



On Two-faced Families of Non-commutative Random Variables

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Abstract. We demonstrate that the notions of bi-free independence and combinatorial-bi-free independence of two-faced families are equivalent using a diagrammatic view of bi-non-crossing partitions. These diagrams produce an operator model on a Fock space suitable for representing any two-faced family of non-commutative random variables. Furthermore, using a Kreweeras complement on bi-non-crossing partitions we establish the expected formulas for the multiplicative convolution of a bi-free pair of two-faced families.

1 Introduction

Free probability for pairs of faces, or simply bi-free probability, was introduced by Voiculescu in [5] as a generalization of the notion of free probability to allow the simultaneous study of “left-handed” and “right-handed” variables. Prior to this work, the left and right actions were only considered separately. Voiculescu demonstrated that many results in free probability, such as the existence of the free cumulants and the free central limit theorem, have direct analogues in the bi-free setting. However, free independence is equivalent to a variety of computational conditions, such as vanishing alternating moments of centered variables or vanishing mixed cumulants. It was shown in [5, Proposition 5.6] that such computational conditions for bi-freeness exist as a collection of universal polynomials on the mixed moments of a bi-free pair of two-faced families, but their explicit formulas were unknown.

Seeking an alternate approach to bi-free probability, Mastnak and Nica [1] defined the (ℓ, r) -cumulant functions, which they predicted to be the universal polynomials of Voiculescu. Such cumulant functions were defined by considering permutations applied to non-crossing diagrams. Taking inspiration from the free case, they defined a pair of two-faced families z' and z'' to be combinatorially-bi-free if all mixed cumulants are zero, and posed the question of whether their definition was equivalent to Voiculescu’s definition of bi-free independence.

In this paper, we will provide an affirmative answer to their question, demonstrating the equivalence of bi-free independence and combinatorial-bi-free independence. Analyzing [1], one can take a diagrammatic view of the desired partitions that is more natural to the study of two-faced families of non-commutative random variables. In

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1290

Section 2, after some preliminaries, we introduce this view via the notion of bi-non-crossing partitions. Such partitions are designed to encapsulate information about whether a variable should be considered on the left or on the right. One main goal of this paper is to demonstrate that bi-non-crossing partitions play the same role in bi-free probability as non-crossing partitions play in free probability.

Following Speicher [4], we introduce the incidence algebra on bi-non-crossing partitions in Section 3. The algebra enables an analysis of left and right variables simultaneously and provides a method of Möbius inversion. This allows us to obtain the bi-free cumulant functions directly.

In Section 4 we will prove our main theorem, Theorem 4.3.1, which demonstrates that the two notions of bi-free independence are equivalent. To do so, we analyze the action of operators on free product spaces as in [5] to obtain explicit descriptions of Voiculescu’s universal polynomials. We give equivalent formulae for these polynomials using the bi-non-crossing Möbius function.

Using the combinatorially-bi-free approach, we will develop further results. In Section 5 we will describe a multiplicative free convolution of two-faced families. By extending the Kreweras complement approach of [3] to bi-non-crossing diagrams, we show that the bi-free cumulants of a product of two-faced families can be written as a convolution of the individual bi-free cumulants.

Finally, in Section 6 we construct an operator model in the linear operators on a Fock space for a two-faced family of non-commutative random variables. This generalizes the model from [2] and provides a bi-free analogue of Voiculescu’s non-commutative R -series.

2 Preliminaries

2.1 Free Probability for Pairs of Faces

Throughout, $z = ((z_i)_{i \in I}, (z_j)_{j \in J})$ will denote a two-faced family in a non-commutative probability space (\mathcal{A}, ϕ) with the left face indexed by I , the right face indexed by J , and I and J disjoint. We will also let z' and z'' be two-faced families, similarly indexed.

Recall that in [5], z' and z'' are said to be *bi-freely independent* (or simply *bi-free*) if there exists a free product $(\mathcal{X}, p, \xi) = (\mathcal{X}', p', \xi') * (\mathcal{X}'', p'', \xi'')$ of vector spaces with specified state-vectors and unital homomorphisms

$$l^\epsilon: \mathbb{C} \langle z_i^\epsilon: i \in I \rangle \rightarrow \mathcal{L}(\mathcal{X}^\epsilon), \quad r^\epsilon: \mathbb{C} \langle z_j^\epsilon: j \in J \rangle \rightarrow \mathcal{L}(\mathcal{X}^\epsilon), \quad \epsilon \in \{', ''\},$$

such that the two-faced families $T^\epsilon = ((\lambda^\epsilon \circ l^\epsilon(z_i^\epsilon))_{i \in I}, (\rho^\epsilon \circ r^\epsilon(z_j^\epsilon))_{j \in J})$ with $\epsilon \in \{', ''\}$ have the same joint distribution in $(\mathcal{L}(\mathcal{X}), \phi)$ as z' and z'' . Here, λ^ϵ and ρ^ϵ are the left and right representations of $\mathcal{L}(\mathcal{X}^\epsilon)$ in $\mathcal{L}(\mathcal{X})$ (cf. [5, Section 1.9]). For $T \in \mathcal{L}(\mathcal{X}^\epsilon)$, we will often suppress the ϵ notation on λ^ϵ , ρ^ϵ , and ϕ^ϵ (the state on $\mathcal{L}(\mathcal{X}^\epsilon)$ induced by p^ϵ), as it will be clear which is meant by noting which vector space T is defined on.

Given $\alpha: \{1, \dots, n\} \rightarrow I \sqcup J$, we will refer to the “ α -moment” of a two-faced family z :

$$\phi_\alpha(z) := \phi(z_{\alpha(1)} \cdots z_{\alpha(n)}).$$

It was shown in [5, Theorem 5.7] that for each α there exists a universal polynomial R_α on indeterminates X_K indexed by non-empty subsets $K \subset \{1, \dots, n\}$ satisfying:

- (i) $R_\alpha = X_{\{1, \dots, n\}} + \tilde{R}_\alpha$, where \tilde{R}_α is a polynomial on indeterminates X_K indexed by non-empty strict subsets $K \subsetneq \{1, \dots, n\}$;
- (ii) R_α and \tilde{R}_α are homogeneous of degree n when X_K is given degree $|K|$;
- (iii) if $R_\alpha(z)$ is R_α evaluated at $X_K = \phi(z_{\alpha(k_1)} \cdots z_{\alpha(k_r)})$ and $K = \{k_1 < \cdots < k_r\}$, then

$$R_\alpha(z' + z'') = R_\alpha(z') + R_\alpha(z''),$$

whenever z' and z'' are bi-free two-faced families.

The number $R_\alpha(z)$ is called the α -cumulant of z . Property (iii) above is referred to as the *cumulant property*.

2.2 Partitions, Ordering, and Non-crossing Partitions

A partition π is a set $\pi = \{V_1, \dots, V_k\}$, where V_1, \dots, V_k (called the *blocks of π*) are non-empty sets satisfying $V_i \cap V_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^k V_i = \{1, \dots, n\}$. We traditionally order the blocks of π so that $\min(V_1) < \cdots < \min(V_k)$. Let $\mathcal{P}(n)$ denote the set of partitions of $\{1, \dots, n\}$.

For $\pi, \sigma \in \mathcal{P}(n)$, we say π is a *refinement* of σ and write $\pi \leq \sigma$ if every block of π is contained in a block of σ . This defines a partial ordering on $\mathcal{P}(n)$ with minimum and maximum elements

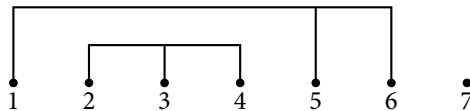
$$0_n := \{\{1\}, \dots, \{n\}\} \quad \text{and} \quad 1_n := \{\{1, \dots, n\}\},$$

respectively. We will also consider the following action of the symmetric group S_n on $\mathcal{P}(n)$: if $\pi = \{V_1, \dots, V_k\} \in \mathcal{P}(n)$ and $s \in S_n$, then

$$s \cdot \pi = \{s(V_1), \dots, s(V_k)\} \in \mathcal{P}(n).$$

Observe that this action is order-preserving.

