Glasgow Math. J. **57** (2015) 211–230. © Glasgow Mathematical Journal Trust 2014.
doi:10.1017/S0017089514000275.

ON THE CONNECTEDNESS OF THE BRANCH LOCI OF NON-ORIENTABLE UNBORDERED KLEIN SURFACES OF LOW GENUS

E. BUJALANCE

Departamento de Matemáticas Fundamentales, UNED, Paseo Senda del Rey 9, 28040-Madrid, Spain
e-mail: ebujalance@mat.uned.es

J. J. ETAYO

Departamento de Álgebra, Facultad de Matemáticas, Universidad Complutense, 28040-Madrid, Spain
e-mail: jetayo@mat.ucm.es

E. MARTÍNEZ

Departamento de Matemáticas Fundamentales, UNED, Paseo Senda del Rey 9, 28040-Madrid, Spain
e-mail: emartinez@mat.uned.es

and B. SZEPIETOWSKI

Institute of Mathematics, University of Gdańsk. Ul. Wita Stwosza 57, 80-952 Gdańsk, Poland
e-mail: blaszep@mat.ug.edu.pl

(Received 20 November 2012; revised 10 April 2014; accepted 15 May 2014)

Abstract. This paper is devoted to determine the connectedness of the branch loci of the moduli space of non-orientable unbordered Klein surfaces. We obtain a result similar to Nielsen's in order to determine topological conjugacy of automorphisms of prime order on such surfaces. Using this result we prove that the branch locus is connected for surfaces of topological genus 4 and 5.

2010 *Mathematics Subject Classification.* Primary 57M60; Secondary 20F05, 20H10, 30F50.

1. Introduction. A non-orientable unbordered surface is called a Klein surface if it is compact and endowed with a dianalytic atlas, see [**1**]. Non-orientable unbordered Klein surfaces are also called non-orientable Riemann surfaces and they correspond to the purely imaginary real algebraic curves.

The study of the moduli space of both Riemann and Klein surfaces is a classic problem. The subspace of surfaces with non-trivial automorphisms is called the branch locus of the moduli space. In the case of Riemann surfaces this subspace was studied by Harvey in [**14**] and Broughton in [**5**]. Its connectedness was studied in several papers, for example in [**3**, **10**, **13**, **15**]. The main result is that the real locus of the moduli space of Riemann surfaces is connected (see [**11**, **12**, **20**]), while the branch locus of the moduli space of Riemann surfaces of genus g is connected only when g is 3, 4, 7, 13, 17, 19 or 59 (see [**2**, **4**]).

In this paper we determine the connectedness of the branch locus for non-orientable unbordered Klein surfaces of topological genera 4 and 5. We do not consider surfaces of genus 3 because they are hyperelliptic and it is already known that the branch locus is connected. In order to make that study in genus 4 and 5 we start from the results on the moduli space and the branch locus of Klein surfaces obtained by Macbeath and Singerman in [**17**], Natanzon in [**18**] and Seppälä in [**20**].

In Section 3, we establish an analog of Nielsen's result on the parameters determining an automorphism of prime order on a given compact Riemann surface up to topological conjugation. Here we are concerned with non-orientable unbordered Klein surfaces. Using this result we obtain in the remaining sections that the branch loci of these surfaces are connected for genus 4 and 5. Before this, in Section 2, we give the preliminaries about NEC groups and Klein surfaces, and explain the technique used to check that the branch locus is connected. In the last sections we consider the surfaces with genera 4 and 5.

2. Preliminaries.

2.1. NEC groups. An NEC group Γ is a discrete subgroup of isometries of the hyperbolic plane $\mathcal{H}$, including orientation-reversing elements, with compact quotient $S = \mathcal{H}/\Gamma$. Each NEC group Γ has associated a signature [**16**]:

$$\sigma(\Gamma) = (g, \pm, [m_1, \ldots, m_r], \{(n_{i,1}, \ldots, n_{i,s_i}),\ i = 1, \ldots, k\}), \tag{1}$$

where $g, k, r, m_i, n_{i,j}$ are integers satisfying $g, k, r \geq 0,\ m_i \geq 2, n_{i,j} \geq 2$. The number g is the topological genus of S. The sign determines the orientability of S. The numbers m_i are the *proper periods*. The brackets $(n_{i,1}, \ldots, n_{i,s_i})$ are the *period-cycles*. The number k of period-cycles is equal to the number of boundary components of S. Numbers $n_{i,j}$ are the periods of the period-cycle $(n_{i,1}, \ldots, n_{i,s_i})$, also called *link-periods*. We will denote by $[-]$ and $(-)$ the cases when $r = 0$ and $s_i = 0$ respectively. Also, we abbreviate by $[(m)^r]$ and $\{(-)^k\}$ the expressions $[m, \overset{r}{\ldots}, m]$ and $\{(-), \overset{k}{\ldots}, (-)\}$ respectively.

The signature determines a presentation [**22**] of Γ by means of generators x_i $(i = 1, \ldots, r)$; e_i $(i = 1, \ldots, k)$; $c_{i,j}$ $(i = 1, \ldots, k; j = 0, \ldots, s_i)$; and a_i, b_i $(i = 1, \ldots, g)$ if σ has sign '+' and d_i $(i = 1, \ldots, g)$ if σ has sign '−'.

These generators satisfy the following relations: $x_i^{m_i} = 1$ $(i = 1, \ldots, r)$; $c_{i,j-1}^2 = c_{i,j}^2 = (c_{i,j-1}c_{i,j})^{n_{i,j}} = 1$ $\quad (i = 1, \ldots, k, j = 1, \ldots, s_i)$; $\quad e_i^{-1}c_{i,0}e_ic_{i,s_i} = 1$ $\quad (i = 1, \ldots, k)$; $\prod_{i=1}^r x_i \prod_{i=1}^k e_i \prod_{i=1}^g (a_ib_ia_i^{-1}b_i^{-1}) = 1$ if σ has sign '+'; and $\prod_{i=1}^r x_i \prod_{i=1}^k e_i \prod_{i=1}^g d_i^2 = 1$ if σ has sign '−'.

The corresponding final relation is called the long relation. The isometries x_i are elliptic, e_i, a_i, b_i are hyperbolic, $c_{i,j}$ are reflections and d_i are glide reflections.

Every NEC group Γ with signature (1) has associated a fundamental region whose area $\mu(\Gamma)$, called the *area of the group*, is:

$$\mu(\Gamma) = 2\pi \left(\eta g + k - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{i,j}}\right) \right), \tag{2}$$

with $\eta = 2$ or 1 depending on whether the sign in the signature is '+' or '−'. An NEC group with signature (1) actually exists if and only if the right-hand side of (2) is greater than 0.

If Γ is a subgroup of an NEC group Λ of finite index N, then also Γ is an NEC group and the following Riemann–Hurwitz formula holds:

$$\mu(\Gamma) = N\mu(\Lambda). \tag{3}$$

Let S be a non-orientable Klein surface without boundary of topological genus $g \geq 3$. Then, by [**19**], there exists an NEC group Γ_g with signature:

$$\sigma(\Gamma_g) = (g, -, [-], \{-\}), \tag{4}$$

such that $S = \mathcal{H}/\Gamma_g$.

2.2. Teichmüller and moduli spaces. Let $\mathcal{G}$ be the group of isometries of the hyperbolic plane $\mathcal{H}$, including reversing-orientation elements. Given an NEC group Γ, we denote by $R(\Gamma)$ the set of monomorphisms $r : \Gamma \to \mathcal{G}$ such that $r(\Gamma)$ is discrete and $\mathcal{H}/\Gamma$ is compact.

Two elements $r_1, r_2 \in R(\Gamma)$ are said to be equivalent if there exists $g \in \mathcal{G}$ such that $r_1(\gamma) = gr_2(\gamma)g^{-1}$ for each $\gamma \in \Gamma$. The quotient space $T(\Gamma)$ is the Teichmüller space of Γ.

When Γ is a Fuchsian group with signature $(g, +, [m_1, \ldots, m_r])$, it is known as that the dimension of $T(\Gamma)$, $d(T(\Gamma))$ is $6(g-1) + 2r$. Let now Γ be a proper NEC group and call Γ^+ its canonical Fuchsian subgroup. Singerman proved in [**21**] that $d(T(\Gamma)) = \frac{1}{2}d(T(\Gamma^+))$. The Teichmüller modular group of Γ, $Mod(\Gamma)$, is the quotient group $Aut(\Gamma)/Inn(\Gamma)$, where $Aut(\Gamma)$ and $Inn(\Gamma)$ denote the full group of automorphisms and the subgroup of inner automorphisms of Γ respectively [**17**]. The moduli space of Γ is the quotient $M(\Gamma) = T(\Gamma)/Mod(\Gamma)$. For the action of the $Mod(\Gamma)$ on the Teichmüller space, we refer the reader to Section 9 in [**17**].

Let K be an NEC subgroup of Γ of finite index. Then following the notation from Harvey in [**14**], let $I(K, \Gamma, H)$ denote the set of all injections $i : K \hookrightarrow \Gamma$ with $\Gamma/i(K) \approx H$. Every such i is determined up to automorphisms of K by a surjection of Γ onto H. Let $\Phi(K, \Gamma, H)$ be the family of all equivalence classes of surjections $\varphi : \Gamma \to H$ with $\ker(\varphi) \approx K$, modulo the actions of $Aut(\Gamma)$ and $Aut(H)$. Note that Φ is finite since Γ is finitely generated.

