C^*-ALGEBRAS ASSOCIATED WITH AMALGAMATED PRODUCTS OF GROUPS

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1. Introduction. Let $\mathbf{V}$ denote the class of discrete groups G which satisfy the following conditions (a), (b) and (c):

(a) $G = (A * B;\ K = \varphi(H))$ is the free product of two groups A and B with the subgroup H amalgamated.

(b) H does not contain the verbal subgroup $A(X^2)$ of A and K does not contain the verbal subgroup $B(X^2)$ of B.

Consequently ([**5**, Problem 4.2.10]), G contains a copy of F_2 freely generated by $x = aba$ and $y = bab$, where $a^2 \notin H$ and $b^2 \notin K$. We now impose furthermore the following mild condition on G.

(c) $(a^{-1}Ha) \cap H = \{e\} = (b^{-1}Kb) \cap K$.

For example, if

$$A = \langle a, c; a^3, c^2, (ac)^2 \rangle$$

and

$$B = \langle b, d; b^3, d^2, (bd)^2 \rangle,$$

the symmetric group on three objects, then the free product of A and B with the cyclic group generated by c and d amalgamated is a group in $\mathbf{V}$.

Let $C_r^*(G)$ denote the C^*-algebra generated by the left regular representation of a discrete group G. If $G = \mathbf{Z} * \mathbf{Z}$, $\mathbf{Z}_2 * \mathbf{Z}_3$ or $G_1 * G_2$, where $\mathbf{Z}$ is the infinite cyclic group, $\mathbf{Z}_2$ the cyclic group of order 2, $\mathbf{Z}_3$ the cyclic group of order 3, and G_1, G_2 are not both of order 2, then it is known that $C_r^*(G)$ is simple and has a unique tracial state ([**7**], [**6**], [**3**]). In this paper, we show that $C_r^*(G)$ is simple and has a unique tracial state if $G \in \mathbf{V}$, thus generalizing the results of [**7**] and [**6**] except when $G = G_1 * G_2$ where G_1 or G_2 only has elements of order 1 or 2. Related work for other classes of groups is treated in [**1**], [**2**].

2. Notation and definitions. A word $R(a, b, c, \ldots)$ which defines the identity element 1 in a group G is called a *relator*. The equation

$$R(a, b, c, \ldots) = S(a, b, c, \ldots)$$

is called a *relation* if the word RS^{-1} is a relator (or equivalently, if R and S define the same element of G).

In a group, the empty word and the words aa^{-1}, $a^{-1}a$, bb^{-1}, $b^{-1}b$, cc^{-1}, $c^{-1}c, \ldots$ are always relators; they are called the *trivial relators*. Suppose $P, Q, R, \ldots$ are any relators of G. We say that the word W is *derivable from* $P, Q, R, \ldots,$ if the following operations, applied a finite number of times, change W into the empty word.

(i) Insertion of one of the words P, P^{-1}, Q, Q^{-1}, R, $R^{-1}, \ldots,$ or one of the trivial relators between any two consecutive symbols of W, or before W, or after W.

(ii) Deletion of one of the words P, P^{-1}, Q, Q^{-1}, R, $R^{-1}, \ldots$ or one of the trivial relators, if it forms a block of consecutive symbols in W.

If every relator is derivable from the relators $P, Q, R, \ldots$, then we call $P, Q, R, \ldots$ a *set of defining relators* for the group G on the generators $a, b, c, \ldots$. If $P, Q, R, \ldots$ is a set of defining relators for the group G on the generators $a, b, c, \ldots$, we call

$$\langle a, b, c, \ldots; P(a, b, c, \ldots), Q(a, b, c, \ldots), R(a, b, c, \ldots), \ldots \rangle$$

a presentation of G and write

$$G = \langle a, b, c, \ldots; P, Q, R, \ldots. \rangle$$

If $a_1, a_2, \ldots, a_n$ are the generating symbols of a group, a word $w(a_1, a_2, \ldots, a_n)$ in $a_1, a_2, \ldots, a_n$ will be denoted by $w(a_\mu)$ for simplicity.