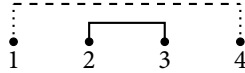
A partition $\pi \in \mathcal{P}(n)$ is said to be *non-crossing* if for any two distinct blocks $V = \{v_1 < \cdots < v_r\}$, $W = \{w_1 < \cdots < w_s\} \in \pi$ we have $v_l < w_1 < v_{l+1}$ if and only if $v_l < w_s < v_{l+1}$ ($l \in \{1, \dots, r-1\}$). The term “non-crossing” refers to the fact that any such partition can be represented as a non-crossing diagram. For example, the non-crossing partition $\{\{1, 5, 6\}, \{2, 3, 4\}, \{7\}\} \in \mathcal{P}(7)$ corresponds to the diagram



We denote the set of non-crossing partitions in $\mathcal{P}(n)$ by $\text{NC}(n)$.

The horizontal segments connecting the nodes of a block $V \in \pi$ will be referred to as the *spine of V* , and the segments connecting the nodes to the spine of V will be referred to as the *ribs of V* . In the following representation of $\{\{1, 4\}, \{2, 3\}\} \in \text{NC}(4)$,

the dashed line is the spine of $\{1, 4\}$, and the dotted lines are its ribs:



For a singleton block $V \in \pi$, $|V| = 1$, the spine of V will simply refer to the corresponding node itself.

2.3 Combinatorial-bi-free Independence

For consistency, we note the following definitions of Mastnak and Nica. Given

$$\chi: \{1, \dots, n\} \longrightarrow \{\ell, r\}$$

let $\{i_1 < \dots < i_p\} = \chi^{-1}(\ell)$ and $\{j_1 < \dots < j_{n-p}\} = \chi^{-1}(r)$ and consider $\sigma_\chi \in S_n$ defined by

$$\sigma_\chi(k) = \begin{cases} i_k & \text{if } k \leq p, \\ j_{n+1-k} & \text{if } k > p. \end{cases}$$

The class of partitions $\mathcal{P}^{(\chi)}(n) \subset \mathcal{P}(n)$ is defined as

$$\mathcal{P}^{(\chi)}(n) := \{\sigma_\chi \cdot \pi \mid \pi \in \text{NC}(n)\}.$$

Definition 2.3.1 ([1, Definition 5.2]) Let (\mathcal{A}, ϕ) be a non-commutative probability space. There exists a family of multilinear functionals

$$(\kappa_\chi: \mathcal{A}^n \rightarrow \mathbb{C})_{n \geq 1, \chi: \{1, \dots, n\} \rightarrow \{\ell, r\}}$$

that are uniquely determined by the requirement

$$\phi(z_1 \cdots z_n) = \sum_{\pi \in \mathcal{P}^{(\chi)}(n)} \left(\prod_{V \in \pi} \kappa_{\chi|_V}((z_1, \dots, z_n)|V) \right)$$

for every $n \geq 1$, $\chi \in \{\ell, r\}^n$, and $z_1, \dots, z_n \in \mathcal{A}$. These κ_χ 's will be called the (ℓ, r) -cumulant functionals of (\mathcal{A}, ϕ) .

Definition 2.3.2 ([1]) Let z' and z'' each be two-faced families in (\mathcal{A}, ϕ) . We say that z' and z'' are *combinatorially-bi-free* if

$$\kappa_\chi(z_{\alpha(1)}^{\epsilon_1}, \dots, z_{\alpha(n)}^{\epsilon_n}) = 0$$

whenever $\alpha: \{1, \dots, n\} \rightarrow I \sqcup J$, $\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}$ is such that $\alpha^{-1}(I) = \chi^{-1}(\{\ell\})$ and $\epsilon \in \{\prime, \prime\prime\}^n$ is non-constant.

Remark 2.3.3 Note that the condition $\alpha^{-1}(I) = \chi^{-1}(\{\ell\})$ completely determines χ , and so we can set

$$\kappa_\alpha(z) := \kappa_\chi(z_{\alpha(1)}, \dots, z_{\alpha(n)}).$$

Then if z' and z'' are combinatorially-bi-free, it is easy to see that

$$\kappa_\alpha(z' + z'') = \kappa_\alpha(z') + \kappa_\alpha(z'');$$

that is, κ_α has the cumulant property.

2.4 Bi-non-crossing Partitions

For $\alpha: \{1, \dots, n\} \rightarrow I \sqcup J$, we let $\{i_1 < \dots < i_p\} = \alpha^{-1}(I)$ and $\{j_1 < \dots < j_{n-p}\} = \alpha^{-1}(J)$ and consider $s_\alpha \in S_n$ defined by

$$s_\alpha(k) = \begin{cases} i_k & \text{if } k \leq p, \\ j_{n+1-k} & \text{if } k > p. \end{cases}$$

We say a partition $\pi \in \mathcal{P}(n)$ is *bi-non-crossing* (with respect to α) if $s_\alpha^{-1} \cdot \pi \in \text{NC}(n)$. We denote the set of such partitions by $\text{BNC}(\alpha)$. The minimum and maximum elements of $\text{BNC}(\alpha)$ are given by $0_\alpha := s_\alpha \cdot 0_n$ and $1_\alpha := s_\alpha \cdot 1_n$, respectively.

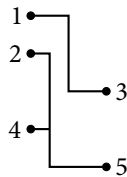
With each partition $\pi \in \text{BNC}(\alpha)$ we can associate a “bi-non-crossing diagram” as follows. For each $k = 1, \dots, n$ place a node labeled k at the position $(-1, n - k)$ if $\alpha(k) \in I$ and at the position $(1, n - k)$ if $\alpha(k) \in J$. Connect nodes whose labels form a block of π similar to how one would for the diagrams associated with $\text{NC}(n)$, except now the spines of blocks are vertically oriented and the ribs extend horizontally from the spine to the left or right, emphasizing the left-right nature of a two-faced family.

Example 2.4.1 If

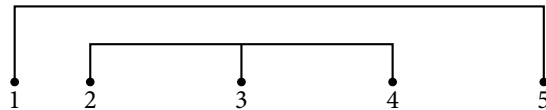
$$\alpha^{-1}(I) = \{1, 2, 4\}, \quad \alpha^{-1}(J) = \{3, 5\}, \quad \text{and}$$

$$\pi = \{ \{1, 3\}, \{2, 4, 5\} \} = s_\alpha \cdot \{ \{1, 5\}, \{2, 3, 4\} \},$$

then the bi-non-crossing diagram associated with π is



That the diagram can always be drawn to be non-crossing is easily seen through its relationship to the diagram of $s_\alpha^{-1} \cdot \pi \in \text{NC}(n)$. Indeed, rotate the line $x = -1$ counter-clockwise a quarter turn about the point $(-1, 0)$, rotate the line $x = 1$ clockwise a quarter turn about the point $(1, 0)$, and adjust the spines and ribs so that they remain connected. Then after relabeling node k as $s_\alpha^{-1}(k)$ the resulting diagram is precisely the one associated with $s_\alpha^{-1} \cdot \pi$ as an element of $\text{NC}(n)$ (modulo some extra space between the nodes). Performing this operation to the above diagram yields



Conversely, given the diagram corresponding to $\sigma \in \text{NC}(n)$, we obtain the diagram for $\pi = s_\alpha \cdot \sigma$ as follows. Initially, the nodes occupy positions $(1, 0), \dots, (n, 0)$, so we

first widen the space between nodes so that node k now occupies position $(s_\alpha(k), 0)$ if $k \leq |\alpha^{-1}(I)|$ and position $(n + 1 - s_\alpha(k), 0)$ if $k > |\alpha^{-1}(I)|$. Given the definition of s_α , it is clear that this does not change the order of the nodes. Next, we rotate the segment from $(1, 0)$ to $(n, 0)$ clockwise a quarter turn about $(n, 0)$; we rotate the segment from $(n + 1, 0)$ to $(2n, 0)$ counter-clockwise a quarter turn about $(n + 1, 0)$, and homotopically vary the spines and ribs so that they remain connected. Relabeling node k as node $s_\alpha(k)$ then yields the diagram corresponding to π .