Given $i : K \hookrightarrow \Gamma$, denote by $T(\hat{\imath})$ the image of the associated isometric embedding $T(\Gamma) \to T(K)$ (see [**17**]). Let i_φ be any injection corresponding to a class $[\varphi] \in \Phi(K, \Gamma, H)$, $i_\varphi : K \to \ker(\varphi)$. We denote by $\Lambda(K, \Gamma, H)$ the set $\{[r] \in T(K) : [r] \in T(\hat{\imath}), i \in I(K, \Gamma, H)\}$. Then,

$$\Lambda(K, \Gamma, H) = \bigcup_{\alpha \in Mod(K)} \alpha \left(\bigcup_{[\varphi] \in \Phi(K,\Gamma,H)} T(\hat{\imath}_\varphi) \right).$$

Let Γ_g be an NEC group with signature $(g, -, [-], \{-\})$. Then $T(\Gamma_g)$ is the Teichmüller space of compact non-orientable Klein surfaces without boundary of topological genus g. We denote by Λ_g the subspace of $T(\Gamma_g)$ corresponding to those surfaces with non-trivial automorphisms. Then, by [**17**], Λ_g consists of those points

https://doi.org/10.1017/S0017089514000275 Published online by Cambridge University Press

which are fixed by some element of $Mod(\Gamma_g)$, which must have finite order. Let n range over all possible orders of elements of $Mod(\Gamma_g)$, in fact, $2 \leq n \leq 2g$ if g is odd, and $2 \leq n \leq 2(g-1)$ if g is even, see [**6**]. For each n, let $\mathcal{F}_{n,g}$ denote the set of isomorphism classes of NEC groups Γ with non-empty $\Phi(\Gamma_g, \Gamma, C_n)$, where C_n is the cyclic group of order n. Then,

$$\Lambda_g = \bigcup_n \bigcup_{\Gamma \in \mathcal{F}_{n,g}} \Lambda(\Gamma_g, \Gamma, C_n).$$

Hence, Λ_g is a countable union of submanifolds of $T(\Gamma_g)$. We call $B_g = \Lambda_g/Mod(\Gamma_g)$. Each set $\Lambda(\Gamma_g, \Gamma, C_n)$ is mapped into itself by $Mod(\Gamma_g)$, and the quotient of this set is made up of connected subsets in a one-to-one correspondence with classes $[\varphi] \in \Phi(\Gamma_g, \Gamma, C_n)$.

Let $\theta_i : \Gamma_i \to G$, $i = 1, 2$ be two epimorphisms with $\ker \theta_i \approx \Gamma_g$, where G is a finite group and Γ_i are NEC groups. We say that θ_1 and θ_2 are *topologically conjugate* if and only if there exist an isomorphism $\psi : \Gamma_1 \to \Gamma_2$ and $\alpha \in Aut(G)$ such that $\alpha\theta_1 = \theta_2\psi$.

Given two primes p and q and two NEC groups Γ' and Γ'', consider two epimorphisms $\theta' : \Gamma' \to C_p$ and $\theta'' : \Gamma' \to C_q$ with $\ker(\theta') \approx \ker(\theta'') \approx \Gamma_g$. Then we say that θ' and θ'' are *connected* if there exist an NEC group Γ^* and an epimorphism $\theta^* : \Gamma^* \to G$, where G is a group of order pq, and there exist two subgroups of G, G' and G'', of orders p and q, satisfying

$$\begin{aligned}
&{\theta^*}^{-1}(G') \approx \Gamma',\\
&{\theta^*}^{-1}(G'') \approx \Gamma'',\\
&\theta^*|_{\theta^{*-1}(G')} \text{ is topologically conjugate to } \theta',\\
&\theta^*|_{\theta^{*-1}(G'')} \text{ is topologically conjugate to } \theta''.
\end{aligned}$$

Since each non-trivial group G contains a subgroup of prime order, for proving that B_g is connected, it suffices to prove that for each primes p, q, each two epimorphisms θ', θ'', there exist a sequence of primes p_i, NEC groups Γ_i and epimorphisms $\theta_i : \Gamma_i \to C_{p_i}$, $1 \leq i \leq k$, such that $\theta_1 = \theta'$, $\theta_k = \theta''$ and θ_i is connected to θ_{i+1} for $i = 1, \ldots, k-1$. This will be done in Sections 4 and 5 for primes 2 and 3 (genus 4) and for 2, 3 and 5 (genus 5). For a similar argument, see for example [**2**] or [**3**].

3. Equivalence. Suppose that $\theta_i : \Gamma \to C_p$, $i = 1, 2$ are two epimorphisms, where Γ is an NEC group, p is prime and $\ker \theta_i$ is isomorphic to the fundamental group of a non-orientable surface. In this section, we give necessary and sufficient conditions for θ_1 and θ_2 to be topologically conjugate.

We list the automorphisms of Γ which will be used. Most of these are from papers [**7**, **14**]. If the sign of the signature of Γ is '+', then we are going to use the following automorphisms of Γ:

ω defined by $\omega(a_1) = a_1b_1$ and the identity on the remaining canonical generators.

ξ defined by $\xi(a_1) = a_1b_1$, $\xi(b_1) = a_1^{-1}$ and the identity on the remaining canonical generators.

ν_j defined by $\nu_j(a_j) = a_{j+1}$, $\nu_j(b_j) = b_{j+1}$, $\nu_j(a_{j+1}) = c_{j+1}^{-1}a_jc_{j+1}$, $\nu_j(b_{j+1}) = c_{j+1}^{-1}b_jc_{j+1}$, where $c_{j+1} = [a_{j+1}, b_{j+1}]$ and the identity on the remaining canonical generators.

μ defined by $\mu(a_1) = a_2a_1$, $\mu(a_2) = b_1a_2b_1^{-1}$, $\mu(b_1) = b_1$, $\mu(b_2) = a_2b_2a_2^{-1}b_1^{-1}$ and $\mu(y) = a_2ya_2^{-1}$ for every canonical generator y different from a_1, b_1, a_2, b_2.

σ defined by $\sigma(x_r) = Ea_1^{-1}E^{-1}x_rEa_1E^{-1}$, $\sigma(a_1) = [a_1^{-1}, E^{-1}x_r^{-1}E]a_1$, $\sigma(b_1) = b_1a_1^{-1}E^{-1}x_rEa_1$, where $E = e_1 \cdots e_k$ and the identity on the remaining canonical generators.

π defined by $\pi(e_k) = a_1^{-1}e_ka_1$, $\pi(c_{k,i}) = a_1^{-1}c_{k,i}a_1$, $\pi(a_1) = [a_1^{-1}, e_k^{-1}]a_1$, $\pi(b_1) = b_1a_1^{-1}e_ka_1$ and the identity on the remaining canonical generators.

If the sign in the signature of Γ is '$-$', then we are going to use the following automorphisms.

α_j defined by $\alpha_j(d_j) = d_j^2d_{j+1}d_j^{-2}$, $\alpha_j(d_{j+1}) = d_j$ and the identity on the remaining canonical generators.

β_j defined by $\beta_j(d_j) = d_jd_{j+1}^{-1}d_j^{-1}$, $\beta_j(d_{j+1}) = d_jd_{j+1}^2$ and the identity on the remaining canonical generators.

γ defined by $\gamma(d_1) = E^{-1}x_rEd_1$, $\gamma(x_r) = x_rEd_1E^{-1}x_r^{-1}Ed_1^{-1}E^{-1}x_r^{-1}$, where $E = e_1 \cdots e_k$ and the identity on the remaining canonical generators.

ε defined for $s_k = 0$ by $\varepsilon(d_1) = e_kd_1$, $\varepsilon(e_k) = e_kd_1e_k^{-1}d_1^{-1}e_k^{-1}$, $\varepsilon(c_{k,0}) = e_kd_1c_{k,0}d_1^{-1}e_k^{-1}$ and the identity on the remaining canonical generators.

η defined for $g \geq 4$ by

$$\eta(d_1) = d_1d_2d_3d_4d_1, \quad \eta(d_2) = (d_3d_4d_1)^{-1},$$
$$\eta(d_3) = d_3d_4d_1d_2d_3, \quad \eta(d_4) = (d_1d_2d_3)^{-1},$$

and $\eta(y) = (d_1d_2d_3d_4)y(d_1d_2d_3d_4)^{-1}$ for every canonical generator different from d_1, d_2, d_3, d_4.

Regardless of the sign in the signature, we are also going to use the following automorphisms.

ρ_i defined by $\rho_i(x_i) = x_ix_{i+1}x_i^{-1}$, $\rho_i(x_{i+1}) = x_i$ and the identity on the remaining canonical generators.

λ_i defined for $s_i = s_{i+1} = 0$ by $\lambda_i(e_i) = e_ie_{i+1}e_i^{-1}$, $\lambda_i(e_{i+1}) = e_i$, $\lambda_i(c_{i,0}) = e_ic_{i+1,0}e_i^{-1}$, $\lambda_i(c_{i+1,0}) = c_{i,0}$ and the identity on the remaining canonical generators.