Let A, B, H, K and G be groups defined as follows:

$$A = \langle a_1, \ldots, a_n; R(a_\mu), \ldots \rangle,$$
$$B = \langle b_1, \ldots, b_m; S(b_\nu), \ldots \rangle,$$
$$G = \langle a_1, \ldots, a_n, b_1, \ldots, b_m; R(a_\mu), \ldots, S(b_\nu), \ldots,$$
$$U_1(a_\mu) = V_1(b_\nu), \ldots, U_q(a_\mu) = V_q(b_\nu) \rangle,$$

H is the subgroup of A generated by $U_1(a_\mu), \ldots, U_q(a_\mu)$, and K is the subgroup of B generated by $V_1(b_\nu), \ldots, V_q(b_\nu)$. Suppose that the mapping $U_i(a_\mu) \to V_i(b_\mu)$ induces an isomorphism φ between H and K. Then we call the group G the *free product of A and B with the subgroups H and K amalgamated under* φ, and denote G by $(A * B; K = \varphi(H))$; for brevity G is often called the *free product of A and B with an amalgamated subgroup H*. The groups A and B are called the *factors of the amalgamation*.

Let $G = (A * B; K = \varphi(H))$. Suppose specific right coset representative systems for $A(\text{mod}\, H)$ and $B(\text{mod}\, K)$ have been selected. Then any element g in G can be represented uniquely as a product $hc_1c_2 \ldots c_r$ called the *reduced form* of g, where $h \in H$, $c_i \notin H$, c_i is a representative from $A(\text{mod}\, H)$ or $B(\text{mod}\, K)$ and c_i, c_{i+1} are not both in A or both in B. The nonnegative integer r is called the (*representative*) *length* $l(g)$ of g. If $g = hc_1c_2 \ldots c_r$ is in the reduced form, then g is said to *begin* with c_1 and *end* with c_r.

Let G be a discrete group and $w(X_\mu)$ a reduced word. Then the *verbal subgroup* $G(w(X_\mu))$ of G associated with the word $w(X_\mu)$ is defined by

$$G(w(X_\mu)) = \langle w(g_\mu); g_\mu \in G \rangle.$$

Let G be a discrete group and let $l^2(G)$ denote the Hilbert space of all complex valued square summable functions on G, with inner product

$$(f, h) = \sum_{w \in G} f(w)\overline{h(w)}.$$

For $g \in G$ and $f \in l^2(G)$ define

$$(U(g)f)(w) = f(g^{-1}w) \qquad (w \in G).$$

Then $U(g)$ is unitary on $l^2(G)$ and the mapping $g \to U(g)$ is the left regular representation of G. Let $\mathscr{A}(G)$ denote the pre-C*-algebra generated by $\{U(g): g \in G\}$ and $C_r^*(G)$ the norm closure of $\mathscr{A}(G)$ in $B(l^2(G))$. There is a natural faithful tracial state τ on $C_r^*(G)$ defined by $\tau(T) = (Te_0, e_0)$, where e_0 is the characteristic function of $\{e\}$. We shall show that $C_r^*(G)$ is simple when $G \in \mathbf{V}$.

3. Simplicity and uniqueness of trace.

THEOREM 3.1. *Let G be a group in* $\mathbf{V}$. *Then*
(i) $C_r^*(G)$ *is nonnuclear*;
(ii) $C_r^*(G)$ *is simple*;
(iii) τ *is the only tracial state on* $C_r^*(G)$.

The proof of the theorem above is based on the techniques of [**7**], [**6**], [**3**] and generalizes [**7**] and [**6**] except for groups of the form $A * B$, where A or B only has elements of order 1 or 2.

To establish the nonnuclearity of $C_r^*(G)$, we need the following lemma.

LEMMA 3.2. ([**5**, *Problem* 4.2.10]). *Let* $G = (A * B;\ K = \varphi(H))$ *where H does not contain the verbal subgroup* $A(X^2)$ *of A and K does not contain the verbal subgroup* $B(X^2)$ *of B; then G contains a free subgroup* F_2 *of rank* 2.