Remark 2.4.2 Given $\alpha: \{1, \dots, n\} \rightarrow I \sqcup J$, define $\chi \in \{\ell, r\}^n$ by $\chi_k = \ell$ if $\alpha(k) \in I$ and $\chi_k = r$ if $\alpha(k) \in J$. Then $\text{BNC}(\alpha)$ is precisely the class of partitions $\mathcal{P}^{(\chi)}(n)$ defined in [1], since s_α defined above is exactly the permutation σ_χ used to define the class $\mathcal{P}^{(\chi)}(n)$. Moreover, the notation $\text{BNC}(\alpha)$ suggests that the lattice of partitions depends on α more than it actually does. In fact, if $\beta: \{1, \dots, n\} \rightarrow I \sqcup J$ is such that $\beta(j)$ and $\alpha(j)$ are in the same face for each $j = 1, \dots, n$, then $\text{BNC}(\alpha) = \text{BNC}(\beta)$. Because of this we can write $\text{BNC}(\chi)$ for $\text{BNC}(\alpha)$. In order to emphasize the diagrammatic viewpoint pervading this paper, we will continue to use the alternate notation of $\text{BNC}(\alpha)$ for this class of partitions.

2.5 Shaded Bi-non-crossing Diagrams and Partitions

Let z' and z'' be a bi-free pair of two-faced families. Let $\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}$ and $\epsilon \in \{', ''\}^n$. We recursively define a collection of diagrams $LR(\chi, \epsilon)$. For $n = 1$, $LR(\chi, \epsilon)$ consists of two parallel, vertical, transparent segments with a single node on the left segment if $\chi(1) = \ell$ or a single node on the right segment if $\chi(1) = r$. We assign a shade to $'$ and $''$ and shade this node the shade associated with ϵ_1 . Then either this node remains isolated, or a rib and spine of the node's shade are drawn connecting to the top of the diagram, between the two segments. For convenience, we will refer to the space between the two vertical segments at the top of a diagram as its *top gap*, through which strings may exit.

For $n > 1$ we define $LR(\chi, \epsilon)$ as follows. Let $\chi_0 = \chi|_{\{2, \dots, n\}}$ and $\epsilon_0 = (\epsilon_2, \dots, \epsilon_n)$. Then a diagram of $LR(\chi, \epsilon)$ is an extension of a diagram $D \in LR(\chi_0, \epsilon_0)$: place an additional ϵ_1 -shaded node p above D , on the left if $\chi(1) = \ell$ and on the right otherwise. Extend any spines from D to the new top gap. If at least one spine was extended and the one nearest p shares its shade, then connect it to p with a rib and optionally terminate the spine at p . Otherwise, either connect p with a rib to a new spine extending to the top gap or leave p isolated.

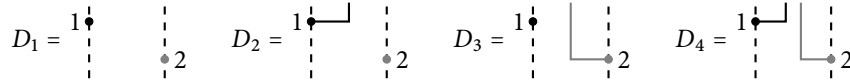
Given its impact on the diagrams, we refer to $\epsilon \in \{', ''\}^n$ as a *choice of shading* or simply a *shading*.

Note that each diagram in $LR(\chi, \epsilon)$ is created from a unique diagram in $LR(\chi_0, \epsilon_0)$ that we can recover by simply erasing the top portion of the diagram. Also, these rules imply that among the chords extending to the top gap, adjacent chords will always be of differing shades. We use the convention where the nodes are labeled numerically from top to bottom.

For $0 \leq k \leq n$, let $LR_k(\chi, \epsilon) \subseteq LR(\chi, \epsilon)$ consist of those diagrams with precisely k chords extending to the top gap. Then $LR(\chi, \epsilon) = \cup_k LR_k(\chi, \epsilon)$.

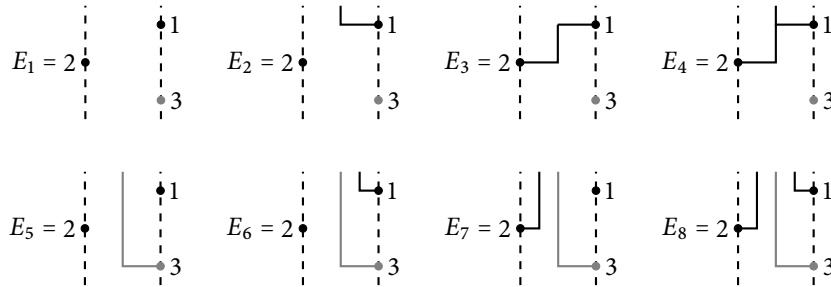
We consider a few examples. In each example, we assign the shade black to ' and the shade grey to '' and have a dashed line in place of the normally transparent left and right segments.

Example 2.5.1 Consider $\chi = (\ell, r)$ and $\epsilon = (''')$. Then $LR(\chi, \epsilon)$ consists of the following diagrams:



Also $LR_0(\chi, \epsilon) = \{D_1\}$, $LR_1(\chi, \epsilon) = \{D_2, D_3\}$, and $LR_2(\chi, \epsilon) = \{D_4\}$.

Example 2.5.2 For a slightly more robust example we consider $\chi = (r, \ell, r)$ and $\epsilon = ('', ''')$. Then $LR(\chi, \epsilon)$ consists of the following diagrams:



Observe in terms of the recursive construction of $LR(\chi, \epsilon)$, the diagram D_k , $k = 1, 2, 3, 4$ from Example 2.5.1 creates diagrams E_{2k-1} and E_{2k} in the present example.

For fixed χ and ϵ we note that each $D \in LR_0(\chi, \epsilon)$ can be associated with a partition $\pi \in \mathcal{P}(n)$ by forming blocks according to which nodes are connected via chords in the diagram. Since $D \in LR_0(\chi, \epsilon)$ is completely determined by the connections between nodes, distinct diagrams yield distinct partitions. Moreover, if $\alpha: \{1, \dots, n\} \rightarrow I \sqcup J$ and we define χ^α by $\chi^\alpha(k) = \ell$ if $\alpha(k) \in I$ and $\chi^\alpha(k) = r$ if $\alpha(k) \in J$, then the partitions we obtain from $LR_0(\chi^\alpha, \epsilon)$ are elements of $\text{BNC}(\alpha)$. We denote by $\text{BNC}(\alpha, \epsilon)$ the partitions obtained from the diagrams in $LR_0(\chi^\alpha, \epsilon)$. It is not hard to see that given the diagram associated with some $\pi \in \text{BNC}(\alpha)$, there exists some shading ϵ such that $\pi \in \text{BNC}(\alpha, \epsilon)$. It then follows that

$$\text{BNC}(\alpha) = \bigcup_{\epsilon \in \{', ''\}^n} \text{BNC}(\alpha, \epsilon).$$

As with $\text{BNC}(\alpha)$, we may denote $\text{BNC}(\alpha, \epsilon)$ by $\text{BNC}(\chi, \epsilon)$ when $\chi = \chi^\alpha$.

Definition 2.5.3 Suppose that V and W are blocks of some $\pi \in \text{BNC}(\chi)$. Then V and W are said to be *piled* if $\max(\min(V), \min(W)) \leq \min(\max(V), \max(W))$. In terms of the diagram corresponding to π , the spines of V and W are not entirely above or below each other; there is some horizontal level at which both are present.

