J. Austral. Math. Soc. Ser. B 40(1999), 392–402

NECESSARY OPTIMALITY CONDITIONS FOR BICRITERION DISCRETE OPTIMAL CONTROL PROBLEMS

X. Q. YANG[1] and K. L. TEO[2]

(Received 16 December 1996; revised 7 October 1997)

Abstract

In management science and system engineering, problems with two incommensurate objectives are often detected. Bicriterion optimization finds an optimal solution for the problems. In this paper it is shown that bicriterion discrete optimal control problems can be solved by using a parametric optimization technique with relaxed convexity assumptions. Some necessary optimality conditions for discrete optimal control problems subject to a linear state difference equation are derived. It is shown that in this case no adjoint equation is required.

1. Introduction

Consider the following bicriterion discrete optimal control (DOC) problem:

$$\text{minimize } h\left(f_1(\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot), z), f_2(\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot), z)\right)$$

subject to the linear state difference equation

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + B\boldsymbol{u}(k) + Cz, \quad k = 0, 1, \dots, N-1, \quad \boldsymbol{x}(0) = \boldsymbol{x}^0(z) \quad (1.1)$$

and the following linear constraints (1.2) and (1.3):

$$\boldsymbol{a}_i^{\mathsf{T}}\boldsymbol{u}(\cdot) + \boldsymbol{b}_i^{\mathsf{T}}\boldsymbol{z} = c_i, \qquad i \in E, \quad (1.2)$$

$$\boldsymbol{a}_j^{\mathsf{T}}\boldsymbol{u}(\cdot) + \boldsymbol{b}_j^{\mathsf{T}}\boldsymbol{z} \le c_j, \qquad j \in I, \quad (1.3)$$

where $\boldsymbol{x} = [x_1, \dots, x_n]^T : \{1, \dots, N\} \longrightarrow \boldsymbol{R}^n$, $\boldsymbol{u} = [u_1, \dots, u_r]^T : \{0, \dots, N-1\} \longrightarrow \boldsymbol{R}^r$ and $z = [z_1, \dots, z_s]^T \in \boldsymbol{R}^s$ are, respectively, the state, control and system parameter vectors, $\boldsymbol{x}^0 = [x_1^0, \dots, x_n^0]^T : \boldsymbol{R}^s \longrightarrow \boldsymbol{R}^n$, and $A_{n\times n}$, $B_{n\times r}$, $C_{n\times s}$ are

[1]Department of Mathematics, University of Western Australia, Nedlands, 6907, W.A., Australia
[2]School of Mathematics, Curtin University of Technology, GPO Box U1987, Perth 6845, Australia

matrices, $\boldsymbol{a}_i, \boldsymbol{a}_j \in \boldsymbol{R}^{nr}, \boldsymbol{b}_i, \boldsymbol{b}_j \in \boldsymbol{R}^{ns}$, and $c_i, c_j \in \boldsymbol{R}, i \in E, j \in I, h : \boldsymbol{R}^2 \to \boldsymbol{R}$ is a real-valued function and $f_i : \boldsymbol{R}^n \times \boldsymbol{R}^r \times \boldsymbol{R}^s \to \boldsymbol{R}$ are real-valued functions ($i = 1, 2$).

This type of bicriterion optimization problems arise in management science and system engineering. Portfolio selection problem (see Cai *et al.* [2]) and fractional program problem (see Craven [5]) are examples of the bicriterion optimization problem. Geoffrion [6] showed that an optimal solution of a bicriterion optimization problem can be obtained by solving a number of parametric optimization problems under a convexity assumption (on f_i) and a monotonicity property (on h). Using the parametric optimization strategy, a continuous minimax convex quadratic optimal control problem was solved by Li [8] and a linear fractional program was solved in Loganathan [9] and Mond [12]. More recently, a bicriterion optimization problem is solved in Marchi [11] using an interactive method without any convexity requirement. It is worth noting that first-order necessary optimality conditions are important and can be applied to check if a candidate point is optimal, see Teo *et al.* [15]. Necessary optimality for a discrete optimal control problem with a nonlinear state difference equation was obtained by Pytlak and Malinowski [13]. This was achieved using a system of adjoint equations.

In this paper, we show that the convexity assumptions (on f_i) used in Geoffrion [6] can be relaxed. This is an application of the weakened scalarization condition, see Jeyakumar and Yang [7] and the references therein. Moreover we show that necessary optimality conditions of various discrete DOC problems subject to a linear state difference equation can be obtained without any adjoint equation. In Section 2 it is shown that an optimal solution of the bicriterion DOC problem can be obtained by solving a number of parametric DOC problems. In Section 3 necessary optimality conditions are derived for the parametric DOC problem where no adjoint equations are present. In Section 4, a necessary optimality condition is established for a bicriterion minimax-max DOC problem. Section 5 concludes the paper.

This paper is in honour of Professor B. D. Craven and Professor B. Mond.

2. Bicriterion and parametric DOC

The control-parameter pair $(\boldsymbol{u}(\cdot), z)$ is said to be feasible if (1.1)-(1.3) are satisfied. Let $\mathscr{F}$ be the set of all feasible control-parameter pairs $(\boldsymbol{u}(\cdot), z)$.

