\}$ are l.s.c. and satisfies $\sum_{i=1}^n f_i \geq_S 0$. Then, there exists $h_i: X \longrightarrow (Y, S)$, $i \in \{1, \ldots, n\}$ continuous such that $h_i \leq_S f_i$ and $\sum_{i=1}^n h_i = 0$.*

PROOF. Since $\sum_{i=1}^{n-1} f_i \geq_S -f_n$, by virtue of Theorem 2.2, (on noting that the finite sum of finite l.s.c. maps is l.s.c.) we may find a continuous mapping w_n such that

$\sum_{i=1}^{n-1} f_i \geq_S -w_n \geq_S -f_n$. Let us define $k_i := f_i + \frac{w_n}{n-1}$. Then, we have $\sum_{i=1}^{n-1} k_i \geq 0$ and, by induction, there exists $\{t_i\}_{i \in \{1,\dots,n-1\}}$ continuous such that $t_i \leq k_i$ and $\sum_{i=1}^{n-1} t_i = 0$. Then, $h_i := t_i - \frac{w_n}{n-1}$, $h_n := w_n$ are as required. ∎

Recall that (Y, S) is (countably) *Daniell* if the infimum of a decreasing (sequence) net in S exists and is the topological limit. A Banach lattice is Daniell exactly when it is Dini, or equivalently when it has weakly compact order intervals [Sch$_2$]. If (Y, S) has countably compact intervals and S is closed and pointed, then it is countably Daniell. It is the case if in particular S has a countably compact base and Y is locally convex, or if Y is locally convex and S is sequentially complete and has a bounded base. For these results, the reader is referred to [Bo].

COROLLARY 2.6. *Let $f: X \longrightarrow (Y^\bullet, S^\bullet)$ be l.s.c. with X paracompact and Y satisfying (i) or (ii) of Corollary 2.3. Then, $f(x_0) = \sup\{h(x_0) : h$ continuous, $h \leq f\}$ and the sup is attained when finite.*

In particular, if Y is a Daniell Banach lattice f is l.s.c., if and only if, f is the supremum of a family of continuous mappings.

PROOF. Apply Corollary 2.3 with $g(x) = -\infty$ for $x \neq x_0$ and $g(x_0) = y_0$ for arbitrary y_0 in Y such that $y_0 \leq f(x_0)$. ∎

COROLLARY 2.7 (TIETZE EXTENSION-TYPE RESULT). *Let A be closed in X with X paracompact, let Y satisfy (i) or (ii) of Corollary 2.3, and let $h: A \subset X \longrightarrow [a,b]_S \subset Y$ be continuous. Then there exists a continuous $\tilde{h}: X \longrightarrow [a,b]_S$ such that $\tilde{h}|_A = h$.*

PROOF. Set $f(x) = h(x)$ if $x \in A$, $f(x) = b$ if $x \notin A$, and $g(x) = h(x)$ if $x \in A$, $g(x) = a$ if $x \notin A$. Then, clearly $g \leq f$, f is l.s.c, and g is u.s.c. Hence, we may apply the Sandwich Theorem 2.2 to conclude. ∎

For the sake of completeness we record the following consequence of Michael's theorem.

LEMMA 2.8. *Let X be paracompact, let $X_0 \subset X$ be a closed subset and let Y_0 be a closed convex subset of Y. Let $f: X_0 \longrightarrow Y_0 \subset Y$ be continuous. Then $\Omega(x) := f(x)$ if $x \in X_0$ and $\Omega(x) := Y_0$ if $x \notin X_0$ is* LSC.

PROOF. Let O be open and set $A := \{x \in X : \Omega(x) \cap O \neq \emptyset\}$. Then clearly,

$$\begin{cases} A = \{x \in X_0 : f(x) \in O \cap Y_0\} \cup \{x \in X : x \notin X_0\} & \text{if } Y_0 \cap O \neq \emptyset \\ A = \emptyset & \text{if } Y_0 \cap O = \emptyset \end{cases}$$

and therefore A is open. ∎

Thence we obtain a more general form of Corollary 2.7.