Given blocks V and W , a third block U *separates* V from W if it is piled with both, and its spine lies between the spines of V and W . Note that V and W need not be

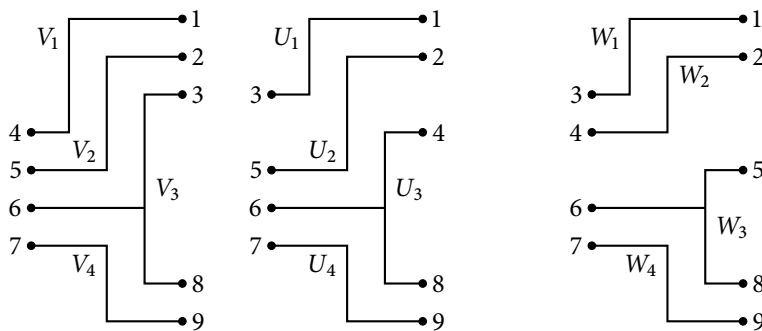
piled with each other to have a separator. Equivalently, U is piled with both V and W , and there are $j, k \in U$ such that

$$s_\alpha^{-1}(V) \subseteq [s_\alpha^{-1}(j), s_\alpha^{-1}(k)] \quad \text{and} \quad s_\alpha^{-1}(W) \cap [s_\alpha^{-1}(j), s_\alpha^{-1}(k)] = \emptyset,$$

or vice versa. Given any three piled blocks, one always separates the other two.

Finally, piled blocks V and W are said to be *tangled* if there is no block that separates them.

Example 2.5.4 Consider the following diagrams.



In the first diagram, V_2 separates V_1 from V_3 , and all three are piled with one another. In the second diagram, U_2 still separates U_1 and U_3 , but U_1 and U_3 are not piled with each other. In the third diagram, there are no separators.

Definition 2.5.5 Suppose $\pi, \sigma \in \text{BNC}(\chi)$ are such that $\pi \leq \sigma$. We say π is a *lateral refinement* of σ and write $\pi \leq_{\text{lat}} \sigma$ if no two piled blocks in π are contained in the same block of σ .

Lateral refinements correspond to making horizontal “cuts” along the spines of blocks of π , between their ribs.

In the notation of Example 2.5.2, E_1 is a lateral refinement of E_3 made by cutting the block $\{1, 2\}$ in between node 1 and node 2.

Lemma 2.5.6 If $\pi \in \text{BNC}(\chi, \epsilon)$, then piled blocks of the same shade in π must be separated. Consequently, if $\sigma \in \text{BNC}(\alpha, \epsilon)$ and $\pi \leq \sigma$, then $\pi \leq_{\text{lat}} \sigma$.

Proof Suppose V_1 and V_2 are piled blocks in $\pi \in \text{BNC}(\chi, \epsilon)$ that have the same shade. Without loss of generality, $k := \max(V_2) < \max(V_1)$. In the construction of the diagram generating π , when node k is placed the nearest spine must be of a different shade, as k begins a new spine. In particular, this spine sits between the spines of V_1 and V_2 , and so its block is a separator.

If two blocks of the same shade in π are piled, the above argument demonstrates that they are separated by a block of a different shade and so cannot be joined in σ . ■

3 The Incident Algebra on Bi-Non-Crossing Partitions

Definition 3.0.1 The lattice of bi-non-crossing partitions is

$$\text{BNC} := \bigcup_{n \geq 1} \bigcup_{\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}} \text{BNC}(\chi),$$

where the lattice structure on $\text{BNC}(\chi)$ is as above.

Given any lattice, there is an algebra of functions associated with the lattice.

Definition 3.0.2 The incident algebra on BNC, denoted $\text{IA}(\text{BNC})$, consists of all functions of the form

$$f: \bigcup_{n \geq 1} \left(\bigcup_{\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}} \text{BNC}(\chi) \times \text{BNC}(\chi) \right) \rightarrow \mathbb{C}$$

such that $f(\pi, \sigma) = 0$ if $\pi \not\leq \sigma$, equipped with pointwise addition and a convolution product defined by

$$(f * g)(\pi, \sigma) = \sum_{\pi \leq \rho \leq \sigma} f(\pi, \rho)g(\rho, \sigma)$$

for all $\pi, \sigma \in \text{BNC}(\chi)$ and $f, g \in \text{IA}(\text{BNC})$.

It is elementary to show that $\text{IA}(\text{BNC})$ is an algebra, and thus $(f * g) * h = f * (g * h)$.

3.1 Multiplicative Functions on the Incident Algebra

In order to construct the notion of multiplicative function on BNC, it is necessary to identify the lattice structure of an interval as a product of full intervals.

Proposition 3.1.1 Let $\pi, \sigma \in \text{BNC}(\chi)$ be such that $\pi \leq \sigma$. The interval

$$[\pi, \sigma] = \{ \rho \in \text{BNC}(\chi) \mid \pi \leq \rho \leq \sigma \}$$

can be associated with a product of full lattices

$$\prod_{j=1}^k \text{BNC}(\beta_k)$$

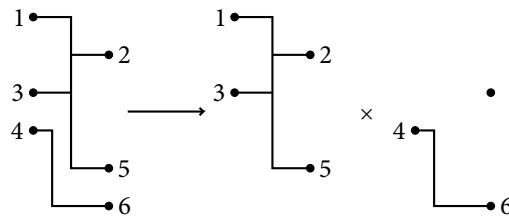
for some $\beta_k: \{1, \dots, m_k\} \rightarrow \{\ell, r\}$ so that the lattice structure is preserved.

Proof The idea behind the decomposition is to take π and σ , view π and σ as elements of $\text{NC}(n)$ by applying s_χ^{-1} , and using the decomposition of intervals in $\text{NC}(n)$ given in [4, Proposition 1] while maintaining the notion of left and right nodes.

First write $\sigma = \{W_1, \dots, W_k\}$. Let π_j and σ_j be the restrictions of π and σ to W_j . Then we decompose $[\pi, \sigma]$ into $\prod_{j=1}^k [\pi_j, \sigma_j]$. Note each σ_j is a full bi-non-crossing partition corresponding to some $\gamma_j: \{1, \dots, n_j\} \rightarrow \{\ell, r\}$, so one can reduce to intervals of the form $[\pi, 1_\chi]$.

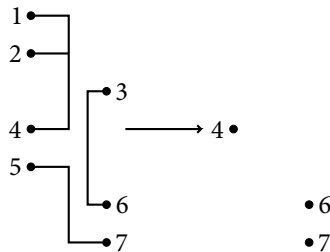
For a fixed $\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}$, a modification to the recursive argument of [4, Proposition 1] under the identification of $\text{BNC}(\chi)$ with $\text{NC}(n)$ will be described. First, viewing $\pi \in \text{NC}(n)$, examine whether π has a block $V = \{k_1 < k_2 < \dots < k_m\}$

containing non-consecutive elements; that is, there exists an index t such that $k_t + 1 \neq k_{t+1}$. If so, the recursive argument of [4, Proposition 1] would decompose $[\pi, 1_\chi]$ into the product of two intervals (removing any trivial intervals that occur): one corresponding to taking $[\pi, 1_\chi]$ and removing all nodes strictly between k_t and k_{t+1} ; and the other corresponding to taking only the nodes strictly between k_t and k_{t+1} and adding an isolated node on the left. The only change made to accommodate BNC is that the isolated node for the second interval should be added to the top left of the bi-non-crossing diagram if the lower of the two nodes of the original diagram corresponding to k_t and k_{t+1} is on the left and otherwise on the top right. For example:



Note that the first term in the product will be ignored as it is a full partition.