DEFINITION 2.1. The feasible control-parameter pair $(\boldsymbol{u}^*(\cdot), z^*)$ of the bicriterion DOC is said to be *optimal* if for any $(\boldsymbol{u}(\cdot), z) \in \mathscr{F}$,

$$h\left(f_1(\boldsymbol{x}^*(\cdot), \boldsymbol{u}^*(\cdot), z^*), f_2(\boldsymbol{x}^*(\cdot), \boldsymbol{u}^*(\cdot), z^*)\right) \le h\left(f_1(\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot), z), f_2(\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot), z)\right),$$

where $\boldsymbol{x}^*(\cdot), \boldsymbol{x}(\cdot)$ are the corresponding (unique) solution of (1.1) with respect to $(\boldsymbol{u}^*(\cdot), z^*)$, $(\boldsymbol{u}(\cdot), z)$, respectively.

DEFINITION 2.2. The feasible control-parameter pair $(\boldsymbol{u}^*(\cdot), z^*)$ of the bicriterion DOC is said to be *efficient* if there exists no other feasible control-parameter pair $(\boldsymbol{u}(\cdot), z)$ such that

$$f_1(\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot), z) < f_1(\boldsymbol{x}^*(\cdot), \boldsymbol{u}^*(\cdot), z^*), \quad f_2(\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot), z) \le f_2(\boldsymbol{x}^*(\cdot), \boldsymbol{u}^*(\cdot), z^*),$$

or

$$f_1(\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot), z) \le f_1(\boldsymbol{x}^*(\cdot), \boldsymbol{u}^*(\cdot), z^*), \quad f_2(\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot), z) < f_2(\boldsymbol{x}^*(\cdot), \boldsymbol{u}^*(\cdot), z^*),$$

where $\boldsymbol{x}^*(\cdot), \boldsymbol{x}(\cdot)$ are the corresponding (unique) solution of (1.1) with respect to $(\boldsymbol{u}^*(\cdot), z^*)$, $(\boldsymbol{u}(\cdot), z)$, respectively — this explanation is often omitted without confusion.

Let $\lambda \in [0, 1]$. The parametric DOC is defined as follows:

$$\begin{aligned}&\text{minimize } \lambda f_1(\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot), z) + (1-\lambda) f_2(\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot), z)\\ &\text{subject to } (\boldsymbol{u}(\cdot), z) \in \mathscr{F}.\end{aligned}$$

For the existence result of a solution for a DOC problem, see Boltyanskii [1]. We assume the following conditions.

(A1). The bicriterion DOC admits an optimal control-parametric pair.
(A2). For each $\lambda \in [0, 1]$, the parametric DOC admits an optimal control-parametric pair.
(A3). h is decreasing with respect to each of its component.

Using a similar method of the proof of Lemma 1 in Geoffrion [6], we have the following.

LEMMA 2.1. *Assume that* (A1)–(A3) *hold. At least one of optimal control-parametric pairs for the bicriterion* DOC *problem is efficient.*

LEMMA 2.2. *The following hold.*

(i) *Let* $\lambda \in (0, 1)$. *If* $(\boldsymbol{u}^*(\cdot), z^*)$ *is an optimal control-parametric pair of the parametric* DOC, *then* $(\boldsymbol{u}^*(\cdot), z^*)$ *is efficient.*

(ii) *Assume that the set*

$$\Omega = \{\boldsymbol{w} = (w_1, w_2) \in \boldsymbol{R}^2 : \exists (\boldsymbol{u}(\cdot), z) \in \mathscr{F}, f_i(\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot), z) < w_i, \forall i = 1, 2\}$$

is convex. If $(\boldsymbol{u}^*(\cdot), z^*)$ *is efficient, then there exists* $\lambda \in [0, 1]$ *such that* $(\boldsymbol{u}^*(\cdot), z^*)$ *is an optimal control-parametric pair of the parametric* DOC.

PROOF. (i) This follows from Definition 2.2 (see also Theorem 10.11 in Yu [16] and Proposition 2.2, Chapter 4, Luc [10]).

(ii) It is evident that

$$\Omega = \{(f_1(x(\cdot), u(\cdot), z), f_2(x(\cdot), u(\cdot), z)) : (u(\cdot), z) \in \mathscr{F}\} + \int R^2_+$$

is convex, hence so is $\Omega' = \Omega - (f_1(x^*(\cdot), u^*(\cdot), z^*), f_2(x^*(\cdot), u^*(\cdot), z^*))$.

It follows that the efficiency that $(0, 0) \notin \Omega'$. Using the separation theorem (see Craven [4]), there exists $p = (p_1, p_2) \in R \setminus \{0\}$ such that

$$p_1 w_1 + p_2 w_2 \geq \sum_{i=1}^{2} p_i f_i(x^*(\cdot), u^*(\cdot), z^*), \qquad \forall w \in \Omega.$$

This implies $p \in R_+ \setminus \{0\}$ and

$$\lambda f_1(x^*(\cdot), u^*(\cdot), z^*) + (1-\lambda) f_2(x^*(\cdot), u^*(\cdot), z^*) \leq \lambda f_1(x(\cdot), u(\cdot), z)) + (1-\lambda) f_2(x(\cdot), u(\cdot), z)), \qquad (u(\cdot), z) \in \mathscr{F},$$

where $\lambda = p_1/\|p\|$, p_1 is the first component of p.