COROLLARY 2.9. *Let X_0 be a closed subset of a paracompact space X, and suppose Y is a Banach space. Let Y_0 be a closed convex subset of Y, and let $h: X_0 \subset X \longrightarrow Y_0 \subset Y$ be continuous. Then there exists a continuous mapping $\tilde{h}: X \longrightarrow Y_0$ such that $\tilde{h}|_{X_0} = h$.*

PROOF. Apply Lemma 2.8 and Michael's selection theorem. ∎

If we apply the preceding result to $Y_0 := Y$, this given unboundedly continuous extensions for a continuous operator defined on a closed subspace.

REMARK 2.10. Let X be paracompact and Y Banach, $\Omega: X \rightrightarrows Y$ be SLSC with closed convex images. Suppose $f: X \longrightarrow (Y^{\bullet}, S^{\bullet})$ is l.s.c. and $g: X \longrightarrow (Y_{\bullet}, S_{\bullet})$ is u.s.c. and $\operatorname{Int} S$ is nonempty. Suppose there exists an arbitrary selection σ of Ω such that

$$\sigma(x) \in \Omega(x) \text{ and } f(x) > \sigma(x) > g(x) \text{ for each } x \text{ in } X.$$

Then there is a continuous selection w of Ω such that

$$w(x) \in \Omega(x) \text{ with } f(x) \geq w(x) \geq g(x) \text{ for each } x \text{ in } X.$$

PROOF. By virtue of Lemmas 1.3 and 1.4, $\Lambda: X \rightrightarrows Y$ given by $\Lambda(x) := \big(f(x) - \operatorname{Int} S\big) \cap \big(g(x) + \operatorname{Int} S\big) \cap \Omega(x)$ is SLSC on X with convex nonempty images, as is $c\ell\Lambda$. By Lemma 1.3 (d), Λ is LSC as also is $c\ell\,\Lambda$. Then apply Michael's selection theorem. ■

A trivial example is produced by setting $\Omega(x) := C$ for C closed convex with nonempty interior in Y.

3. **Open interpolation results.**

LEMMA 3.1. *Let X be paracompact. If $\Omega: X \rightrightarrows Y$ is* SLSC *and convex-valued with* $\operatorname{Int}\Omega(x) \neq \emptyset$ *for each $x \in X$, then* $\operatorname{Int}\Omega$ *admits a continuous selection. In particular, if is* Ω SLSC *with convex open nonempty images then Ω admits a continuous selection.*

PROOF. Set $O(y) = \{x \in X : y \in \operatorname{Int}\Omega(x)\}$. By paracompactness, we may find a locally finite refinement $\{V_i\}_{i\in I}$ of $O(y)$ for $y \in Y$ and a subordinate partition of the unity denoted by $\{p_i\}_{i\in I}$. For each i in I select $y(i)$ such that $V_i \subset O\big(y(i)\big)$ and let $f(x) := \sum_{i\in I} p_i(x)y(i)$. Then f is continuous and if $x \in V_i$ then $y(i) \in \operatorname{Int}\Omega(x)$. Thus, since $\operatorname{Int}\Omega(x)$ is convex and $\sum_{i\in I} p_i(x) = 1$ we derive that $f(x)$ necessarily belongs to $\operatorname{Int}\Omega(x)$. ■

COROLLARY 3.2. *Suppose X is paracompact, $\Omega: X \rightrightarrows Y$ has convex images with nonempty interior and is* SLSC. *For each $y_0 \in \operatorname{Int}\Omega(x_0)$ and each given open neighbourhood $N(y_0)$, there exists a continuous selection h of Ω such that $h(x) \in \operatorname{Int}\Omega(x)$ and $h(x_0) \in N$.*

PROOF. Let $\Omega_N(x) := \Omega(x)$ if $x \neq x_0$ and $\Omega_N(x) := N(y_0) \cap \operatorname{Int}\Omega(x_0)$ if $x = x_0$. Then Ω_N is also SLSC and therefore we may apply Lemma 3.1 to derive the result. ■

THEOREM 3.3 (DOWKER-TYPE RESULT [DU; P. 171]). *Let X be paracompact and let Y be an ordered topological vector space with $\operatorname{Int} S \neq \emptyset$. Assume that $f: X \longrightarrow (Y^{\bullet}, S^{\bullet})$ is l.s.c. and $g_1: X \longrightarrow (Y_{\bullet}, S_{\bullet})$ $i \in \{1, \ldots, n\}$ are u.s.c. such that $g_i(x) <_S f(x)$ for each $x \in X$. Then, there exists a continuous mapping $h: X \longrightarrow Y$ such that $g_i(x) <_S h(x) <_S f(x)$ for each $x \in X$ and each $i \in \{1, \ldots, n\}$.*