THEOREM 1. *Suppose that p is an odd prime, Γ is an NEC group of signature $(g; -; [(p)^r]; \{-\})$ and $\theta_i \colon \Gamma \to C_p$ for $i = 1, 2$ are two epimorphisms with* $\ker \theta_i$ *isomorphic to the fundamental group of a non-orientable surface. Then θ_1 and θ_2 are topologically conjugate if and only if $(\theta_2(x_1), \ldots, \theta_2(x_r))$ is a permutation of $((\theta_1(x_1))^{\varepsilon_1 a}, \ldots, (\theta_1(x_r))^{\varepsilon_r a})$ for some $a \in \{1, \ldots, p-1\}$ and $\varepsilon_j \in \{-1, 1\}$, $j = 1, \ldots, r$.*

Proof. The 'only if' part follows from the fact that every automorphism of Γ maps x_i to a conjugate of some x_j or x_j^{-1}, see [**16**]. We are going to prove the 'if' part.

Suppose that $r > 0$. By using the automorphisms ρ_i, γ we may obtain a new system of canonical generators $x'_1, \ldots, x'_r, d'_1, \ldots, d'_g$ for which $\theta_2(x'_i) = (\theta_1(x'_i))^a$ for $i = 1, \ldots, r$. By using the automorphisms α_j we can permute the generators d'_j so that $\theta_1(d'_j) = 1$ for $j \geq m$ and $\theta_1(d'_j) \neq 1$ for $j < m$. Suppose that $m > 2$. There exists b such that $\theta_1(d'_1)(\theta(x'_r))^b = 1$. With the automorphism $(\gamma\alpha_1\gamma\alpha_1)^b$ we obtain a new system of generators $x''_1, \ldots, x''_r, d''_1, \ldots, d''_g$ such that $\theta_1(x''_i) = \theta_1(x'_i)$ for $i = 1, \ldots, r$, $\theta_1(d''_j) = 1$ for $j \geq m$ and $\theta_1(d''_1) = 1$. Reordering the d''_j, we have a new system of generators such that $\theta_1(d''_j) = 1$ for $j \geq m-1$. By repeating this process we can obtain a system of generators, where $\theta_1(d''_j) = 1$ for $j > 1$ and $\theta_1(d''_1)$ is determined by

$$\theta_1(x''_1 \cdots x''_r(d''_1)^2 \cdots (d''_g)^2) = 1.$$

https://doi.org/10.1017/S0017089514000275 Published online by Cambridge University Press

Analogously we can find a system of generators $x_1''', \dots, x_r''', d_1''', \dots, x_g'''$ such that $\theta_2(x_i''') = \theta_2(x_i')$ for $i = 1, \dots, r$ and $\theta_2(d_j''') = 1$ for $j > 1$. It follows that there exists $\psi \in Aut(\Gamma)$ and $\alpha \in Aut(C_p)$ such that $\theta_2\psi = \alpha\theta_1$.

Suppose that $r = 0$. By using the automorphisms α_j and β_j we can find a new system of generators $d_1', \dots, d_g'$ such that $\theta_1(d_i') = X$ for $i > 1$, where $X \neq 1$, and $\theta_1(d_1')$ is determined by the long relation (cf [**7**, proof of Theorem 3]). Since we can do the same for θ_2, there exist $\psi \in Aut(\Gamma)$ and $\alpha \in Aut(C_p)$ such that $\theta_2\psi = \alpha\theta_1$. □

The next result can be obtained from [**8**]. However, we give here a direct proof.

THEOREM 2. *Suppose that* Γ *is an NEC group of signature* $(g; \pm; [(2)^r]; \{(-)^k\})$ *and* $\theta_i \colon \Gamma \to C_2$ *for* $i = 1, 2$ *are two epimorphisms with* $\ker\theta_i$ *isomorphic to the fundamental group of a non-orientable surface. Let* X *be the generator of* C_2 *and*

$$n_i = \#\{j \in \{1, \dots, k\} \mid \theta_i(e_j) = X\} \quad \textit{for } i = 1, 2.$$

Then θ_1 *and* θ_2 *are topologically conjugate if and only if*

(1) $n_1 = n_2$*, and if* $r = n_1 = n_2 = 0$ *and the sign is '*$-$*' then*
(2) $\theta_1(d_1 \cdots d_g) = \theta_2(d_1 \cdots d_g)$ *and*
(3) $\theta_1(d_1) = \cdots = \theta_1(d_g) = 1$ *if and only if* $\theta_2(d_1) = \cdots = \theta_2(d_g) = 1$.

Proof. Let $\theta\colon \Gamma \to C_2$ be an epimorphism with $\ker\theta$ isomorphic to the fundamental group of a non-orientable surface, and $n = \#\{j \in \{1, \dots, k\} \mid \theta_i(e_j) = X\}$.

Case 1. $r + n > 0$.

We are going to show that in this case θ is determined by n up to an automorphism of Γ. Since $\ker\theta$ is torsion-free, we have $\theta(x_i) = X$ for $i = 1, \dots, r$ and $\theta(c_{j,0}) = X$ for $j = 1, \dots, k$. The automorphisms λ_i permute the boundary generators e_i up to conjugation, so we may assume that $\theta(e_i) = 1$ for $i \leq k - n$ and $\theta(e_i) = X$ for $i > k - n$.

Subcase 1(a) The sign of the signature is '$-$'. By using the automorphisms α_j and ε if $n > 0$ or γ if $r > 0$, we can obtain a new system of canonical generators such that $\theta(d_i) = 1$ for $i = 1, \dots, g$.

Subcase 1(b) The sign of the signature is '$+$'. Suppose that $\theta(a_i) = \theta(b_i) = 1$ for $i = 1, \dots, g$. Then by using the automorphism π if $n > 0$ or σ if $r > 0$, we obtain a new system of canonical generators such that $\theta(b_1) = X$. Therefore, we may assume that $\theta(a_i) = X$ or $\theta(b_i) = X$ for some i and by the automorphisms ν_i we may assume that $i = 1$. There exists $\psi \in Aut(\Gamma)$ such that $\theta(\psi(a_1)) = \theta(\psi(b_1)) = X$ and $\theta(\psi(y)) = \theta(y)$ for every canonical generator y different from a_1 and b_1. Indeed, if $\theta(a_1) = 1$ and $\theta(b_1) = X$, then $\psi = \omega$, whereas if $\theta(a_1) = X$ and $\theta(b_1) = 1$, then $\psi = \xi$. Therefore, we may assume that $\theta(a_i) = \theta(b_i) = X$ for $i \leq m$ and $\theta(a_i) = \theta(b_i) = 1$ for $i > m$ for some $m > 0$. Suppose that $m < g$. Then after reordering the generators by the automorphisms ν_i we may assume that $\theta(a_1) = \theta(b_1) = 1$ and $\theta(a_2) = \theta(b_2) = X$. Now for $\psi = \xi\mu$ we have $\theta(\psi(a_1)) = \theta(\psi(b_1)) = X$ and $\theta(\psi(y)) = \theta(y)$ for every canonical generator y different from a_1 and b_1. Repeating this process we arrive at a new system of canonical generators such that $\theta(a_i) = \theta(b_i) = X$ for $i = 1, \dots, g$.

Case 2. $r = n = 0$.

Subcase 2(a) The sign of the signature is '$+$'. Since $\mathcal{H}/\ker\theta$ is non-orientable, $\theta(a_i) = X$ or $\theta(b_i) = X$ for some i. Proceeding as in subcase 1(b), we can find a system of canonical generators such that $\theta(a_i) = \theta(b_i) = X$ for $i = 1, \dots, g$.

https://doi.org/10.1017/S0017089514000275 Published online by Cambridge University Press

Subcase 2(b) The sign of the signature is '$-$'. Let

$$m = \#\{j \in \{1, \dots, g\} \mid \theta_1(d_j) = X\}.$$

The condition $m = 0$ clearly determines θ. We are going to show that if $m > 0$, then θ is determined up to an automorphism of Γ by m mod 2. Since $\mathcal{H}/\ker\theta$ is non-orientable, $m < g$. Suppose that $m \geq 3$. Reordering the d_is we may assume that $\theta(d_1) = \theta(d_2) = \theta(d_3) = X$ and $\theta(d_4) = 1$. Then $\theta(\eta(d_1)) = \theta(\eta(d_2)) = \theta(\eta(d_3)) = 1$, $\theta(\eta(d_4)) = X$ and $\theta(\eta(y)) = \theta(y)$ for every canonical generator y different from d_1, d_2, d_3, d_4. Repeating this process we can find a system of canonical generators such that if m was odd then $\theta(d_1) = X$ and $\theta(d_i) = 1$ for $i \geq 2$, and if m was even then $\theta(d_1) = \theta(d_2) = X$ and $\theta(d_i) = 1$ for $i \geq 3$.