REMARK 2.1. It is clear that if for each $i = 1, 2$, f_i, $i = 1, 2$ is a convex function of $(x(\cdot), u(\cdot), z)$, then Ω is convex. However, there are examples that Ω is convex, but f_i is not convex, see Jeyakumar and Yang [7].

THEOREM 2.1. *Assume that* (A1) – (A3) *hold and that the set*

$$\Omega = \{w = (w_1, w_2) \in R^2 : \exists (u(\cdot), z) \in \mathscr{F}, f_i(x(\cdot), u(\cdot), z) < w_i, \forall i = 1, 2\}$$

is convex. Then an optimal control-parametric pair of the bicriterion DOC *problem is found among the optimal control-parametric pairs of the parametric* DOC *problem for some* $\lambda \in [0, 1]$.

PROOF. Follows from Lemmas 2.1 and 2.2.

3. Necessary optimality conditions of the parametric DOC

To obtain necessary optimality conditions for the parametric DOC problem, a necessary optimality condition for a general scalar DOC problem subject to a linear difference equation is first derived. Consider the following scalar DOC problem:

$$\text{minimize } \phi(x(\cdot), u(\cdot), z)$$
$$\text{subject to } (u, z) \in \mathscr{F}.$$

https://doi.org/10.1017/S0334270000010973 Published online by Cambridge University Press

Let $g : \boldsymbol{R}^n \longrightarrow \boldsymbol{R}$ be a locally Lipschitz function. The Clarke subgradient of g at $\boldsymbol{x} \in \boldsymbol{R}^n$ is defined by

$$\partial^\circ g(\boldsymbol{x}) = \{\boldsymbol{x}^* \in \boldsymbol{R}^n : g^\circ(\boldsymbol{x}; \boldsymbol{y}) \geq \boldsymbol{x}^{*T}\boldsymbol{u}, \forall \boldsymbol{y} \in \boldsymbol{R}^n\},$$

where

$$g^\circ(\boldsymbol{x}; \boldsymbol{y}) = \limsup_{\boldsymbol{x}' \to \boldsymbol{x}, s \downarrow 0} \frac{g(\boldsymbol{x}' + s\boldsymbol{y}) - g(\boldsymbol{x}')}{s}$$

is the Clarke generalized directional derivative of g at $\boldsymbol{x}$ in the direction $\boldsymbol{y}$. Some properties of $g^\circ(\boldsymbol{x}; \boldsymbol{y})$ are summarized below (see Clarke [3]).

PROPOSITION 3.1. *We have*

(i) $g^\circ(\boldsymbol{x}; \boldsymbol{y}) = \max_{\boldsymbol{x}^* \in \partial^\circ g(\boldsymbol{x})} \boldsymbol{x}^{*\top}\boldsymbol{y}$.

(ii) *Let* $g(\boldsymbol{x}) = G(f(\boldsymbol{x}))$ *where* $f : \boldsymbol{R}^n \to \boldsymbol{R}^m$, $G : \boldsymbol{R}^m \to \boldsymbol{R}$. *Let*

$$P(\boldsymbol{x}) = \mathrm{co}\{\boldsymbol{p} \in \boldsymbol{R}^n : \boldsymbol{p} = [f^1, \ldots, f^m] \cdot W, \quad f^i \in \partial^\circ f_i(\boldsymbol{x}), \quad W \in \partial^\circ G(f(\boldsymbol{x}))\}.$$

Then

$$\partial^\circ g(\boldsymbol{x}) \subseteq P(\boldsymbol{x}).$$

(iii) *g is said to be subdifferentially regular at $\boldsymbol{x}$ if the directional derivative $g'(\boldsymbol{x}; \boldsymbol{y})$ exists and*

$$g'(\boldsymbol{x}; \boldsymbol{y}) = g^\circ(\boldsymbol{x}; \boldsymbol{y}), \quad \forall \boldsymbol{y} \in \boldsymbol{R}^n.$$

The necessary optimality conditions for a scalar DOC problem subject to a *nonlinear* state difference equation was derived by virtue of adjoint equations in [13]. We will derive necessary optimality conditions of the parametric DOC using the special structure of the linear state difference equation. More specifically, no adjoint equation is required for the linear state difference equation (1.1). This characterization can be observed from the following subgradient formula of ϕ.