Proof. Recall that $A(X^2)$ is the subgroup of A generated by the squares g^2 of all elements g of A. Hence if H does not contain $A(X^2)$, there exists an element $a \in A$ such that a^2 is not in H. Similarly, we can choose an element b in B so that b^2 is not in K. We shall show that aba and bab freely generate a subgroup of G of rank 2.

Let $a = h_1 a_1$, where $h_1 \in H$ and a_1 is the coset representative of a in $A(\operatorname{mod} H)$. Then $ba = b \,.\, h_1 a_1 = b\varphi(h_1) \,.\, a_1 = k_1 b_1 \,.\, a_1$, where $k_1 \in K$ and b_1 is the coset representative of $b\varphi(h_1)$ in B (mod K). Hence

$$aba = a \,.\, k_1 b_1 \,.\, a_1 = a\varphi^{-1}(k_1) \,.\, b_1 \,.\, a_1 = h_2 a_2 \,.\, b_1 a_1,$$

where $h_2 \in H$ and a_2 is the coset representative of $a\varphi^{-1}(k_1)$ in $A(\operatorname{mod} H)$. Hence aba begins and ends with a coset representative from A. First we assume that $n > 0$ and suppose we already know that $(aba)^k$ begins and ends with a coset representative from A whenever $1 \le k < n$. Then

$$\begin{aligned}(aba)^n = (aba)^{n-1} \,.\, (aba) &= (h_2 a_2 b_1 a_1)^{n-1} \,.\, (h_2 a_2 b_1 a_1) \\ &= (h_2 a_2 b_1 a_1) \ldots (h_2 a_2 b_1 a_1)(h_2 a_2 b_1 a_1) \\ &= h_2 a_2 b_1 a_1 \ldots h_2 a_2 b_1 \,.\, (a_1 h_2 a_2) b_1 a_1.\end{aligned}$$

If $a_1 h_2 a_2$ is in H and $b_1 \varphi(a_1 h_2 a_2) b_1$ is in K then $a_2 \varphi^{-1}[b_1 \varphi(a_1 h_2 a_2) b_1] a_1$ cannot be in H since the reduced form of $(aba)^2$ must begin and end in a coset representative from A by the hypothesis that a^2 is not in H. On the other hand if $a_1 h_2 a_2$ is not in H, we replace it with $h_3 a_3$, where $h_3 \in H$ and a_3 is the coset representative of $a_1 h_2 a_2$ in $A(\operatorname{mod} H)$. Finally, if $a_1 h_2 a_2$ is in H but $b_1 \varphi(a_1 h_2 a_2) b_1$ is not in K, then we replace the latter with $k_1 b_2$, where k_1 is in K and b_2 is the coset representative of $b_1\varphi(a_1h_2a_2)b_1$ in $B(\text{mod } K)$. This completes the inductive step. This conclusion is valid for all $n \neq 0$ since $(aba)^{-n} = (a^{-1}b^{-1}a^{-1})^n$, and a^2 is not in H. Similarly $(bab)^n$ starts and ends with a coset representative from B if $n \neq 0$.

Now consider the subgroup N generated by aba and bab. We are now in a position to show that this subgroup is freely generated by aba and bab. Let $x = aba$ and $y = bab$. Then every element of N other than the identity element has positive length. Hence x and y freely generate N. This completes the proof of the lemma.

Thus G is not amenable by the above lemma and 4.4.22, 4.4.21 of [**8**]. Hence $C_r^*(G)$ is nonnuclear by Theorem 4.2 of [**4**]. As in [**7**] and [**6**], parts (ii) and (iii) of Theorem 3.1 are direct consequences of the following two lemmas, the first of which is a variant of Lemma 2.1 in [**3**] with $G \in \mathbf{V}$.