This recursive process eventually terminates, leaving only partitions π such that the blocks of $\sigma_\chi^{-1} \cdot \pi$ are intervals. For such a bi-non-crossing partition, we associate the zero bi-non-crossing partition corresponding to keeping only the lowest node of each block. For example:



Thus, we have reduced $[\pi, \sigma]$ to products of full lattices in BNC. ■

Note that as in [4, Proposition 1] we make no claim that this association is unique. However, this ambiguity does not affect the following computations.

Definition 3.1.2 A function $f \in \text{IA}(\text{BNC})$ is said to be *multiplicative* if whenever $\pi, \sigma \in \text{BNC}(\chi)$ are such that

$$[\pi, \sigma] \leftrightarrow \prod_{j=1}^k \text{BNC}(\beta_k)$$

for some $\beta_k: \{1, \dots, m_k\} \rightarrow \{\ell, r\}$, then

$$f(\pi, \sigma) = \prod_{j=1}^k f(0_{\beta_k}, 1_{\beta_k}).$$

For a multiplicative function $f \in \text{IA}(\text{BNC})$, we will call the collection

$$\{f([0_\chi, 1_\chi]) \mid n \geq 1, \chi: \{1, \dots, n\} \rightarrow \{\ell, r\}\} \subseteq \mathbb{C}$$

the *multiplicative net* associated with f . Note that for any net

$$\Lambda = \{a_\chi \mid n \geq 1, \chi: \{1, \dots, n\} \rightarrow \{\ell, r\}\} \subseteq \mathbb{C}$$

there is precisely one multiplicative function f with multiplicative sequence Λ .

Lemma 3.1.3 *If $f, g \in \text{IA}(\text{BNC})$ are multiplicative, then $f * g$ is multiplicative.*

See [4, Proposition 2] for a proof of the above.

Remark 3.1.4 There are three special multiplicative functions to consider; namely,

$$\delta_{\text{BNC}}(\pi, \sigma) = \begin{cases} 1 & \text{if } \pi = \sigma, \\ 0 & \text{otherwise,} \end{cases}$$

which is called the delta function on BNC and is the identity element in $\text{IA}(\text{BNC})$,

$$\zeta_{\text{BNC}}(\pi, \sigma) = \begin{cases} 1 & \text{if } \pi \leq \sigma, \\ 0 & \text{otherwise,} \end{cases}$$

which is called the zeta function on BNC, and μ_{BNC} , which is called the Möbius function on BNC and which is defined such that

$$\mu_{\text{BNC}} * \zeta_{\text{BNC}} = \zeta_{\text{BNC}} * \mu_{\text{BNC}} = \delta_{\text{BNC}}$$

(as it is clear that ζ_{BNC} a left and right (and thereby a two-sided) inverse can be recursively defined). It is clear that δ_{BNC} is multiplicative with $\delta_{\text{BNC}}(0_\chi, 1_\chi)$ being one if $n = 1$ and zero otherwise, and ζ_{BNC} is multiplicative with $\zeta_{\text{BNC}}(0_\chi, 1_\chi) = 1$ for all χ . In addition, one can verify that μ_{BNC} is multiplicative and for any $\pi, \sigma \in \text{BNC}(\chi)$,

$$\mu_{\text{BNC}}(\pi, \sigma) = \mu(s_\chi^{-1} \cdot \pi, s_\chi^{-1} \cdot \sigma),$$

where μ is the Möbius function in [4].

Remark 3.1.5 To consolidate the above with Subsection 2.3, for T_1, \dots, T_n in a non-commutative probability space (\mathcal{A}, ϕ) and $\pi \in \text{BNC}(\chi)$ where $\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}$ and $V_t = \{k_{t,1} < \dots < k_{t,m_t}\}$ for $t \in \{1, \dots, k\}$ being the blocks of π , we define

$$\phi_\pi(T_1, \dots, T_n) := \prod_{t=1}^k \phi(T_{k_{t,1}} \cdots T_{k_{t,m_t}})$$

and

$$\kappa_\pi(T_1, \dots, T_n) := \sum_{\sigma \in \text{BNC}(\chi), \sigma \leq \pi} \phi_\sigma(T_1, \dots, T_n) \mu_{\text{BNC}}(\sigma, \pi).$$

Then, as in [4], one can show that

$$\kappa_\pi(T_1, \dots, T_n) = \prod_{t=1}^k \kappa_{\pi|_{V_t}}(T_{k_{t,1}} \cdots T_{k_{t,m_t}}),$$

where $\kappa_{\pi|V_t}$ should be thought of as the (single block) partition induced by the block V_t of π , and

$$\phi(T_1 \dots T_n) = \sum_{\pi \in \text{BNC}(\chi)} \kappa_{\pi}(T_1, \dots, T_n).$$

In particular, $\kappa_{1_{\chi}} = \kappa_{\chi}$ are the bi-free cumulant functions of [1, Definition 5.2].

For a two-faced family $z = ((z_i)_{i \in I}, (z_j)_{j \in J})$, $\alpha: \{1, \dots, n\} \rightarrow I \sqcup J$, and $\pi \in \text{BNC}(\alpha)$ we denote

$$\phi_{\pi}(z) := \phi_{\pi}(z_{\alpha(1)}, \dots, z_{\alpha(n)}) \quad \text{and} \quad \kappa_{\pi}(z) := \kappa_{\pi}(z_{\alpha(1)}, \dots, z_{\alpha(n)}).$$

In particular, $\phi_{1_{\alpha}}(z) = \phi_{\alpha}(z)$ and $\kappa_{1_{\alpha}}(z) = \kappa_{\alpha}(z)$. When the faces consist of a single element each, say z_{ℓ} and z_r , we define the above quantities for $\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}$ replacing α . In this case we let $m_z, \kappa_z \in \text{IA}(\text{BNC})$ be the multiplicative functions with multiplicative nets $(\phi_{\chi}(z))_{\chi}$ and $(\kappa_{\chi}(z))_{\chi}$, respectively. We call m_z the *moment function* and κ_z the *bi-free cumulant function*. Thus, the formulae $m_z * \mu_{\text{BNC}} = \kappa_z$ and $\kappa_z * \zeta_{\text{BNC}} = m_z$ are obtained.

4 Unifying Bi-free Independence

4.1 Computing Bi-free Moments

We will demonstrate how the partitions of $\text{BNC}(\chi, \epsilon)$ may be used to compute joint moments of a bi-free pair of two-faced families.

Fix $\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}$ and a shading $\epsilon \in \{', ''\}^n$, and let $T_k \in \mathcal{L}(\mathcal{X}^{\epsilon_k})$. Given $D \in \text{LR}(\chi, \epsilon)$, we will assign a vector weight $\psi(D; T_1, \dots, T_n) \in \mathcal{X}$ to D . Define $\mu \in \{\lambda, \rho\}^n$ by $\mu_j = \lambda$ if $\chi(j) = \ell$ and $\mu_j = \rho$ if $\chi(j) = r$. Let $V = \{k_1 < \dots < k_r\}$ be a block in D and let $\epsilon(V) := \epsilon_{k_1} = \dots = \epsilon_{k_r}$. If the spine of V is not connected to the top gap, then V contributes a scalar factor of

$$\psi(V; T_1, \dots, T_n) := \psi^{\epsilon(V)}(T_{k_1}(1 - p^{\epsilon(V)})T_{k_2} \dots (1 - p^{\epsilon(V)})T_{k_r} \xi^{\epsilon(V)})$$

to $\psi(D; T_1, \dots, T_n)$. If the spine does reach the top gap, then it contributes a vector factor of

$$\psi(V; T_1, \dots, T_n) := (1 - p^{\epsilon(V)})T_{k_1}(1 - p^{\epsilon(V)})T_{k_2} \dots (1 - p^{\epsilon(V)})T_{k_r} \xi^{\epsilon(V)}.$$

Then $\psi(D; T_1, \dots, T_n)$ is the product of the scalar factors and the tensor product of the vector factors where the order in the tensor product is determined by the left to right order of the spines reaching the top gap. If all contributions are scalar factors then we multiply this with the state-vector ξ , thinking of it as the “empty tensor word.”