THEOREM 3.1. *The subgradient of $\phi : \boldsymbol{R}^{rN+s} \longrightarrow \boldsymbol{R}$ is calculated by*

$$\begin{aligned}
&\partial^\circ \phi(\boldsymbol{x}(\cdot, \boldsymbol{u}(\cdot), z), \boldsymbol{u}(\cdot), z) \subset \{(\boldsymbol{g}_0, \ldots, \boldsymbol{g}_{N-1}, \boldsymbol{h}_0) : \boldsymbol{g}_k \\
&\qquad = \sum_{l=k+1}^{N} A^{l-k-1} B \boldsymbol{g}_{\boldsymbol{x}(l)} + \boldsymbol{g}_{\boldsymbol{u}(k)}, k = 0, \ldots, N-1, \\
&\boldsymbol{h}_0 = \sum_{k=1}^{N} \left[A^k \frac{\partial \boldsymbol{x}^\circ(z)}{\partial z} + (A^{k-1} + A^{k-2} + \cdots + I) C \right] \boldsymbol{g}_{\boldsymbol{x}(k)} + \boldsymbol{g}_z, \\
&(\boldsymbol{g}_{\boldsymbol{x}(1)}, \ldots, \boldsymbol{g}_{\boldsymbol{x}(N)}, \boldsymbol{g}_{\boldsymbol{u}(0)}, \ldots, \boldsymbol{g}_{\boldsymbol{u}(N-1)}, \boldsymbol{g}_z) \in \partial^\circ \phi(\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot), z)\} := \Phi,
\end{aligned}$$

where $\boldsymbol{x}(\cdot, \boldsymbol{u}(\cdot), z)$ denotes the solution of (1.2)–(1.3) *with respect to* $(\boldsymbol{u}(\cdot), z)$.

PROOF. By Proposition 2.3.15 in Clarke [3],

$$\partial^\circ \phi(\boldsymbol{x}(\cdot, \boldsymbol{u}(\cdot), z), \boldsymbol{u}(\cdot), z) \subseteq \partial_u^\circ \phi(\boldsymbol{x}(\cdot, \boldsymbol{u}(\cdot), z), \boldsymbol{u}(\cdot), z) \times \partial_z^\circ \phi(\boldsymbol{x}(\cdot, \boldsymbol{u}(\cdot), z), \boldsymbol{u}(\cdot), z).$$

The state trajectory $\boldsymbol{x}(\cdot, \boldsymbol{u}(\cdot), z)$ as a function of $(\boldsymbol{u}(\cdot), z)$ is differentiable. By induction, gradients of the function $\boldsymbol{x}(\cdot, \boldsymbol{u}(\cdot), z)$ can be calculated by

$$\begin{aligned} \partial \boldsymbol{x}(k, \boldsymbol{u}(\cdot), z)/\partial \boldsymbol{u}(l) &= 0, \qquad k = 0, 1, \dots, l, \\ \partial \boldsymbol{x}(k, \boldsymbol{u}(\cdot), z)/\partial \boldsymbol{u}(l) &= A^{k-l-1}B, \qquad k = l+1, \dots, N, \end{aligned}$$

where $A^0 = I$, the identity matrix.

$$\begin{gathered} \partial \boldsymbol{x}(0)/\partial z = \partial \boldsymbol{x}^0(z)\partial z, \\ \partial \boldsymbol{x}(k, \boldsymbol{u}(\cdot), z)/\partial z = A^k \partial \boldsymbol{x}^0(z)/\partial z + (A^{k-1} + A^{k-2} + \cdots + I)C, \quad k = 1, \dots, N. \end{gathered}$$

Let $(g_0, \dots, g_{N-1})^\mathsf{T} \in \partial_u^\circ \phi(\boldsymbol{x}(\cdot, \boldsymbol{u}(\cdot), z), \boldsymbol{u}(\cdot), z)$. By using the chain rule,

$$\begin{aligned} &(g_0, \dots, g_{N-1})^\mathsf{T} \\ &= \sum_{k=1}^{N} g_{x(k)} \left[\frac{\partial \boldsymbol{x}(k, \boldsymbol{u}(\cdot), z)}{\partial \boldsymbol{u}(0)}, \dots, \frac{\partial \boldsymbol{x}(k, \boldsymbol{u}(\cdot), z)}{\partial \boldsymbol{u}(N-1)} \right]^\mathsf{T} + \sum_{k=0}^{N-1} g_{u(k)} [0, \dots, 1, \dots, 0]^\mathsf{T}. \end{aligned}$$

Let $\boldsymbol{h}_0 \in \partial_z^\circ \phi(\boldsymbol{x}(\cdot, \boldsymbol{u}(\cdot), z), \boldsymbol{u}(\cdot), z)$. Then

$$\boldsymbol{h}_0 = \sum_{k=1}^{N} g_{x(k)} \frac{\partial \boldsymbol{x}(k, \boldsymbol{u}(\cdot), z)}{\partial z} + g_z.$$

Thus

$$\begin{aligned} g_l &= \sum_{k=1}^{N} g_{x(k)} \frac{\partial \boldsymbol{x}(k, \boldsymbol{u}(\cdot), z)}{\partial \boldsymbol{u}(l)} + g_{u(l)} \\ &= \sum_{k=1}^{N} A^{k-l-1} B g_{x(k)} + g_{u(l)}, \; l = 0, \dots, N-1, \\ \boldsymbol{h}_0 &= \sum_{k=1}^{N} g_{x(k)} \frac{\partial \boldsymbol{x}(k, \boldsymbol{u}(\cdot), z)}{\partial z} + g_z \\ &= \sum_{k=1}^{N} \left[A^k \frac{\partial \boldsymbol{x}^\circ(z)}{\partial z} + (A^{k-1} + A^{k-2} + \cdots + I)C \right] g_{x(k)} + g_z. \end{aligned}$$