We have proved that the conditions (1)–(3) are sufficient for θ_1 and θ_2 to be topologically conjugate. Now we are going to prove that they are also necessary. The necessity of condition (1) follows from the fact that every automorphism of Γ maps e_i to a conjugate of some e_j or e_j^{-1}, see [**16**]. Suppose that the sign is '$-$', $r = n_1 = n_2 = 0$ and $\theta_2 = \theta_1\psi$ for some $\psi \in Aut(\Gamma)$. Let K be the smallest normal subgroup of Γ containing $e_1, \dots, e_k$ and the commutator subgroup $[\Gamma, \Gamma]$. Let $H = \Gamma/K$ and $\overline{c_i}, \overline{d_j} \in H$ be the projections of $c_{i,0}$, d_j respectively. As a $\mathbb{Z}$-module, H has the presentation

$$\langle \overline{c_1}, \dots, \overline{c_k}, \overline{d_1}, \dots, \overline{d_g} \mid 2\overline{c_1} = \cdots = 2\overline{c_k} = 2(\overline{d_1} + \cdots + \overline{d_g}) = 0\rangle.$$

Observe that $\psi(K) = K$ and $K \subseteq \ker\theta_i$. If $\psi^*\colon H \to H$, $\theta_i^*\colon H \to C_2$ are the induced maps, then $\theta_2^* = \theta_1^*\psi^*$. Let $h = \overline{d_1} + \cdots + \overline{d_g}$. Note that ψ^* permutes the $\overline{c_i}$'s and $\psi^*(h) = h + \overline{c_{i_1}} + \cdots + \overline{c_{i_{2s}}}$ for some $s \geq 0$ and $\{i_1, \dots, i_{2s}\} \subseteq \{1, \dots, k\}$. Thus, $\theta_2^*(h) = \theta_1^*\psi^*(h) = \theta_1^*(h)$. But $\theta_i^*(h) = \theta_i(d_1 \cdots d_g)$, and so we have proved the necessity of condition (2). To prove the necessity of condition (3), suppose that $\theta_1(d_1) = \cdots = \theta_1(d_g) = 1$ and $\theta_2(d_i) = X$ for some $i \in \{1, \dots, g\}$ and $\theta_2 = \theta_1\psi$ for some $\psi \in Aut(\Gamma)$. By condition (2) and proof of subcase 2(b), we may assume that $\theta_2(d_1) = \theta_2(d_2) = X$ and $\theta_2(d_i) = 1$ for $i \geq 3$. For $i = 1, 2$, let $H_i = \ker\theta_i/[\ker\theta_i, \ker\theta_i]$ (H_i is isomorphic to the first homology group with integral coefficients of the surface $\mathcal{H}/\ker\theta_i$) and let $p_i\colon \ker\theta_i \to H_i$ be the canonical projection. Let $\overline{e_i} = p_1(e_i)$ for $i = 1, \dots, k$, $\overline{c_l} = p_1(c_{k,0}c_{l,0})$ for $l = 1, \dots, k-1$, $\overline{d_j} = p_1(d_j)$, $\overline{f_j} = p_1(c_{k,0}d_jc_{k,0})$ for $j = 1, \dots, g$. A presentation for $\ker\theta_1$ may be obtained by the Reidemeister–Schreier procedure, then by taking abelianization we obtain that H_1 has a presentation (as a $\mathbb{Z}$-module) with generators $\overline{e_i}, \overline{c_l}, \overline{d_j}, \overline{f_j}$ satisfying the following two defining relations:

$$\begin{aligned} \overline{e_1} + \cdots + \overline{e_k} &= -2(\overline{d_1} + \cdots + \overline{d_g}), \\ 2(\overline{d_1} + \cdots + \overline{d_g} - (\overline{f_1} + \cdots + \overline{f_g})) &= 0. \end{aligned}$$

Let $\widetilde{e_i} = p_2(e_i)$ for $i = 1, \dots, k$, $\widetilde{c_l} = p_2(c_{k,0}c_{l,0})$ for $l = 1, \dots, k-1$, $\widetilde{d_j} = p_2(d_j)$, $\widetilde{f_j} = p_2(c_{k,0}d_jc_{k,0})$ for $j = 3, \dots, g$, $a_1 = p_2(d_1c_{k,0})$, $b_1 = p_2(c_{k,0}d_1)$, $a_2 = p_2(d_2c_{k,0})$, $b_2 = p_2(c_{k,0}d_2)$. Then H_2 has a presentation with generators $\widetilde{e_i}, \widetilde{c_l}, \widetilde{d_j}, \widetilde{f_j}, a_1, b_1, a_2, b_2$ satisfying the following two defining relations:

$$\begin{aligned} \widetilde{e_1} + \cdots + \widetilde{e_k} &= -(a_1 + b_1 + a_2 + b_2 + 2(\widetilde{d_3} + \cdots + \widetilde{d_g})), \\ 2(\widetilde{d_3} + \cdots + \widetilde{d_g} - (\widetilde{f_3} + \cdots + \widetilde{f_g})) &= 0. \end{aligned}$$

Table 1. Groups G and Γ for genus 4

G	Number	Γ
C_2	1	$(0, +, [2, 2, 2, 2], \{(-)\})$
	2	$(0, +, [2, 2], \{(-), (-)\})$
	3	$(0, +, [-], \{(-), (-), (-)\})$
	4	$(1, -, [2, 2, 2, 2], \{-\})$
	5	$(1, -, [2, 2], \{(-)\})$
	6	$(1, -, [-], \{(-), (-)\})$
	7	$(2, -, [2, 2], \{-\})$
	8	$(2, -, [-], \{(-)\})$
	9	$(3, -, [-], \{-\})$
	10	$(1, +, [-], \{(-)\})$
C_3		$(2, -, [3], \{-\})$
$C_2 \times C_2$	1	$(0, +, [2, 2, 2], \{(-)\})$
	2	$(0, +, [2, 2], \{(2, 2)\})$
	3	$(0, +, [2], \{(2, 2, 2, 2)\})$
	4	$(0, +, [-], \{(2, 2, 2, 2, 2, 2)\})$
	5	$(1, -, [2, 2, 2], \{-\})$
	6	$(1, -, [2], \{(-)\})$
	7	$(0, +, [2], \{(-)(-)\})$
D_3		$(0, +, [2, 2], \{(3)\})$
		$(0, +, [-], \{(-), (3)\})$
		$(1, -, [-], \{(3)\})$

Consider the groups $H_1' = H_1 \otimes C_2$, $H_2' = H_2 \otimes C_2$ (H_i' is isomorphic to the first homology group of $\mathcal{H}/\ker\theta_i$ with coefficients in C_2). For every $a \in H_i$, we denote by $[a]$ the element $a \otimes X \in H_i'$. Let $h_1 = [\overline{e_1} + \cdots + \overline{e_k}]$, $h_2 = [\widetilde{e_1} + \cdots + \widetilde{e_k}]$. Note that $h_1 = 0$ and $h_2 \neq 0$. On the other hand, we claim that $\psi^*(h_2) = h_1$, where $\psi^* : H_2' \to H_1'$ is the isomorphism induced by ψ. To see this, note that for every i, $\psi(e_i) = ye_j^{\pm 1}y^{-1}$ for some j and some $y \in \Gamma$. If $y \in \ker\theta_1$ then $\psi^*[\widetilde{e_i}] = [\overline{e_j}]$, and if $y \notin \ker\theta_1$ then $yc_{j,0} \in \ker\theta_1$ and since $\psi(e_i) = yc_{j,0}e_j^{\pm 1}(yc_{j,0})^{-1}$, we also have $\psi^*[\widetilde{e_i}] = [\overline{e_j}]$. Thus, $\psi^*(h_2) = h_1$, which is a contradiction proving that θ_1 and θ_2 are not topologically conjugate and the condition (3) is necessary. □

4. Surfaces of topological genus 4. As seen in Section 2.2, we need only to study the cyclic automorphism groups C_p for p prime. From **[6]** for $g = 4$, the only such primes p are 2 and 3. According to this we are going to consider the groups C_2, C_3, $C_2 \times C_2$ and D_3. For each one of these groups, say G, we take from **[9]** the list of groups Γ such that there exists an epimorphism $\theta : \Gamma \to G$ with $\ker(\theta) = \Gamma_4$. We give this list in Table 1.

First we consider the 10 signatures of Γ corresponding to group C_2. We are going to define all epimorphisms $\theta : \Gamma \to C_2$ with kernel Γ_4, and to determine by using Theorem 2 whether two epimorphisms from the same NEC group are topologically conjugate.

Signature 1: $(0, +, [2, 2, 2, 2], \{(-)\})$.

The unique epimorphism is given by

$$\theta : (x_1, x_2, x_3, x_4, e_1, c_{1,0}) \to (X, X, X, X, 1, X).$$

Signature 2: $(0, +, [2, 2], \{(-), (-)\})$.

https://doi.org/10.1017/S0017089514000275 Published online by Cambridge University Press

There are two possible epimorphisms:

$$\theta_1 : (x_1, x_2, e_1, e_2, c_{1,0}, c_{2,0}) \to (X, X, 1, 1, X, X),$$
$$\theta_2 : (x_1, x_2, e_1, e_2, c_{1,0}, c_{2,0}) \to (X, X, X, X, X, X).$$

By Theorem 2, the epimorphisms θ_1 and θ_2 are non-conjugate because $n_1 = 0$ and $n_2 = 2$. We call them cases 2.1 and 2.2.

Signature 3: $(0, +, [-], \{(-), (-), (-)\})$.

The unique epimorphism is given by

$$\theta : (e_1, e_2, e_3, c_{1,0}, c_{2,0}, c_{3,0}) \to (X, X, 1, X, X, X).$$

Signature 4: $(1, -, [2, 2, 2, 2], \{-\})$.

The epimorphism is given by

$$\theta : (d_1, x_1, x_2, x_3, x_4) \to (1 \text{ or } X, X, X, X, X),$$

where the image of d_1 is irrelevant according to Theorem 2 since $r \neq 0$.

Signature 5: $(1, -, [2, 2], \{(-)\})$.

The epimorphism is given by

$$\theta : (d_1, x_1, x_2, e_1, c_{1,0}) \to (1 \text{ or } X, X, X, 1, X),$$

again the image of d_1 is irrelevant.

Signature 6: $(1, -, [-], \{(-), (-)\})$.