Recalling Example 2.5.2, we see that

$$\psi(E_3; T_1, T_2, T_3) = \psi'(T_1(1 - p')T_2\xi')\psi''(T_3\xi'')\xi,$$

while

$$\psi(E_8; T_1, T_2, T_3) = (1 - p')T_2\xi' \otimes (1 - p'')T_3\xi'' \otimes (1 - p')T_1\xi'.$$

Proposition 4.1.1 Fix $\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}$ and a shading $\epsilon \in \{', ''\}^n$. Let $\mu \in \{\lambda, \rho\}^n$ be as above. If $T_j \in \mathcal{L}(\mathcal{X}^{\epsilon_j})$ for $j = 1, \dots, n$, then

$$(4.1) \quad \mu_1(T_1) \dots \mu_n(T_n)\xi = \sum_{D \in \text{LR}(\chi, \epsilon)} \psi(D; T_1, \dots, T_n).$$

Moreover,

$$(4.2) \quad \phi(\mu_1(T_1) \cdots \mu_n(T_n)) = \sum_{\pi \in \text{BNC}(\chi)} \left[\sum_{\substack{\sigma \in \text{BNC}(\chi, \epsilon) \\ \sigma \succeq_{\text{lat}} \pi}} (-1)^{|\pi| - |\sigma|} \right] \phi_{\pi}(T_1, \dots, T_n).$$

Proof We establish (4.1) via induction on n . The base case is clear, so we assume the formula holds for $n - 1$ operators and apply it as

$$\mu_2(T_2) \cdots \mu_n(T_n) \xi = \sum_{D \in LR(\chi_0, \epsilon_0)} \psi(D; T_2, \dots, T_n),$$

where $\chi_0 = \chi|_{\{2, \dots, n\}}$ and $\epsilon_0 = (\epsilon_2, \dots, \epsilon_n)$. Fix a $D \in LR(\chi_0, \epsilon_0)$ and assume $\mu_1 = \lambda$. Either there is a leftmost spine in D of the shade ϵ_1 reaching the top gap, or there is not (meaning either the nearest spine is the wrong shade or that D has no spines reaching the top gap). In the former case, writing $\psi(D; T_2, \dots, T_n)$ as $x_1 \otimes \cdots \otimes x_m$, this implies $x_1 \in \mathcal{X}^{\epsilon_1}$. Hence,

$$\begin{aligned} & \lambda(T_1)x_1 \otimes \cdots \otimes x_m \\ &= \psi(T_1(1 - p^{\epsilon_1})x_1)x_2 \otimes \cdots \otimes x_m + (1 - p^{\epsilon_1})T_1(1 - p^{\epsilon_1})x_1 \otimes x_2 \otimes \cdots \otimes x_m \\ &= \psi(D_1; T_1, \dots, T_n) + \psi(D_2; T_1, \dots, T_n), \end{aligned}$$

where $D_1, D_2 \in LR(\chi, \epsilon)$ are the diagrams constructed from D by adding a rib and, respectively, terminating the leftmost spine in D at the new top node or extending the leftmost spine in D .

If there is no leftmost spine of the same shade as ϵ_1 , then $\psi(D; T_2, \dots, T_n)$ can be written in the same way as before except $x_1 \notin \mathcal{X}^{\epsilon_1}$ (if D has no spines reaching the top gap then this is simply a scalar multiple of ξ). Hence

$$\begin{aligned} \lambda(T_1)x_1 \otimes \cdots \otimes x_m &= \psi^{\epsilon_1}(T_1 \xi^{\epsilon_1})x_1 \otimes \cdots \otimes x_m + (1 - p^{\epsilon_1})T_1 \xi^{\epsilon_1} \otimes x_1 \otimes \cdots \otimes x_m \\ &= \psi(E_1; T_1, \dots, T_n) + \psi(E_2; T_1, \dots, T_n), \end{aligned}$$

where $E_1, E_2 \in LR(\chi, \epsilon)$ are the diagrams constructed from D by, respectively, leaving the new top node isolated or adding a new rib and spine.

Since every $D \in LR(\chi, \epsilon)$ is constructed from exactly one diagram in $LR(\chi_0, \epsilon_0)$, we have

$$\lambda(T_1)\mu_2(T_2) \cdots \mu_n(T_n) \xi = \sum_{D \in LR(\chi, \epsilon)} \psi(D; T_1, \dots, T_n).$$

The case $\mu_1 = \rho$ is exactly the same upon replacing “leftmost” with “rightmost” and the considerations about x_1 with ones about x_m .

Now, $\phi(\mu_1(T_1) \cdots \mu_n(T_n))$ is given by applying ψ to the left side of (4.1). So only the terms on the right whose vector parts are ξ will survive, that is, the terms corresponding to $E \in LR_0(\chi, \epsilon)$. Fix such a diagram and let $\sigma \in \text{BNC}(\chi, \epsilon)$ be the corresponding partition. We examine

$$\psi(E; T_1, \dots, T_n) = \prod_{W \in \sigma} \psi(W; T_1, \dots, T_n).$$

For $W = \{l_1 < \dots < l_s\} \in \sigma$, we have

$$\begin{aligned} \psi(W; T_1, \dots, T_n) &= \psi^{\epsilon(W)}(T_{l_1}(1 - p^{\epsilon(W)})T_{l_2} \dots (1 - p^{\epsilon(V)})T_{l_s} \xi^{\epsilon(V)}) \xi \\ &= \sum_{1 \leq q_1 < \dots < q_m \leq s-1} (-1)^m \phi^{\epsilon(W)}(T_{l_1} \dots T_{l_{q_1}}) \dots \phi^{\epsilon(V)}(T_{l_{q_{m+1}}} \dots T_{l_s}) \xi. \end{aligned}$$

Each term in the last sum corresponds to a lateral refinement $\pi_W = \{V_1, \dots, V_{m+1}\}$ of W , weighted by $(-1)^{|\pi_W| - |W|}$. As any lateral refinement of σ is simply a collection of lateral refinements of its individual blocks, we see that $\pi = \bigcup_{W \in \sigma} \pi_W$ is a lateral refinement of σ . The overall weight associated to π is $\prod_{W \in \sigma} (-1)^{|\pi_W| - |W|} = (-1)^{|\pi| - |\sigma|}$. Thus we obtain

$$\psi(E; T_1, \dots, T_n) = \sum_{\substack{\pi \in \text{BNC}(\chi) \\ \pi \leq_{\text{lat}} \sigma}} (-1)^{|\pi| - |\sigma|} \phi_{\pi}(T_1, \dots, T_n).$$

Summing over $E \in LR_0(\chi, \epsilon)$ (or equivalently $\sigma \in \text{BNC}(\chi, \epsilon)$) and reversing the order of the two summations yields (4.2). ■

Corollary 4.1.2 *Let z' and z'' be a pair of two-faced families in (\mathcal{A}, ϕ) . Then z' and z'' are bi-free if and only if for every map $\alpha: \{1, \dots, n\} \rightarrow I \sqcup J$ and $\epsilon \in \{', ''\}^n$ we have*

$$(4.3) \quad \phi_{\alpha}(z^{\epsilon}) = \sum_{\pi \in \text{BNC}(\alpha)} \left[\sum_{\substack{\sigma \in \text{BNC}(\alpha, \epsilon) \\ \sigma \geq_{\text{lat}} \pi}} (-1)^{|\pi| - |\sigma|} \right] \phi_{\pi}(z^{\epsilon}),$$

where $z^{\epsilon} = (z_{\alpha(1)}^{\epsilon_1}, \dots, z_{\alpha(n)}^{\epsilon_n})$.

Proof If z' and z'' are bi-free then this immediately follows by applying the previous proposition to the representation guaranteed by the definition of bi-freeness.