The proof is complete.

https://doi.org/10.1017/S0334270000010973 Published online by Cambridge University Press

Assume that $\phi(\boldsymbol{x}(\cdot,\boldsymbol{u}(\cdot),z),\boldsymbol{u}(\cdot),z)$ is subdifferentially regular at $(\boldsymbol{u}(\cdot),z)$. Then from Proposition 3.1 (i) and (iii),

$$\phi'(\boldsymbol{x}(\cdot,\boldsymbol{u}(\cdot),z),\boldsymbol{u}(\cdot),z;(\delta\boldsymbol{u}(\cdot),\delta z)) = \max_{g^*\in\partial^\circ\phi(\boldsymbol{x}(\cdot,\boldsymbol{u}(\cdot),z),\boldsymbol{u}(\cdot),z)} \boldsymbol{g}^{*\top}(\delta\boldsymbol{u}(\cdot),\delta z).$$

It follows from Theorem 3.1 that

$$\begin{aligned}
&\phi'(\boldsymbol{x}(\cdot,\boldsymbol{u}(\cdot),z),\boldsymbol{u}(\cdot),z;(\delta\boldsymbol{u}(\cdot),\delta z))\\
&\quad\le \max_{(g_0,\dots,g_{N-1},h_0)\in\Phi} \sum_{k=0}^{N-1} \boldsymbol{g}_k^\top\delta\boldsymbol{u}(\cdot) + \boldsymbol{h}_0^\top\delta z\\
&\quad= \max_{\tilde g}\left\{\sum_{k=0}^{N-1}\left(\sum_{l=k+1}^{N} A^{l-k-1}B\boldsymbol{g}_{x(l)} + \boldsymbol{g}_{u(l)}\right)^\top \delta\boldsymbol{u}(\cdot)+\right.\\
&\qquad\left.\sum_{k=1}^{N}\left(\left[A^k\frac{\partial \boldsymbol{x}^\circ(z)}{\partial z} + (A^{k-1}+A^{k-2}+\dots+I)C\right]\boldsymbol{g}_{x(k)} + \boldsymbol{g}_z\right)^\top\delta z\right\},
\end{aligned}$$

where

$$\tilde g = (\boldsymbol{g}_{x(1)},\dots,\boldsymbol{g}_{x(N)},\boldsymbol{g}_{u(0)},\dots,\boldsymbol{g}_{u(N-1)},\boldsymbol{g}_z) \in \partial^\circ\phi(\boldsymbol{x}(\cdot),\boldsymbol{u}(\cdot),z).$$

Define

$$\begin{aligned}
CE(\boldsymbol{u}(\cdot),z) &= \{(\delta\boldsymbol{u}(\cdot),\delta z)\in \boldsymbol{R}^{mN+s} : \boldsymbol{a}_i^\top\delta\boldsymbol{u}(\cdot) + \boldsymbol{b}_i^\top\delta z = 0,\ i\in E\},\\
CI(\boldsymbol{u}(\cdot),z) &= \{(\delta\boldsymbol{u}(\cdot),\delta z)\in \boldsymbol{R}^{mN+s} : \boldsymbol{a}_j^\top\delta\boldsymbol{u}(\cdot) + \boldsymbol{b}_j^\top\delta z \le 0,\ j\in I(\delta\boldsymbol{u}(\cdot),\delta z)\},
\end{aligned}$$

where $I(\delta\boldsymbol{u}(\cdot),\delta z) = \{j\in I : \boldsymbol{a}_j^\top\boldsymbol{u}(\cdot) + \boldsymbol{b}_j^\top z = c_j\}$.

THEOREM 3.2. *Assume that $\phi(\boldsymbol{x}(\cdot,\boldsymbol{u}(\cdot),z),\boldsymbol{u}(\cdot),z)$ is subdifferentially regular at $(\boldsymbol{u}^*(\cdot),z^*)$. If $(\boldsymbol{u}^*(\cdot),z^*)$ is optimal of the scalar* DOC *problem, then*

$$\begin{aligned}
\inf_{(\delta u(\cdot),\delta z)}\max_{\tilde g}&\left\{\sum_{k=0}^{N-1}\left(\sum_{l=k+1}^{N} A^{l-k-1}B\boldsymbol{g}_{x(l)} + \boldsymbol{g}_{u(k)}\right)^\top \delta\boldsymbol{u}(\cdot)+\right.\\
&\left.\sum_{k=1}^{N}\left(\left[A^k\frac{\partial \boldsymbol{x}^\circ(z)}{\partial z} + (A^{k-1}+A^{k-2}+\dots+I)C\right]\boldsymbol{g}_{x(k)} + \boldsymbol{g}_z\right)^\top\delta z\right\} \ge 0,
\end{aligned}$$

where

$$\begin{aligned}
&(\delta\boldsymbol{u}(\cdot),\delta z)\in CI(\boldsymbol{u}^*(\cdot),z^*)\cap CE(\boldsymbol{u}^*(\cdot),z^*),\\
&\tilde g = (\boldsymbol{g}_{x(1)},\dots,\boldsymbol{g}_{x(N)},\boldsymbol{g}_{u(0)},\dots,\boldsymbol{g}_{u(N-1)},\boldsymbol{g}_z) \in \partial^\circ\phi(\boldsymbol{x}^*(\cdot),\boldsymbol{u}^*(\cdot),z^*).
\end{aligned}$$

PROOF. This follows from Theorem 3.1 above and Theorem 3.2 in Pytlak and Malinowski [13] and is omitted.