There are two kinds of epimorphisms given by

$$\theta_1 : (d_1, e_1, e_2, c_{1,0}, c_{2,0}) \to (1 \text{ or } X, X, X, X, X),$$
$$\theta_2 : (d_1, e_1, e_2, c_{1,0}, c_{2,0}) \to (1, 1, 1, X, X).$$

The choice of $\theta_1(d_1)$ is irrelevant because $n_1 \neq 0$. On the other hand, θ_2 is not topologically conjugate to θ_1 because $n_1 \neq n_2$. We denote them as cases 6.1 and 6.2.

Signature 7: $(2, -, [2, 2], \{-\})$.

The epimorphism is given by

$$\theta : (d_1, d_2, x_1, x_2) \to (1 \text{ or } X, 1 \text{ or } X, X, X),$$

and as above the choice for the images of d_1 and d_2 are irrelevant.

Signature 8: $(2, -, [-], \{(-)\})$.

The epimorphism is given by

$$\theta : (d_1, d_2, e_1, c_{1,0}) \to (1, 1 \text{ or } X,\ 1, X),$$

where the choice for the image of d_2 is irrelevant.

Signature 9: $(3, -, [-], \{-\})$.

There are three possible epimorphisms given by

$$\theta_1 : (d_1, d_2, d_3) \to (X, 1, 1),$$
$$\theta_2 : (d_1, d_2, d_3) \to (X, X, X),$$
$$\theta_3 : (d_1, d_2, d_3) \to (X, 1, X).$$

We can see that $\theta_1(d_1d_2d_3) = \theta_2(d_1d_2d_3) = X$, while $\theta_3(d_1d_2d_3) = 1$. Hence, by Theorem 2, keeping in mind that $r = k = 0$, there are two kinds of epimorphisms whose representatives are θ_1 and θ_3. We denote them as cases 9.1 and 9.2.

Signature 10: $(1, +, [-], \{(-)\})$.

The epimorphism is given by

$$\theta : (a_1, b_1, e_1, c_{1,0}) \to (X, 1 \text{ or } X,\ 1, X),$$

where again the choice for the image of b_1 is irrelevant.

So we have 13 different cases denoted by 1, 2.1, 2.2, 3, 4, 5, 6.1, 6.2, 7, 8, 9.1, 9.2 and 10. For each of these there exists an epimorphism $\theta : \Gamma \to C_2$ with $\ker(\theta) = \Gamma_4$. According to the construction at the end of the Section 2.2, we are going to look for epimorphisms $\theta^* : \Gamma^* \to C_2 \times C_2$ which connect any pair of epimorphisms.

We have in Table 1 the possible signatures for Γ^*. We call Γ_i^* the NEC group with signature labeled i in Table 1. We shall run through these signatures until we connect the 13 cases. We exhibit the epimorphism θ^* in each case and distinguish carefully the situations 2.1 and 2.2, 6.1 and 6.2 and 9.1 and 9.2.

We denote the generators of Γ^* with the corresponding starred letters. The generators of $C_2 \times C_2$ are denoted by X and Y.

First we take the group Γ_1^* with signature $(0, +, [2, 2, 2], \{(-)\})$. Define the epimorphism $\theta^* : \Gamma_1^* \to C_2 \times C_2$ by

$$(x_1^*, x_2^*, x_3^*, e_1^*, c_{1,0}^*) \to (X, X, Y, Y, X).$$

Then ${\theta^*}^{-1}(\langle X\rangle)$ is an NEC group with signature 1, ${\theta^*}^{-1}(\langle Y\rangle)$ is an NEC group with signature 7 and ${\theta^*}^{-1}(\langle XY\rangle)$ is an NEC group with signature 9. For the last one we must check whether we are in case 9.1 or 9.2. The generators of group ${\theta^*}^{-1}(\langle XY\rangle)$ with signature $(3, -, [-], \{-\})$ can be expressed in terms of the generators of Γ_1^* as

$$\begin{aligned} d_1 &= x_1^* c_{1,0}^*, \\ d_2 &= c_{1,0}^* x_2^*, \\ d_3 &= x_2^* c_{1,0}^* x_3^* x_2^*. \end{aligned}$$

Hence, $\theta^*(d_1) = 1$, $\theta^*(d_2) = 1$ and $\theta^*(d_3) = XY$. So this is case 9.1. As a consequence, cases 1, 7 and 9.1 are connected,

$$\boxed{1} - \boxed{7} - \boxed{9.1}$$

Now we deal with the group Γ_2^* with signature $(0, +, [2, 2], \{(2, 2)\})$. Define the epimorphism $\theta^* : \Gamma_2^* \to C_2 \times C_2$ by

$$(x_1^*, x_2^*, e_1^*, c_{1,0}^*, c_{1,1}^*, c_{1,2}^*) \to (X,\ Y,\ XY,\ X,\ Y,\ X).$$

Then ${\theta^*}^{-1}(\langle X\rangle)$ is an NEC group with signature 5, and ${\theta^*}^{-1}(\langle XY\rangle)$ is an NEC group with signature 7. Hence, case 5 is also connected with the previous list,

$$\boxed{1} - \boxed{7} - \boxed{9.1} - \boxed{5}$$

We now define a new epimorphism from the same group Γ_2^* by

$$(x_1^*, x_2^*, e_1^*, c_{1,0}^*, c_{1,1}^*, c_{1,2}^*) \to (X, X, 1, X,\ Y, X).$$

https://doi.org/10.1017/S0017089514000275 Published online by Cambridge University Press

For this epimorphism we have that ${\theta^*}^{-1}(\langle X\rangle)$ is an NEC group with signature 1, and ${\theta^*}^{-1}(\langle Y\rangle)$ is an NEC group with signature 8. Case 8 is connected with the previous ones,

$$\boxed{1} - \boxed{7} - \boxed{9.1} - \boxed{5} - \boxed{8}$$

Let Γ_3^* be an NEC group with signature $(0, +, [2], \{(2, 2, 2, 2\})$. Define $\theta^* : \Gamma_3^* \to C_2 \times C_2$ by

$$(x_1^*, e_1^*, c_{1,0}^*, c_{1,1}^*, c_{1,2}^*, c_{1,3}^*, c_{1,4}^*) \to (X, X, X, Y, X, Y, X).$$

Then ${\theta^*}^{-1}(\langle X\rangle)$ is an NEC group with signature 2, ${\theta^*}^{-1}(\langle Y\rangle)$ is an NEC group with signature 6 and ${\theta^*}^{-1}(\langle XY\rangle)$ is an NEC group with signature 4. We must distinguish which is the respective case for signatures 2 and 6.

We express the generators of ${\theta^*}^{-1}(\langle X\rangle)$ in terms of the generators of Γ_3^* as

$$\begin{aligned} x_1 &= x_1^*, \\ x_2 &= c_{1,3}^* x_1^* c_{1,3}^*, \\ e_1 &= c_{1,3}^* c_{1,1}^*, \\ e_2 &= c_{1,1}^* x_1^* c_{1,3}^* x_1^*, \\ c_{1,0} &= c_{1,2}^*, \\ c_{2,0} &= c_{1,0}^*. \end{aligned}$$

Then $\theta^*(e_1) = 1$ and $\theta^*(e_2) = 1$. Hence, this is case 2.1.

We now deal with ${\theta^*}^{-1}(\langle Y\rangle)$. Its generators can be expressed as

$$\begin{aligned} d_1 &= e_1^* c_{1,0}^*, \\ e_1 &= c_{1,0}^* c_{1,2}^*, \\ e_2 &= c_{1,2}^* c_{1,4}^*, \\ c_{1,0} &= c_{1,1}^*, \\ c_{2,0} &= c_{1,3}^*. \end{aligned}$$

Then $\theta^*(e_1) = 1$, $\theta^*(e_2) = 1$ and so this is case 6.2. So we have seen that cases 2.1, 4 and 6.2 are connected. We have up to now

$$\boxed{1} - \boxed{7} - \boxed{9.1} - \boxed{5} - \boxed{8}$$

and

$$\boxed{2.1} - \boxed{4} - \boxed{6.2}$$

We now define a new epimorphism from the same group Γ^* by

$$(x_1^*, e_1^*, c_{1,0}^*, c_{1,1}^*, c_{1,2}^*, c_{1,3}^*, c_{1,4}^*) \to (X, X, X, Y, XY, Y, X).$$

Then ${\theta^*}^{-1}(\langle X\rangle)$ has signature 1, and ${\theta^*}^{-1}(\langle Y\rangle)$ has signature 6. We now check whether this is case 6.1 or 6.2. Since $\theta^*(e_1) = \theta^*(e_2) = Y$, this is case 6.1. Hence, this case is connected with case 1,

$$\boxed{1} - \boxed{7} - \boxed{9.1} - \boxed{5} - \boxed{8} - \boxed{6.1}$$

Take now Γ_4^* with signature $(0, +, [-], \{(2, 2, 2, 2, 2, 2)\})$. Define $\theta^* : \Gamma_4^* \to C_2 \times C_2$ by

$$(c_{1,0}^*, c_{1,1}^*, c_{1,2}^*, c_{1,3}^*, c_{1,4}^*, c_{1,5}^*, c_{1,6}^*) \to (X, XY, X, Y, X, Y, X).$$

Then ${\theta^*}^{-1}(\langle X\rangle)$ has signature 3, and ${\theta^*}^{-1}(\langle XY\rangle)$ has signature 1 so that case 3 is connected with 1,

$$\boxed{1} - \boxed{7} - \boxed{9.1} - \boxed{5} - \boxed{8} - \boxed{6.1} - \boxed{3}$$

We now define another epimorphism $\theta^* : \Gamma_4^* \to C_2 \times C_2$ by

$$(c_{1,0}^*, c_{1,1}^*, c_{1,2}^*, c_{1,3}^*, c_{1,4}^*, c_{1,5}^*, c_{1,6}^*) \to (X, XY, X, Y, XY, Y, X).$$

Then both ${\theta^*}^{-1}(\langle Y\rangle)$ and ${\theta^*}^{-1}(\langle XY\rangle)$ have signature 2. Consider first ${\theta^*}^{-1}(\langle Y\rangle)$. Its generators can be expressed by

$$\begin{aligned}
x_1 &= c_{1,0}^* c_{1,1}^*,\\
x_2 &= c_{1,1}^* c_{1,2}^*,\\
e_1 &= c_{1,2}^* c_{1,4}^*,\\
e_2 &= c_{1,4}^* c_{1,6}^*,\\
c_{1,0} &= c_{1,3}^*,\\
c_{2,0} &= c_{1,5}^*.
\end{aligned}$$

Then $\theta^*(e_1) = \theta^*(e_2) = Y$ and so this is case 2.2.