LEMMA 3.3. *Suppose $w_i \in G$, $w_i \neq e$ for $i = 1, 2, \ldots, m$. Then there is an integer n such that $x^n w_i x^{-n}$ (when written in reduced form) begins and ends with a coset representative from A for each $i = 1, 2, \ldots, m$.*

Proof. If $w \in G - \{e\}$ and $l(w) = 0$, then $w \in H - \{e\}$ and $xwx^{-1} = abawa^{-1}b^{-1}a^{-1}$. By (c) $awa^{-1} \in A - H$. Hence xwx^{-1} begins and ends with a coset representative from $A(\text{mod } H)$.

If $l(w) = 1$, then $w \in A - H$ or $w \in B - K$. If $w \in A - H$ and $awa^{-1} \in H - \{e\}$, then by (c) $bawa^{-1}b^{-1} \in B - K$ and so the reduced form of $abawa^{-1}b^{-1}a^{-1}$ begins and ends with a coset representative from $A(\text{mod } H)$. On the other hand, if $w \in A - H$ and $awa^{-1} \in A - H$, then x^2wx^{-2} begins and ends with a coset representation from $A(\text{mod } H)$. To complete the length one case, we note that if $w \in B - K$ then x^2wx^{-2} begins and ends with a and a^{-1} respectively and so the reduced form has the desired property.

If $l(w) = 2$, we show that x^3wx^{-3} begins and ends with a coset representative from $A(\text{mod } H)$. We suppose first of all that $w = hpq$, where $p \in A - H$, $q \in B - K$ and $h \in H$. If furthermore $ahp \in A - H$, the desired conclusion is clearly true. However, if $ahp \in H$, there are three possibilities: $bahpq = e$, $bahpq \in H - \{e\}$ or $bahpq \in B - K$. In the first case, $x^3wx^{-3} = x^2b^{-1}a^{-1}x^{-1}$ which has the desired property. In the second case, we have $abahpq \in A - H$ and so $l(abahpq) = 1$; thus $x^2 . abahpq . x^{-2}$ begins with a and ends with a^{-1} by the length one case. Hence x^3wx^{-3} has the required property since $a^2 \notin H$ by the choice of a. The third case is quite clear. To complete the length 2 case, we note that if $w = h'q'p'$ with $h' \in H$, $q' \in B - K$ and $p' \in A - H$ we consider $p'a^{-1}$ instead of ahp.

For the inductive step, we assume that the reduced form of $x^{l(w)+1}wx^{-(l(w)+1)}$ begins and ends with a coset representative from A whenever $1 < l(w) \leq s$, and consider the case $l(w) = s + 1$.

If $l(w)$ is odd, say $s + 1 = 2n + 1$ and w has a reduced form $hw_1w_2 \ldots w_{2n+1}$, where w_1, w_{2n+1} are both in $B - K$, then it is clear that $x^{s+2}wx^{-(s+2)}$ has the required property.

Next we suppose w has a reduced form $w = hw_1w_2 \ldots w_{2n+1}$, where w_1, w_{2n+1} are in $A - H$. If $ahw_1 \in H$, then $bahw_1 \in B - K$ and there are three possibilities: $bahw_1w_2 = e$, $bahw_1w_2 \in H - e$ or $bahw_1w_2 \in B - K$. In the first case we have xhw_1w_2 is an element of

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$\{e\}$, $H-\{e\}$, or $A-H$ and so the induction hypothesis takes care of this case. In the second case we consider xhw_1w_2 and use the induction hypothesis. In the third case, we consider $w_{2n+1}a^{-1}$. If $w_{2n+1}a^{-1}$ is in $A-H$, it is clear that $x^{2n+2}wx^{-(2n+2)}$ begins with a and ends with a^{-1}. If $w_{2n+1}a^{-1}=e$, then $w_{2n}\,.\,w_{2n+1}a^{-1}b^{-1}=w_{2n}b^{-1}$ and so if $w_{2n}b^{-1}$ is in $B-K$, no further consolidation can take place. However, if $w_{2n}b^{-1}$ is in H then $w_{2n}b^{-1}a^{-1}$ is in $A-H$ and we can use the induction hypothesis. Finally, if $w_{2n+1}a^{-1}\in H-\{e\}$, then $w_{2n+1}a^{-1}b^{-1}\in B-K$ and no further consolidation can take place, and so $x^{2n+2}wx^{-(2n+2)}$ begins with a and ends with a^{-1}. This completes the proof for the odd length case.