THEOREM 3.3. *Assume that* $f_i(\boldsymbol{x}(\cdot,\boldsymbol{u}(\cdot),z),\boldsymbol{u}(\cdot),z)$, $i=1,2$ *is subdifferentially regular at* $(\boldsymbol{u}^*(\cdot),z^*)$. *Let* $\lambda\in[0,1]$. *If* $(\boldsymbol{u}^*(\cdot),z^*)$ *is optimal of the parametric* DOC *problem, then*

$$\inf_{(\delta\boldsymbol{u}(\cdot),\delta z)}\max_{\tilde{g}^\lambda}\left\{\sum_{k=0}^{N-1}\left(\sum_{l=k+1}^{N}A^{l-k-1}Bg^\lambda_{x(l)}+g^\lambda_{u(k)}\right)^{\mathrm{T}}\delta\boldsymbol{u}(\cdot)+\right.$$
$$\left.\sum_{k=1}^{N}\left(\left[A^k\frac{\partial \boldsymbol{x}^\circ(z)}{\partial z}+(A^{k-1}+A^{k-2}+\cdots+I)C\right]g^\lambda_{x(k)}+g^\lambda_z\right)^{\mathrm{T}}\delta z\right\}\geq 0,$$

where

$$\begin{aligned}
&(\delta\boldsymbol{u}(\cdot),\delta z)\in CI(\boldsymbol{u}^*(\cdot),z^*)\cap CE(\boldsymbol{u}^*(\cdot),z^*),\\
&\tilde{g}^\lambda=\lambda\tilde{g}^1+(1-\lambda)\tilde{g}^2,\\
&\tilde{g}^1=(g^1_{x(1)},\dots,g^1_{x(N)},g^1_{u(0)},\dots,g^1_{u(N-1)},g^1_z)\in\partial^\circ f_1(\boldsymbol{x}^*(\cdot),\boldsymbol{u}^*(\cdot),z^*),\\
&\tilde{g}^2=(g^2_{x(1)},\dots,g^2_{x(N)},g^2_{u(0)},\dots,g^2_{u(N-1)},g^2_z)\in\partial^\circ f_2(\boldsymbol{x}^*(\cdot),\boldsymbol{u}^*(\cdot),z^*).
\end{aligned}$$

PROOF. Now

$$\begin{aligned}
\phi(\boldsymbol{x}(\cdot),\boldsymbol{u}(\cdot),z)&=\lambda f_1(\boldsymbol{x}(\cdot),\boldsymbol{u}(\cdot),z)+(1-\lambda)f_2(\boldsymbol{x}(\cdot),\boldsymbol{u}(\cdot),z),\\
\phi(\boldsymbol{x}(\cdot,\boldsymbol{u}(\cdot),z),\boldsymbol{u}(\cdot),z)&=\lambda f_1(\boldsymbol{x}(\cdot,\boldsymbol{u}(\cdot),z),\boldsymbol{u}(\cdot),z)\\
&\quad+(1-\lambda)f_2(\boldsymbol{x}(\cdot,\boldsymbol{u}(\cdot),z),\boldsymbol{u}(\cdot),z).
\end{aligned}$$

From Proposition 3.1 (ii) and the subdifferential regularity assumption on f_i, $i=1,2$, we have that $\phi(\boldsymbol{x}(\cdot,\boldsymbol{u}(\cdot),z),\boldsymbol{u}(\cdot),z)$ is subdifferentially regular at $(\boldsymbol{u}^*(\cdot),z^*)$ and

$$\partial^\circ\phi(\boldsymbol{x}^*(\cdot),\boldsymbol{u}^*(\cdot),z^*)\subseteq\lambda\partial^\circ f_1(\boldsymbol{x}^*(\cdot),\boldsymbol{u}^*(\cdot),z^*)+(1-\lambda)\partial^\circ f_2(\boldsymbol{x}^*(\cdot),\boldsymbol{u}^*(\cdot),z^*).$$

The result follows from Theorem 3.2.

4. Necessary optimality conditions of a minimax-max DOC

Consider the minimax-max DOC problem

$$\text{minimize}\ \max\left\{\max_{1\leq k\leq N}c_1(k,\boldsymbol{x}(k),\boldsymbol{u}(k),z),\ \max_{1\leq k\leq N}c_2(k,\boldsymbol{x}(k),\boldsymbol{u}(k),z)\right\}$$
$$\text{subject to}\ (\boldsymbol{u},z)\in\mathscr{F},$$

where $c_i : \{1, \dots, N\} \times \boldsymbol{R}^n \times \boldsymbol{R}^r \times \boldsymbol{R}^s \to \boldsymbol{R}$, $i = 1, 2$ is differentiable with respect to each of its component. This model has applications in the modelling of an economic system using different objective functions, see Cai *et al.* [2] and Sandblom *et al.* [14]. If there is only one criterion c_1, the minimax-max DOC problem is reduced to the minimax DOC.