Conversely, suppose z' and z'' satisfy (4.3) for each α and ϵ . As in the proof of [5, Proposition 2.9], we consider the universal representations of z' and z'' . That the joint representation in their free product is the same as the joint representation of z' and z'' follows precisely from (4.3). ■

4.2 Summation Considerations

For $\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}$, $\epsilon \in \{', ''\}^n$, and $\pi \in \text{BNC}(\chi)$, we will write $\pi \leq \epsilon$ where we think of ϵ as the induced partition in $\mathcal{P}(n)$.

Proposition 4.2.1 *Let $\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}$ and $\epsilon \in \{', ''\}^n$. Then for every $\pi \in \text{BNC}(\chi)$ such that $\pi \leq \epsilon$,*

$$\sum_{\substack{\sigma \in \text{BNC}(\chi, \epsilon) \\ \sigma \geq_{\text{lat}} \pi}} (-1)^{|\pi| - |\sigma|} = \sum_{\substack{\sigma \in \text{BNC}(\chi) \\ \pi \leq \sigma \leq \epsilon}} \mu_{\text{BNC}}(\pi, \sigma).$$

To prove Proposition 4.2.1 we will appeal to free probability to handle the following case and reduce all others to it.

Lemma 4.2.2 *Let $\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}$ with $\chi \equiv \ell$ and $\epsilon \in \{', ''\}^n$. Then for every $\pi \in \text{BNC}(\chi)$ such that $\pi \leq \epsilon$,*

$$\sum_{\substack{\sigma \in \text{BNC}(\chi, \epsilon) \\ \sigma \geq_{\text{lat}} \pi}} (-1)^{|\pi| - |\sigma|} = \sum_{\substack{\sigma \in \text{BNC}(\chi) \\ \pi \leq \sigma \leq \epsilon}} \mu_{\text{BNC}}(\pi, \sigma).$$

Proof Let $\{X'_1, \dots, X'_n\}$ and $\{X''_1, \dots, X''_n\}$ be freely independent sets. Note by [5, Proposition 2.15b] these sets can be viewed as a bi-free pair of two faced families X' and X'' with trivial right faces. Hence, by Corollary 4.1.2,

$$\phi(X_1^{\epsilon_1} \cdots X_n^{\epsilon_n}) = \sum_{\pi \in \text{BNC}(\chi)} \left(\sum_{\substack{\sigma \in \text{BNC}(\chi, \epsilon) \\ \sigma \geq_{\text{lat}} \pi}} (-1)^{|\pi| - |\sigma|} \right) \phi_{\pi}(X_1^{\epsilon_1}, \dots, X_n^{\epsilon_n}).$$

Since $\chi \equiv \ell$, $\text{BNC}(\chi) = \text{NC}(n)$. Thus, since $\{X'_1, \dots, X'_n\}$ and $\{X''_1, \dots, X''_n\}$ are free,

$$\begin{aligned} \phi(X_1^{\epsilon_1} \cdots X_n^{\epsilon_n}) &= \sum_{\sigma \in \text{BNC}(\chi)} \kappa_{\sigma}(X_1^{\epsilon_1}, \dots, X_n^{\epsilon_n}) \\ &= \sum_{\substack{\sigma \in \text{BNC}(\chi) \\ \sigma \leq \epsilon}} \kappa_{\sigma}(X_1^{\epsilon_1}, \dots, X_n^{\epsilon_n}) \\ &= \sum_{\substack{\sigma \in \text{BNC}(\chi) \\ \sigma \leq \epsilon}} \sum_{\substack{\pi \in \text{BNC}(\chi) \\ \pi \leq \sigma}} \mu(\pi, \sigma) \phi_{\pi}(X_1^{\epsilon_1}, \dots, X_n^{\epsilon_n}) \\ &= \sum_{\substack{\pi \in \text{BNC}(\chi) \\ \pi \leq \epsilon}} \left(\sum_{\substack{\sigma \in \text{BNC}(\chi) \\ \pi \leq \sigma \leq \epsilon}} \mu(\pi, \sigma) \right) \phi_{\pi}(X_1^{\epsilon_1}, \dots, X_n^{\epsilon_n}). \end{aligned}$$

Since these expressions agree for any selection of $\{X'_1, \dots, X'_n\}$ and $\{X''_1, \dots, X''_n\}$ that are freely independent, by selecting $\{X'_1, \dots, X'_n\}$ and $\{X''_1, \dots, X''_n\}$ that are free and such that $\phi_{\pi}(X_1^{\epsilon_1}, \dots, X_n^{\epsilon_n})$ is non-zero for precisely one π , the desired sums are obtained to be equal (as $\mu = \mu_{\text{BNC}}$ in this setting). ■

We will use Lemma 4.2.2 to show that the desired equations in Proposition 4.2.1 hold. To do so, we will show that an arbitrary bi-non-crossing partition can be obtained by a sequence of steps, preserving the summations in Proposition 4.2.1, applied to a partition with all left nodes.

Lemma 4.2.3 *Let $\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}$ with $\chi(n) = \ell$, $\epsilon \in \{', ''\}^n$, and $\pi \in \text{BNC}(\chi)$ be such that $\pi \leq \epsilon$. Let $\widehat{\chi}: \{1, \dots, n\} \rightarrow \{\ell, r\}$ be such that*

$$\widehat{\chi}(t) = \begin{cases} \chi(t) & \text{if } t \neq n, \\ r & \text{if } t = n, \end{cases}$$

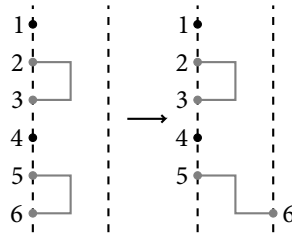
and let $\widehat{\pi} \in \text{BNC}(\widehat{\chi})$ be the unique shaded bi-non-crossing partition with the same blocks as π (note $\widehat{\pi} \leq \epsilon$ by construction). Then

$$\sum_{\substack{\sigma \in \text{BNC}(\chi, \epsilon) \\ \sigma \geq_{\text{lat}} \pi}} (-1)^{|\pi| - |\sigma|} = \sum_{\substack{\widehat{\sigma} \in \text{BNC}(\widehat{\chi}, \epsilon) \\ \widehat{\sigma} \geq_{\text{lat}} \widehat{\pi}}} (-1)^{|\widehat{\pi}| - |\widehat{\sigma}|}$$

and

$$\sum_{\substack{\sigma \in \text{BNC}(\chi) \\ \pi \leq \sigma \leq \epsilon}} \mu_{\text{BNC}}(\pi, \sigma) = \sum_{\substack{\widehat{\sigma} \in \text{BNC}(\widehat{\chi}) \\ \widehat{\pi} \leq \widehat{\sigma} \leq \epsilon}} \mu_{\text{BNC}}(\widehat{\pi}, \widehat{\sigma}).$$

Proof It is clear that the operator that takes an element $\sigma \in \text{BNC}(\chi, \epsilon)$ and constructs an element $\widehat{\sigma} \in \text{BNC}(\widehat{\chi}, \epsilon)$ with the same blocks as σ corresponds to taking the bottom node of σ that is on the left and placing this node on the right (keeping all strings connected). For example, consider the following diagrams.