Suppose now that $l(w)$ is even, say $s+1=2k$, $k\geq 2$, and that w has the reduced form $w=hw_1w_2\ldots w_{2k}$, where $w_1\in A-H$ and $w_{2k}\in B-K$. If $ahw_1\in A-H$, then no further consolidation can take place. However if $ahw_1\in H$, then $baw_1\in B-K$ and again no further consolidation can take place.

Finally we note that the case in which $w=hw_1w_2\ldots w_{2k}$ with $w_1\in B-K$ and $w_{2k}\in A-H$ is similarly treated by considering $w_{2k}a^{-1}$. This concludes the proof of the inductive step. The lemma now follows by setting $n=\max\{l(w_i)+1\colon i=1, 2, \ldots, m\}$.

LEMMA 3.4. *Let G be in* **V**. *Suppose*

$$T=\sum_{i=1}^{m}[\alpha_iU(w_i)+\bar{\alpha}_iU(w_i)^{-1}],$$

where α_i are complex numbers and w_i are nonidentity elements of G. Then there exist $t_r\in G$, $r=1, 2, \ldots, n$, such that

$$\left\|\frac{1}{n}\sum_{r=1}^{n}U(t_r)TU(t_r^{-1})\right\|\leq\frac{2}{\sqrt{n}}\,\|T\|\,.$$

Proof. By Lemma 3.3, there exists an integer k such that $x^kw_ix^{-k}$ begins and ends with coset representatives from $A(\text{mod}\,H)$ for $i=1, 2, \ldots, m$. For $r=1, 2, \ldots, n$, let $t_r=x^ryx^k$, where as before $y=bab$. Let S_r denote the set of words w in G such that $x^{-r}w$ begins with a coset representative from B. Then $\{S_r\colon r=1, 2, \ldots, n\}$ are pairwise disjoint; and if $z\in G-S_r$, then $y^{-1}x^{-r}z$ begins with a coset representative from $B(\text{mod}\,K)$. Consider $l^2(S_r)$ as a closed subspace of $l^2(G)$ in the natural way. Let E_r denote the Hermitian projection associated with $l^2(S_r)$. Since the S_r are pairwise disjoint, it follows that the E_r are pairwise orthogonal. Given a function f in $l^2(G)$ with support in $G-S_r$, let z be an element of $G-S_r$; then $y^{-1}x^{-r}z$ begins with a coset representative from $B(\text{mod}\,K)$. Hence

$$t_rw_it_r^{-1}z=x^ry\,.\,x^kw_ix^{-k}\,.\,y^{-1}x^{-r}z$$

begins with a coset representative from $A(\text{mod}\,H)$ since no reduction can take place between x^{-k} and y^{-1} or between y and x^k. Thus $(I-E_r)U(t_rw_it_r^{-1})(I-E_r)$ is zero for $r\geq 1$.

The rest of the proof of the lemma can now be completed as in [**6**, p. 214]. Indeed, if T is any bounded operator on a Hilbert space H and P is a projection such that

$(I-P)T(I-P)=0$, then $\|(Tf, f)\| \leq 2\|T\|\,\|Pf\|$ for all f in H with $\|f\| \leq 1$. Then we apply this to the operator U and the projections E_r to deduce that for every f in the unit ball of $l^2(G)$,

$$\left(\frac{1}{n}\sum_{r=1}^{n} U(t_r)TU(t_r^{-1})f, f\right) \leq \frac{1}{n}\sum_{r=1}^{n} 2\,\|T\|\,\|E_r f\| \leq \frac{2}{\sqrt{n}}\|T\|.$$

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