The parametric DOC of the minimax-max DOC problem is a generalized parametric minimax DOC problem:

$$\text{minimize } \lambda \max_{1\le k\le N} c_1(k, \boldsymbol{x}(k), \boldsymbol{u}(k), z) + (1-\lambda) \max_{1\le k\le N} c_2(k, \boldsymbol{x}(k), \boldsymbol{u}(k), z),$$
$$\text{subject to } (\boldsymbol{u}, z) \in \mathscr{F}.$$

The necessary optimality condition of the generalized parametric minimax DOC problem can be derived from Theorem 3.2 as shown below.

THEOREM 4.1. *Let* $\lambda \in [0, 1]$. *If* $(\boldsymbol{u}^*(\cdot), z^*)$ *is optimal for the generalized parametric minimax* DOC *problem, then*

$$\inf_{(\delta\boldsymbol{u}(\cdot),\delta z)} \max_{\boldsymbol{\eta}^1,\boldsymbol{\eta}^2,\boldsymbol{\xi}^1,\boldsymbol{\xi}^2} \left\{ \sum_{k=0}^{N-1} \left(\sum_{l=k+1}^{N} A^{l-k-1} B \sum_{i=1}^{2} \lambda_i \eta_l^i \boldsymbol{v}_{\boldsymbol{x}(l)}^i + \sum_{i=1}^{2} \lambda_i \eta_l^i \boldsymbol{w}_{\boldsymbol{u}(k)}^i \right)^{\mathsf{T}} \delta\boldsymbol{u}(\cdot) + \sum_{k=1}^{N} \left(\left[A^k \frac{\partial \boldsymbol{x}^\circ(z)}{\partial z} + (A^{k-1} + A^{k-2} + \cdots + I) C \right] \sum_{i=1}^{2} \lambda_i \eta_k^i \boldsymbol{v}_{\boldsymbol{x}(k)}^i + \sum_{i=1}^{2} \lambda_i \sum_{k=1}^{N} \eta_k^i s_{zk}^i \right)^{\mathsf{T}} \delta z \right\} \ge 0,$$

where

$$(\delta\boldsymbol{u}(\cdot), \delta z) \in CI(\boldsymbol{u}^*(\cdot), z^*) \cap CE(\boldsymbol{u}^*(\cdot), z^*),$$
$$\boldsymbol{w}_{\boldsymbol{u}(k-1)}^i = \partial c_i(\boldsymbol{x}(k), \boldsymbol{u}(k), z)/\partial \boldsymbol{u}(k),$$
$$\boldsymbol{v}_{\boldsymbol{x}(k)}^i = \partial c_i(\boldsymbol{x}(k), \boldsymbol{u}(k), z)/\partial \boldsymbol{x}(k),$$
$$s_{zk}^i = \partial c_i(\boldsymbol{x}(k), \boldsymbol{u}(k), z)/\partial z,$$
$$\sum_{k=1}^{N} \eta_{ik} = 1,\ \eta_{ik} \ge 0,\ \lambda_1 = \lambda,\ \lambda_2 = 1 - \lambda.$$

PROOF. Now

$$\phi(\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot), z) = \sum_{i=1}^{2} \lambda_i \max_{1\le k\le N} c_i(\boldsymbol{x}(k), \boldsymbol{u}(k), z).$$

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It is clear $\phi(\boldsymbol{x}(\cdot, \boldsymbol{u}(\cdot), z), \boldsymbol{u}(\cdot), z)$ is subdifferentially regular at $(\boldsymbol{u}^*(\cdot), z^*)$. We need to appropriately work out $\partial^\circ\phi(\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot), z)$. From Proposition 3.1 (ii), we have

$$\partial^\circ\phi(\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot), z) \subseteq \left\{ \sum_{i=1}^{2} \lambda_i \sum_{l=1}^{N} \eta_k^i \boldsymbol{\xi}_l^i : \boldsymbol{\eta}^1, \boldsymbol{\eta}^2, \boldsymbol{\xi}^1, \boldsymbol{\xi}^2 \right\},$$

where $\boldsymbol{\eta}^i \in \boldsymbol{R}^N (i = 1, 2)$ is such that $\sum_{k=1}^N \eta_k^i = 1$, $\eta_k^i \geq 0$ and $\boldsymbol{\xi}_k^i \in \partial^\circ c_i(\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot), z)$ $(i = 1, 2)$ which satisfies

$$\begin{aligned}
\boldsymbol{\xi}_k^i &= (0, \dots, \boldsymbol{v}_{x(k)}^i, \dots, 0, 0, \dots, \boldsymbol{w}_{u(k-1)}^i, \dots, 0, s_{zk}^i) \in \boldsymbol{R}^{(n+r)N+s}, \\
\boldsymbol{v}_{x(k)}^i &= \partial c_i(\boldsymbol{x}(k), \boldsymbol{u}(k), z)/\partial \boldsymbol{x}(k), \quad \boldsymbol{w}_{u(k-1)}^i = \partial c_i(\boldsymbol{x}(k), \boldsymbol{u}(k), z)/\partial \boldsymbol{u}(k), \\
s_{zk}^i &= \partial c_i(\boldsymbol{x}(k), \boldsymbol{u}(k), z)/\partial z,
\end{aligned}$$