Take now ${\theta^*}^{-1}(\langle XY\rangle)$. Its generators are expressed by

$$\begin{aligned}
x_1 &= c_{1,5}^* c_{1,3}^* c_{1,2}^* c_{1,5}^*,\\
x_2 &= c_{1,5}^* c_{1,6}^*,\\
e_1 &= c_{1,0}^* c_{1,2}^*,\\
e_2 &= c_{1,3}^* c_{1,5}^*,\\
c_{1,0} &= c_{1,1}^*,\\
c_{2,0} &= c_{1,4}^*.
\end{aligned}$$

Hence, $\theta^*(e_1) = \theta^*(e_2) = 1$, and this is case 2.1 so that both cases are connected,

$$\boxed{2.1} - \boxed{4} - \boxed{6.2} - \boxed{2.2}$$

Now we have the group Γ_7^* with signature $(0, +, [2], \{(-), (-)\})$ and define $\theta^* : \Gamma_7^* \to C_2 \times C_2$ by

$$(x_1^*, e_1^*, e_2^*, c_{1,0}^*, c_{2,0}^*) \to (Y, 1, Y, Y, X).$$

https://doi.org/10.1017/S0017089514000275 Published online by Cambridge University Press

Then ${\theta^*}^{-1}(\langle X\rangle)$ has signature 10, ${\theta^*}^{-1}(\langle Y\rangle)$ has signature 2 and finally ${\theta^*}^{-1}(\langle XY\rangle)$ has signature 9.

We are going to express the generators of ${\theta^*}^{-1}(\langle Y\rangle)$ in terms of the generators of Γ_7^* by

$$\begin{aligned}
x_1 &= x_1^*,\\
x_2 &= c_{2,0}^* x_1^* c_{2,0}^*,\\
e_1 &= c_{2,0}^* e_1^* c_{2,0}^*,\\
e_2 &= {e_1^*}^{-1},\\
c_{1,0} &= c_{2,0}^*,\\
c_{2,0} &= c_{1,0}^*.
\end{aligned}$$

Hence, $\theta^*(e_1) = \theta^*(e_2) = 1$, so this is case 2.1.

We now deal with ${\theta^*}^{-1}(\langle XY\rangle)$. Its generators can be expressed as

$$\begin{aligned}
d_1 &= x_1^* c_{1,0}^*,\\
d_2 &= c_{1,0}^* x_1^* e_1^*,\\
d_3 &= e_2^* c_{2,0}^*.
\end{aligned}$$

Then $\theta^*(d_1) = \theta^*(d_2) = 1$ and $\theta^*(d_3) = XY$. So this is case 9.1. Thus, cases 9.1 and 10 are connected with 2.1,

$$\begin{tikzcd}
\boxed{1} \arrow[r, dash] & \boxed{7} \arrow[r, dash] & \boxed{9.1} \arrow[r, dash] \arrow[d, dash] & \boxed{5} \arrow[r, dash] & \boxed{8} \arrow[r, dash] & \boxed{6.1} \arrow[r, dash] & \boxed{3} \\
 & & \boxed{2.1} \arrow[r, dash] \arrow[d, dash] & \boxed{4} \arrow[r, dash] & \boxed{6.2} \arrow[r, dash] & \boxed{2.2} & \\
 & & \boxed{10} & & & &
\end{tikzcd}$$

Up to now all cases are connected except for 9.2. We are dealing with this last case. We define a new epimorphism $\theta^* : \Gamma_7^* \to C_2 \times C_2$ by

$$(x_1, e_1, e_2, c_{1,0}, c_{2,0}) \to (Y, XY, X, Y, Y).$$

Then ${\theta^*}^{-1}(\langle Y\rangle)$ has signature 2 and ${\theta^*}^{-1}(\langle XY\rangle)$ has signature 9. We must decide if we have the case 9.1 or 9.2. The generators of the group ${\theta^*}^{-1}(\langle XY\rangle)$ can be expressed as above. Then $\theta^*(d_1) = 1$, $\theta^*(d_2) = \theta^*(d_3) = XY$, and so this is case 9.2. Hence, case 9.2 is connected with one of the cases of signature 2 and we are done. We have finished to check that of all cases appearing for the prime $p = 2$ are connected.

The following step deals with the other prime $p = 3$. According to Table 1, the unique signature is $(2, -, [3], \{-\})$. We denote by Γ a group with this signature. We are going to see that there are two epimorphisms $\theta : \Gamma \to C_3$ with kernel Γ_4 and check that they are conjugate by Theorem 1. Call X the generator of C_3. The epimorphisms

Table 2. Groups G and Γ for genus 5

G	Number	Γ
C_2	1	$(0, +, [2, 2, 2, 2, 2], \{(-)\})$
	2	$(0, +, [2, 2, 2], \{(-), (-)\})$
	3	$(0, +, [2], \{(-), (-), (-)\})$
	4	$(1, -, [2, 2, 2], \{(-)\})$
	5	$(1, -, [2], \{(-), (-)\})$
	6	$(2, -, [2], \{(-)\})$
	7	$(1, +, [2], \{(-)\})$
C_3	1	$(1, -, [3, 3, 3], \{-\})$
	2	$(3, -, [-], \{-\})$
$C_2 \times C_2$	1	$(0, +, [2, 2], \{(2, 2, 2)\})$
	2	$(0, +, [2], \{(2, 2, 2, 2, 2)\})$
	3	$(0, +, [-], \{(2, 2, 2, 2, 2, 2, 2)\})$
	4	$(0, +, [-], \{(2, 2, 2), (-)\})$
	5	$(1, -, [-], \{(2, 2, 2)\})$
C_5		$(1, -, [5, 5], \{-\})$
D_3	1	$(0, +, [2, 3], \{(3)\})$
	2	$(0, +, [2], \{(3, 3, 3)\})$
	3	$(0, +, [2, 2, 2], \{(-)\})$
	4	$(0, +, [2], \{(-), (-)\})$
	5	$(1, -, [2], \{(-)\})$
D_5	1	$(0, +, [2, 5], \{(-)\})$
	2	$(0, +, [2], \{(5, 5)\})$

are defined by

$$(d_1, d_2, x_1) \to (X, 1, X) \quad \text{or} \quad (d_1, d_2, x_1) \to (X^2, X^2, X)$$

and because the images of x_1 are the same, both epimorphisms are conjugate.

By [**9**], the action of Γ extends to a group Γ^* such that there exists an epimorphism $\theta^* : \Gamma^* \to D_3$ with kernel Γ_4. Let $X, Y \in D_3$ have orders 2 and 3. Then ${\theta^*}^{-1}(\langle X\rangle)$ is one of the above studied groups corresponding to C_2, while ${\theta^*}^{-1}(\langle Y\rangle)$ is Γ. So the unique case corresponding to C_3 is connected to one of the cases of C_2, and then to all of them. We have finished to prove the following.

THEOREM 3. *The space B_4 is connected.*

5. Surfaces of topological genus 5. We are going to consider the groups C_p for p prime acting on surfaces of genus $g = 5$. From [**6**], the values of p are 2, 3 or 5. Hence, we consider the groups C_2, C_3, C_5, $C_2 \times C_2$, D_3 and D_5. For each of these groups, say G, the list of groups Γ such that there exists an epimorphism $\theta : \Gamma \to G$ with $\ker(\theta) = \Gamma_5$ is taken from [**9**] and collected in Table 2.

We are going to consider the seven signatures of Γ corresponding to the group C_2. We determine all epimorphisms $\theta : \Gamma \to C_2$ with kernel Γ_5 which are topologically non-conjugate. As in the previous section, we call X the generator of C_2.

Signature 1: $(0, +, [2, 2, 2, 2, 2], \{(-)\})$.