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PROOF. Set $\Omega(x) := \bigcap_{i=1}^{n}\left[\left(f(x) - \operatorname{Int} S\right) \cap \left(g_i(x) + \operatorname{Int} S\right)\right]$. Then, Ω has convex open nonempty images and is SLSC by virtue of Lemmas 1.4 and 1.3 (a). Apply Lemma 3.1 to derive the result. ∎

Note that Theorem 3.3 does not require that Y be Banach or lattice-like, while Theorem 2.2 does not require that the cone have nonempty interior.

We recall that $f: X \longrightarrow Y$ is *Baire class* 1 if f is the limit (pointwise) of a sequence of continuous mappings.

COROLLARY 3.4. *Suppose X is paracompact and (Y, S) is countably Daniell with* $\operatorname{Int} S$ *non-empty. If f, $-g$ are l.s.c. and satisfies $g \leq f$, then there exists a mapping h which is Baire class 1, such that $g \leq_S h \leq_S f$.*

PROOF. Given $e \in \operatorname{Int} S$, set $f_0 := f$ and $f_n := f + \frac{e}{n}$. We may always define recursively a sequence $\{h_n; n \in \mathbb{N}\}$ of continuous mappings such that

$$g <_S h_{n+1} <_S f_n \text{ and } g <_S h_{n+1} <_S h_n.$$

Then, $h := \inf_{n \in \mathbb{N}} h_n$ exists, is Baire class 1 and satisfies $g \leq_S h \leq_S f$. ∎

Finally, we show that in the complete latticial setting (with interior) Hahn-Katetov-Tong's interpolation theorem may be also derived directly from our open selection results.

COROLLARY 3.5. *Let S be a normal, complete latticial ordering cone with nonempty interior in Y, let X be a paracompact space. Suppose $f: X \longrightarrow (Y^\bullet, S^\bullet)$ is l.s.c. and $g: X \longrightarrow (Y_\bullet, S_\bullet)$ is u.s.c. with $g \leq_S f$. Then there exists a continuous mapping h such that $g \leq_S f \leq_S h$.*

PROOF. The proof follows the lines of Jameson [Ja; p. 123]. Let e be in $\operatorname{Int} S$. We build a sequence $\{f_n; n \in \mathbb{N}\}$ such that both of the following conditions hold:

$$(*) \qquad \|f_n - f_{n-1}\| \leq \left(\frac{2}{3}\right)^n$$

and

$$(**) \qquad g - \left(\frac{2}{3}\right)^n e <_s f_n <_S h + \left(\frac{2}{3}\right)^n e.$$

Indeed, since $g \leq_S f$ and e is in $\operatorname{Int} S$, we have $g - \frac{2}{3}e <_S h + \frac{2}{3}e$ and Dowker's Theorem 3.3 ensures the existence of a continuous mapping f_1 such that

$$g - \frac{2}{3}e <_S f_1 <_S h + \frac{2}{3}e.$$

Assume now that $f_1, f_2, \ldots, f_n$ have been defined so that the preceding conditions $(*)$ and $(**)$ hold.

Since

$$g <_S f_n + \left(\frac{2}{3}\right)^n e \text{ and } f_n <_S h + \left(\frac{2}{3}\right)^n e$$

we have

$$g \vee f_n <_S h \wedge f_n + \left(\frac{2}{3}\right)^n e.$$

Again, using Theorem 3.3, we obtain a continuous mapping f_{n+1} such that

$$g \vee f_n <_S f_{n+1} <_S h \wedge f_n + \left(\frac{2}{3}\right)^n e.$$

Therefore we get,

$$g - \left(\frac{2}{3}\right)^{n+1} e <_S f_{n+1} <_S h + \left(\frac{2}{3}\right)^{n+1} e$$

and the desired conditions $(*)$ and $(**)$. By $(*)$ and the completeness of Y it follows that $\{f_n; n \in \mathbb{N}\}$ converges uniformly to a continuous f. Now $(**)$ yields $g \leq_S f \leq_S h$, as desired. ∎

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