since $c_i(\boldsymbol{x}(k), \boldsymbol{u}(k), z)$ only depends on $\boldsymbol{x}(k)$, $\boldsymbol{u}(k)$ and z. So

$$\sum_{k=1}^{N} \eta_k^i \boldsymbol{\xi}_k^i = \left(\eta_1^i \boldsymbol{v}_{x(1)}^i, \dots, \eta_N^i \boldsymbol{v}_{x(N)}^i, \eta_1^i \boldsymbol{w}_{u(0)}^i, \dots, \eta_N^i \boldsymbol{w}_{u(N-1)}^i, \sum_{k=1}^{N} \eta_k^i s_{zk}^i \right).$$

Thus

$$\begin{aligned}
\sum_{i=1}^{2} \lambda_i \sum_{k=1}^{N} \eta_k^i \boldsymbol{\xi}_k^i = \Bigg(&\sum_{i=1}^{2} \lambda_i \eta_1^i \boldsymbol{v}_{x(1)}^i, \dots, \sum_{i=1}^{2} \lambda_i \eta_N^i \boldsymbol{v}_{x(N)}^i, \sum_{i=1}^{2} \lambda_i \eta_1^i \boldsymbol{w}_{u(0)}^i, \dots, \\
&\sum_{i=1}^{2} \lambda_i \eta_N^i \boldsymbol{w}_{u(N-1)}^i, \sum_{i=1}^{2} \lambda_i \sum_{k=1}^{N} \eta_k^i s_{zk}^i \Bigg).
\end{aligned}$$

Then the conclusion follows from Theorem 3.2.

5. Conclusion

It was shown in this paper that a solution of a bicriterion DOC problem can be approached by solving a number of parametric DOC problems if the set Ω defined in Theorem 2.1 is convex. This is a much weaker condition than the convexity on each f_i. Necessary optimality conditions for various DOC problems were derived. It was shown that no adjoint equation is required for DOC problems subject to a linear state difference equation, while the adjoint equation is needed for DOC problems subject to a nonlinear state difference equation.

Numerical experiments using the technique outlined in this paper and some further generalization along this line (e.g., the efficient solution of a bicriterion DOC problem is defined by a closed convex cone) will be reported elsewhere.

Acknowledgement

This work was partially supported by the Australian Research Council.

References

[1] V. G. Boltyanskii, *Optimal Control of Discrete Systems* (John Wiley & Sons, New York, Toronto, 1978).

[2] X. Cai, K. L. Teo, X. Q. Yang and X. Y. Zhou, "Portfolio optimization under l_∞ risk measure", in *35th IEEE Conference on Decision and Control, Kobe, Japan*, 1996.

[3] F. H. Clarke, *Optimization and Nonsmooth Analysis* (John Wiley & Sons, New York, 1983).

[4] B. D. Craven, *Mathematical Programming and Control Theory* (Chapman and Hall, London, 1978).

[5] B. D. Craven, *Fractional Programming* (Heldermann Verlag, Berlin, 1988).

[6] A. M. Geoffrion, "Solving bicriteria mathematical programs", *Operations Research* **15** (1967) 39–54.

[7] V. Jeyakumar and X. Q. Yang, "Convex composite multi-objective nonsmooth programming", *Mathematical Programming* **59** (1993) 325–343.

[8] D. Li, "On the minimax solution of multiple linear-quadratic problems", *IEEE Transactions on Automatic Control* **35** (1990) 1153–1156.

[9] G. V. Loganathan, "Benefit-cost problems: a bi-criteria approach", *Engineering Optimization* **13** (1988) 225–234.

[10] D. T. Luc, *Theory of Vector Optimization*, Lecture Notes in Economics and Mathematical Systems (Springer, Berlin, 1989).

[11] A. Marchi, *On the relationships between bicriteria problems and nonlinear programming*, Lecture Notes in Economics and Mathematical Systems, 405 (Springer, Berlin, 1994).

[12] B. Mond, "On algorithmic equivalence in linear fractional programming", *Mathematics of Computation* **37** (1981) 185–187.

[13] R. Pytlak and K. Malinowski, "Optimal scheduling of reservior releases during flood: deterministic optimization problem", *J. Optim. Theory Appl.* **61** (1989) 409–431.

[14] C. L. Sandblom, H. A. Eiselt and K. O. Jornsten, "Discrete-time optimal control of an economic system using different objective functions", *Optimal Control Applications & Methods* **8** (1987) 253–269.

[15] K. L. Teo, C. J. Goh and K. H. Wong, *A Unified Computational Approach to Optimal Control Problems* (Longman Scientific and Technical, New York, 1990).

[16] P. L. Yu, *Multiple-Criteria Decision Making* (Plenum Press, New York, 1985).

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