The unique epimorphism is given by

$$\theta : (x_1, x_2, x_3, x_4, x_5, e_1, c_{1,0}) \to (X, X, X, X, X, X, X).$$

Signature 2: $(0, +, [2, 2, 2], \{(-), (-)\})$.

https://doi.org/10.1017/S0017089514000275 Published online by Cambridge University Press

The unique epimorphism is given by

$$\theta : (x_1, x_2, x_3, e_1, e_2, c_{1,0}, c_{2,0}) \to (X, X, X, X, 1, X, X).$$

Signature 3: $(0, +, [2], \{(-), (-), (-)\})$.
There are two possible epimorphisms given by

$$\theta_1 : (x_1, e_1, e_2, e_3, c_{1,0}, c_{2,0}, c_{3,0}) \to (X, X, X, X, X, X, X),$$
$$\theta_2 : (x_1, e_1, e_2, e_3, c_{1,0}, c_{2,0}, c_{3,0}) \to (X, X, 1, 1, X, X, X).$$

By Theorem 2, the epimorphisms θ_1 and θ_2 are non-conjugate because $n_1 = 2$ and $n_2 = 0$. We call these cases 3.1 and 3.2.

Signature 4: $(1, -, [2, 2, 2], \{(-)\})$.
The unique epimorphism is given by

$$\theta : (d_1, x_1, x_2, x_3, e_1, c_{1,0}) \to (1 \text{ or } X, X, X, X, X, X).$$

The choice of the image of d_1 is irrelevant according to Theorem 2 because $r \neq 0$.

Signature 5: $(1, -, [2], \{(-), (-)\})$.
The unique epimorphism is given by

$$\theta : (d_1, x_1, e_1, e_2, c_{1,0}, c_{2,0}) \to (1 \text{ or } X, X, X, 1, X, X).$$

Again, the choice of the image of d_1 is irrelevant.

Signature 6: $(2, -, [2], \{(-)\})$.
The unique epimorphism is given by

$$\theta : (d_1, d_2, x_1, e_1, c_{1,0}) \to (1 \text{ or } X, 1 \text{ or } X, X, X, X).$$

As in the previous cases, the choice of the images of d_1 and d_2 are irrelevant.

Signature 7: $(1, +, [2], \{(-)\})$.
The unique epimorphism is given by

$$\theta : (a_1, b_1, x_1, e_1, c_{1,0}) \to (1 \text{ or } X, 1 \text{ or } X, X, X, X).$$

Again, the choice of the images of a_1 and b_1 are irrelevant.

Hence, we have eight different cases denoted by 1, 2, 3.1, 3.2, 4, 5, 6 and 7. For each of these there exists an epimorphism $\theta : \Gamma \to C_2$ with $\ker(\theta) = \Gamma_5$. We are going to look for epimorphisms $\theta^* : \Gamma^* \to C_2 \times C_2$ which connect all epimorphisms θ.

The possible signatures for Γ^* appear in Table 2. We call Γ_i^* the NEC group with signature labeled i in Table 2 for the group $C_2 \times C_2$. We denote the generators of Γ^* with the corresponding starred letters, and call X and Y the generators of $C_2 \times C_2$.

First we take the group Γ_1^* with signature $(0, +, [2, 2], \{(2, 2, 2)\})$. Define an epimorphism $\theta^* : \Gamma_1^* \to C_2 \times C_2$ by

$$(x_1^*, x_2^*, e_1^*, c_{1,0}^*, c_{1,1}^*, c_{1,2}^*, c_{1,3}^*) \to (X, X, 1, X, Y, XY, X).$$

Then $\theta^{*^{-1}}(\langle X \rangle)$ is an NEC group with signature 1, and $\theta^{*^{-1}}(\langle Y \rangle)$ is an NEC group with signature 6. Hence, cases 1 and 6 are connected,

$$\boxed{1} - \boxed{6}$$

https://doi.org/10.1017/S0017089514000275 Published online by Cambridge University Press

We define a different epimorphism from the same group Γ_1^* by

$$(x_1^*, x_2^*, e_1^*, c_{1,0}^*, c_{1,1}^*, c_{1,2}^*, c_{1,3}^*) \to (X, Y, XY, X, Y, XY, X).$$

For this epimorphism we have that $\theta^{*^{-1}}(\langle X\rangle)$ is an NEC group with signature 4, and $\theta^{*^{-1}}(\langle XY\rangle)$ is an NEC group with signature 6. Hence, case 4 is connected with the previous ones,

$$\boxed{1} - \boxed{6} - \boxed{4}$$

Now we take Γ_2^* as an NEC group with signature $(0, +, [2], \{(2, 2, 2, 2, 2)\})$. Define $\theta^* : \Gamma_2 \to C_2 \times C_2$ by

$$(x_1^*, e_1^*, c_{1,0}^*, c_{1,1}^*, c_{1,2}^*, c_{1,3}^*, c_{1,4}^*, c_{1,5}^*) \to (X, X, Y, X, Y, XY, X, Y).$$

Then $\theta^{*^{-1}}(\langle X\rangle)$ is an NEC group with signature 2, $\theta^{*^{-1}}(\langle Y\rangle)$ is an NEC group with signature 5 and $\theta^{*^{-1}}(\langle XY\rangle)$ is an NEC group with signature 4. Hence, cases 2 and 5 are connected with the three previous ones,

$$\boxed{1} - \boxed{6} - \boxed{4} - \boxed{2} - \boxed{5}$$

Now, let Γ_3^* be an NEC group with signature $(0, +, [-], \{(2, 2, 2, 2, 2, 2, 2)\})$. Define an epimorphism θ^* from this group onto $C_2 \times C_2$ by

$$(c_{1,0}^*, c_{1,1}^*, c_{1,2}^*, c_{1,3}^*, c_{1,4}^*, c_{1,5}^*, c_{1,6}^*, c_{1,7}^*) \to (X, XY, X, XY, Y, X, Y, X).$$

Then $\theta^{*^{-1}}(\langle X\rangle)$ is an NEC group with signature 3, and $\theta^{*^{-1}}(\langle Y\rangle)$ is an NEC group with signature 2. We must distinguish whether this is case 3.1 or 3.2. We express the generators of $\theta^{*^{-1}}(\langle X\rangle)$ as follows:

$$\begin{aligned}
x_1 &= c_{1,3}^* c_{1,4}^*,\\
e_1 &= c_{1,4}^* c_{1,6}^*,\\
e_2 &= c_{1,6}^* c_{1,1}^*,\\
e_3 &= c_{1,1}^* c_{1,3}^*,\\
c_{1,0} &= c_{1,5}^*,\\
c_{2,0} &= c_{1,7}^*,\\
c_{3,0} &= c_{1,2}^*.
\end{aligned}$$

Then $\theta^*(e_1) = 1$, $\theta^*(e_2) = X$ and $\theta^*(e_3) = 1$. So this is case 3.2 and is connected with case 2,

$$\boxed{1} - \boxed{6} - \boxed{4} - \boxed{2} - \boxed{5} - \boxed{3.2}$$

We now consider an NEC group Γ_4^* with signature $(0, +, [-], \{(2, 2, 2), (-)\})$. Define an epimorphism θ^* from this group onto $C_2 \times C_2$ by

$$(e_1^*, e_2^*, c_{1,0}^*, c_{1,1}^*, c_{1,2}^*, c_{1,3}^*, c_{2,0}^*) \to (X, X, X, XY, Y, X, X).$$

Then $\theta^{*^{-1}}(\langle X\rangle)$ is an NEC group with signature 3, and $\theta^{*^{-1}}(\langle Y\rangle)$ is an NEC group with signature 7. We must distinguish whether the former is case 3.1 or 3.2. We express the generators of $\theta^{*^{-1}}(\langle X\rangle)$ as follows:

$$\begin{aligned}
x_1 &= c^*_{1,2}c^*_{1,1},\\
e_1 &= c^*_{1,1}e^*_1c^*_{1,2}e_1^{*^{-1}},\\
e_2 &= e_2^{*^{-1}},\\
e_3 &= c^*_{1,2}e^*_2c^*_{1,2},\\
c_{1,0} &= c^*_{1,0},\\
c_{2,0} &= c^*_{2,0},\\
c_{3,0} &= c^*_{1,2}c^*_{2,0}c^*_{1,2}.
\end{aligned}$$

Then $\theta^*(e_1) = \theta^*(e_2) = \theta^*(e_3) = X$. So this is case 3.1 and is connected with Case 7,

$$\boxed{7} - \boxed{3.1}$$

Take an NEC group Γ^*_5 with signature $(1, -, [-], \{(2, 2, 2)\})$. Define $\theta^* : \Gamma^*_5 \to C_2 \times C_2$ by

$$(d^*_1, e^*_1, c^*_{1,0}, c^*_{1,1}, c^*_{1,2}, c^*_{1,3}) \to (XY, 1, X, Y, XY, X).$$

Then $\theta^{*^{-1}}(\langle X\rangle)$ is an NEC group with signature 7 and $\theta^{*^{-1}}(\langle XY\rangle)$ is an NEC group with signature 6. Hence, case 7 (and also case 3.1) is connected with the previous ones,

$$\boxed{1} - \boxed{6} - \boxed{4} - \boxed{2} - \boxed{5} - \boxed{3.2} - \boxed{7} - \boxed{3.1}$$

We have proven that all cases for prime $p = 2$ are connected.

We now deal with the prime $p = 3$. We denote by X the generator of C_3. According to Table 2 we must consider two signatures.

Signature 1: $(1, -, [3, 3, 3], \{-\})$.

There are three possible epimorphisms given by

$$\begin{aligned}
\theta_1 &: (d_1, x_1, x_2, x_3) \to (1, X, X, X),\\
\theta_2 &: (d_1, x_1, x_2, x_3) \to (X, X, X, X^2),\\
\theta_3 &: (d_1, x_1, x_2, x_3) \to (X^2, X, X^2, X^2).
\end{aligned}$$

In order to apply Theorem 1, observe that the respective images of the elliptic elements are $\{X, X, X\}$, $\{X, X, X^2\}$ and $\{X, X^2, X^2\}$. Since $X^2 = X^{-1}$, the epimorphism θ_1 is topologically conjugate to both θ_2 and θ_3, and so these are all equivalent.

Signature 2: $(3, -, [-], \{-\})$.

There are two possible epimorphisms given by

$$\begin{aligned}
\theta_1 &: (d_1, d_2, d_3) \to (X, X, X),\\
\theta_2 &: (d_1, d_2, d_3) \to (1, X, X^2).
\end{aligned}$$

Since $r = 0$, both of these are topologically conjugate by Theorem 1.

We are going to extend both actions to D_3, and so these two cases will be connected to those corresponding to $p = 2$. For that we consider an NEC group Γ_2^* with signature $(0, +, [2], \{(3, 3, 3)\})$, which is the second signature in Table 2 corresponding to group D_3. We denote by X and Y the generators of D_3 satisfying $X^2 = Y^2 = (XY)^3 = 1$. We define an epimorphism $\theta^* : \Gamma_2^* \to D_3$ by

$$(x_1^*, e_1^*, c_{1,0}^*, c_{1,1}^*, c_{1,2}^*, c_{1,3}^*) \to (Y, Y, X, Y, X, YXY).$$

Then ${\theta^*}^{-1}(\langle XY \rangle)$ has signature 1, and ${\theta^*}^{-1}(\langle X \rangle)$ is a group corresponding to C_2. So case 1 is connected with a case of C_2 and so with all of them.

We are now going to make the same with signature 2. Call Γ_5^* an NEC group with signature $(1, -, [2], \{(-)\})$, the fifth signature in Table 2 corresponding to D_3. Define $\theta^* : \Gamma_5^* \to D_3$ by

$$(d_1^*, x_1^*, e_1^*, c_{1,0}^*) \to (XY, X, Y, Y).$$

Then ${\theta^*}^{-1}(\langle XY \rangle)$ has signature 2 and ${\theta^*}^{-1}(\langle X \rangle)$ is a group corresponding to C_2. So case 2 is connected with a case of C_2 and so with all of them.

Finally, we deal with the prime $p = 5$. There exists a unique signature for group C_5 in Table 2, which is $(1, -, [5, 5], \{-\})$. Call Γ an NEC group with this signature. Applying Theorem 1 as in the previous case, we see that there are two kinds of topologically non-conjugate epimorphisms from Γ onto C_5 represented by

$$\theta_1 : (d_1, x_1, x_2) \to (X^4, X, X),$$
$$\theta_2 : (d_1, x_1, x_2) \to (X, X, X^2),$$

where X denotes a generator of C_5. In order to keep the same notation, we call them cases 1.1 and 1.2. We are going to connect both cases to those of C_2 by extending the action to D_5. As usual we call X and Y the generators of D_5 satisfying $X^2 = Y^2 = (XY)^5 = 1$.

First, let Γ_1^* be an NEC group with signature $(0, +, [2, 5], \{(-)\})$, the first signature for D_5 in Table 2. Define $\theta^* : \Gamma_1^* \to D_5$ by

$$(x_1^*, x_2^*, e_1^*, c_{1,0}^*) \to (X, XY, Y, Y).$$

We consider the group ${\theta^*}^{-1}(\langle XY \rangle)$. Its generators can be expressed in terms of the generators of Γ_1^* by

$$\begin{aligned} d_1 &= x_1^* c_{1,0}^*, \\ x_1 &= c_{1,0}^* x_2^* c_{1,0}^*, \\ x_2 &= x_2^*. \end{aligned}$$

Then $\theta^*(x_1) = YX = (XY)^{-1}$ and $\theta^*(x_2) = XY$, hence by applying Theorem 1, this epimorphism is topologically conjugate to θ_1 and this is case 1.1. Since ${\theta^*}^{-1}(\langle X \rangle)$ is an NEC group corresponding to C_2, case 1.1 is connected with those corresponding to C_2.

https://doi.org/10.1017/S0017089514000275 Published online by Cambridge University Press

Take now Γ_2^* an NEC group with signature $(0, +, [2], \{(5, 5)\})$, the second signature for D_5 in Table 2. Define $\theta^* : \Gamma_2^* \to D_5$ by

$$(x_1^*, e_1^*, c_{1,0}^*, c_{1,1}^*, c_{1,2}^*) \to (XYXYX, XYXYX, X, XYX, Y).$$

We consider the group $\theta^{*^{-1}}(\langle XY \rangle)$. Its generators can be expressed in terms of the generators of Γ_2^* by

$$\begin{aligned} d_1 &= x_1^* c_{1,0}^*, \\ x_1 &= c_{1,0}^* c_{1,1}^*, \\ x_2 &= c_{1,1}^* c_{1,2}^*. \end{aligned}$$

Then $\theta^*(x_1) = YX = (XY)^{-1}$ and $\theta^*(x_2) = (XY)^2$. So this epimorphism is topologically conjugate to θ_2, and this is case 1.2. Arguing as above, this case is connected to those corresponding to C_2.

We have finished to connect all possibilities for the primes $p = 2$, 3 and 5, and so we have proved the following.

THEOREM 4. *The space B_5 is connected.*

ACKNOWLEDGEMENTS. The authors wish to thank the referee for his/her careful reading and suggestions on additional references in order to enlarge the context of the paper. The first and third authors are partially supported by MTM2011-23092, the second author by UCM910444 and MTM2011-22435 and the fourth one by NN201 366436 and NCN 2012/05/B/ST1/02171.

REFERENCES

1. N. L. Alling and N. Greenleaf, *Foundations of the theory of Klein surfaces*, Lecture Notes in Mathematics, vol. 219 (Springer-Verlag, Berlin, Germany, 1971).

2. G. Bartolini, A. F. Costa and M. Izquierdo, On the connectivity of branch loci of moduli spaces, *Ann. Acad. Sci. Fenn. Math*. **38** (2013), 245–258.

3. G. Bartolini, A. F. Costa, M. Izquierdo and A. M. Porto, On the connectedness of the branch locus of the moduli space of Riemann surfaces, *Rev. R. Acad. Cienc. Ser. A. Math.* **104** (2010), 81–86.

4. G. Bartolini and M. Izquierdo, On the connectedness of the branch locus of the moduli space of Riemann surfaces of low genus, *Proc. Am. Math. Soc*. **140** (2012), 35–45.

5. S. A. Broughton, The equisymmetric stratification of the moduli space and the Krull dimension of mapping class groups, *Topology Appl.* **37** (1990), 101–113.

6. E. Bujalance, Cyclic groups of automorphisms of compact non-orientable Klein surfaces without boundary, *Pac. J. Math.* **109** (1983), 279–289.

7. E. Bujalance and A. F. Costa, Orientation reversing automorphisms of Riemann surfaces, *Illinois J. Math.* **38** (1994), 616–623.

8. E. Bujalance, A. F. Costa, S. Natanzon and D. Singerman, Involutions of compact Klein surfaces, *Math. Z.* **211** (1992), 461–478.

9. E. Bujalance, J. J. Etayo and E. Martínez, The full group of automorphisms of non-orientable unbordered Klein surfaces of topological genus 3, 4 and 5, *Rev. Mat. Complut*. **27** (2014), 305–326.

10. J. A. Bujalance, A. F. Costa and A. M. Porto, On the connectedness of the locus of real elliptic-hyperelliptic Riemann surfaces, *Int. J. Math.* **20** (2009), 1069–1080.

11. P. Buser, M. Seppälä and R. Silhol, Triangulations and moduli spaces of Riemann surfaces with group actions, *Manuscr. Math*. **88** (1995), 209–224.

12. A. F. Costa and M. Izquierdo, On the connectedness of the locus of real Riemann surfaces, *Ann. Acad. Fenn. Math.* **27** (2002), 341–356.

13. A. F. Costa and M. Izquierdo, On the connectedness of the branch locus of the moduli space of Riemann surfaces of genus 4, *Glasgow Math. J.* **52** (2010), 401–408.

14. W. J. Harvey, On branch loci in Teichmüller space, *Trans. Am. Math. Soc.* **153** (1971), 387–399.

15. R. S. Kulkarni, Isolated points in the branch locus of the moduli space of compact Riemann surfaces, *Ann. Acad. Fenn. Sci. Math.* **16** (1991), 71–81.

16. A. M. Macbeath, The classification of non-Euclidean crystallographic groups, *Can. J. Math.* **19** (1967), 1192–1205.

17. A. M. Macbeath and D. Singerman, Spaces of subgroups and Teichmüller space, *Proc. London Math. Soc.* **31** (1975), 211–256.

18. S. M. Natanzon, Klein surfaces, *Russ. Math. Surv.* **45** (1990), 43–108.

19. R. Preston, *Projective structures and fundamental domains on compact Klein surfaces*, PhD Thesis (University of Texas, Austin, TX, 1975).

20. M. Seppälä, Moduli spaces of stable real algebraic curves, *Ann. Sci. Éc. Norm. Sup. (4)* **24** (1991), 519–544.

21. D. Singerman, Symmetries of Riemann surfaces with large automorphism group, *Math. Ann.* **210** (1974), 17–32.

22. H. C. Wilkie, On non-Euclidean crystallographic groups, *Math. Z.* **91** (1966), 87–102.