Such an operation is clearly a bijection, maps $\text{BNC}(\chi, \epsilon)$ to $\text{BNC}(\widehat{\chi}, \epsilon)$, $(-1)^{|\pi|-|\sigma|} = (-1)^{|\widehat{\pi}|-|\widehat{\sigma}|}$, and $\sigma \geq_{\text{lat}} \pi$ if and only if $\widehat{\sigma} \geq_{\text{lat}} \widehat{\pi}$. Hence, the first equation holds. Similarly, by Remarks 3.1.4, it is clear that the second equation holds. ■

Lemma 4.2.4 Let $\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}$ be such that there exists a $k \in \{1, \dots, n-1\}$ such that $\chi(k) = \ell$ and $\chi(k+1) = r$, $\epsilon \in \{', ''\}^n$, and let $\pi \in \text{BNC}(\chi)$ be such that $\pi \leq \epsilon$. Fix $k \in \{1, \dots, n-1\}$. Let $\widehat{\epsilon} \in \{', ''\}^n$ be such that

$$\widehat{\epsilon}_t = \begin{cases} \epsilon_t & \text{if } t \notin \{k, k+1\}, \\ \epsilon_k & \text{if } t = k+1, \\ \epsilon_{k+1} & \text{if } t = k, \end{cases}$$

let $\widehat{\chi}: \{1, \dots, n\} \rightarrow \{\ell, r\}$ be such that

$$\widehat{\chi}(t) = \begin{cases} \chi(t) & \text{if } t \notin \{k, k+1\}, \\ \chi(k) & \text{if } t = k+1, \\ \chi(k+1) & \text{if } t = k, \end{cases}$$

and let $\widehat{\pi} \in \text{BNC}(\widehat{\chi})$ be the unique shaded bi-non-crossing partition obtained by interchanging k and $k+1$ in π (note $\widehat{\pi} \leq \widehat{\epsilon}$ by construction). Then

$$\sum_{\substack{\sigma \in \text{BNC}(\chi, \epsilon) \\ \sigma \geq_{\text{lat}} \pi}} (-1)^{|\pi|-|\sigma|} = \sum_{\substack{\widehat{\sigma} \in \text{BNC}(\widehat{\chi}, \widehat{\epsilon}) \\ \widehat{\sigma} \geq_{\text{lat}} \widehat{\pi}}} (-1)^{|\widehat{\pi}|-|\widehat{\sigma}|}$$

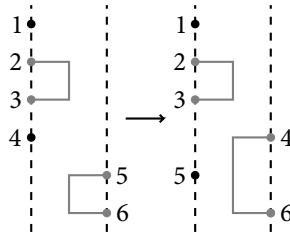
and

$$\sum_{\substack{\sigma \in \text{BNC}(\chi) \\ \pi \leq \sigma \leq \epsilon}} \mu_{\text{BNC}}(\pi, \sigma) = \sum_{\substack{\widehat{\sigma} \in \text{BNC}(\widehat{\chi}) \\ \widehat{\pi} \leq \widehat{\sigma} \leq \epsilon}} \mu_{\text{BNC}}(\widehat{\pi}, \widehat{\sigma}).$$

Proof Since the operation that takes an element $\sigma \in \text{BNC}(\chi)$ with $\sigma \leq \epsilon$ and produces an element $\widehat{\sigma} \in \text{BNC}(\widehat{\chi})$ with $\widehat{\sigma} \leq \widehat{\epsilon}$ by interchanging k and $k+1$ in σ is a

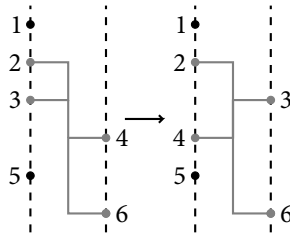
bijection, and since $\mu_{\text{BNC}}(\pi, \sigma) = \mu_{\text{BNC}}(\widehat{\pi}, \widehat{\sigma})$ by Remarks 3.1.4, the second equation clearly holds.

To prove the first equation holds, we break the discussion into several cases. For the first case, suppose $\epsilon_k \neq \epsilon_{k+1}$; that is, the nodes we desire to change the orders of are of different shades. For example, see the following diagrams where $k = 4$.



In this case, it is clear that the operation described above that takes $\sigma \in \text{BNC}(\chi)$ to $\widehat{\sigma} \in \text{BNC}(\widehat{\chi})$ is a bijection that maps $\text{BNC}(\chi, \epsilon)$ to $\text{BNC}(\widehat{\chi}, \widehat{\epsilon})$, is such that $(-1)^{|\pi|-|\sigma|} = (-1)^{|\widehat{\pi}|-|\widehat{\sigma}|}$, and is such that $\sigma \geq_{\text{lat}} \pi$ if and only if $\widehat{\sigma} \geq_{\text{lat}} \widehat{\pi}$. Hence, the first equation holds in this case.

Otherwise, $\epsilon_k = \epsilon_{k+1}$. Suppose k and $k + 1$ are in the same block of π . For example, consider the following diagrams where $k = 3$.



It is again clear that the same identifications hold as the previous case, and thus the first equation holds in this case. Hence, we have reduced to the case that k and $k + 1$ are in different blocks of the same shade.

Let V_1 and V_2 be the blocks in π of k and $k + 1$ respectively. Note that V_1 contains a left node and V_2 contains a right node and the sum on the left-hand side of the first equation is

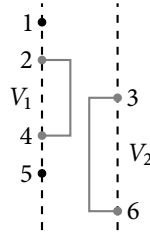
$$\sum_{\substack{\sigma \in \text{BNC}(\chi, \epsilon) \\ \sigma \geq_{\text{lat}} \pi \\ k, k+1 \text{ in separated blocks of } \sigma}} (-1)^{|\pi|-|\sigma|} + \sum_{\substack{\sigma \in \text{BNC}(\chi, \epsilon) \\ \sigma \geq_{\text{lat}} \pi \\ k, k+1 \text{ not in separated blocks of } \sigma}} (-1)^{|\pi|-|\sigma|}.$$

We claim that

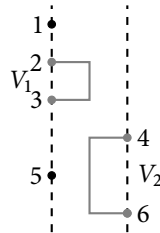
$$\sum_{\substack{\sigma \in \text{BNC}(\chi, \epsilon) \\ \sigma \geq_{\text{lat}} \pi \\ k, k+1 \text{ not in separated blocks of } \sigma}} (-1)^{|\pi|-|\sigma|} = 0.$$

Indeed, we will split the discussion into two cases: when V_1 and V_2 are piled and when they are not. For an example where V_1 and V_2 are piled, consider the following

diagram.



If V_1 and V_2 are piled, it is easy to see that any $\sigma \in \text{BNC}(\chi, \epsilon)$ such that $\pi \leq \sigma$ and k and $k + 1$ are not in separated blocks of σ must be such that V_1 and V_2 are contained in the same block of σ . However, this implies that π is not a lateral refinement of σ , as joining piled blocks cannot be undone by a lateral refinement. Hence the sum is zero in this case. Otherwise, suppose V_1 and V_2 are not piled. For an example where V_1 and V_2 are not piled, consider the following diagram.



This implies that k is the lowest element of V_1 in the bi-non-crossing diagram of π and $k + 1$ is the highest element of V_2 . If $\sigma \in \text{BNC}(\chi, \epsilon)$ is such that k and $k + 1$ are not in separated blocks of σ and $\sigma \geq \pi$, then if k and $k + 1$ are in the same block of σ , let $\sigma' \leq_{\text{lat}} \sigma$ splitting the block containing k and $k + 1$ in between these nodes (note $\sigma' \in \text{BNC}(\chi, \epsilon)$). Otherwise, k and $k + 1$ are not in the same block of σ , so letting $\sigma' \geq_{\text{lat}} \sigma$ be the partition made by joining the blocks containing k and $k + 1$ together also forms a partition in $\text{BNC}(\chi, \epsilon)$. In either case $(-1)^{|\pi|-|\sigma|} + (-1)^{|\pi|-|\sigma'|} = 0$. Note that the correspondance between σ and σ' in each case is one-to-one, and thus the sum is zero.

Similar arguments show that

$$\sum_{\substack{\widehat{\sigma} \in \text{BNC}(\widehat{\chi}, \widehat{\epsilon}) \\ \widehat{\sigma} \geq_{\text{lat}} \widehat{\pi}}} (-1)^{|\widehat{\pi}|-|\widehat{\sigma}|} = \sum_{\substack{\widehat{\sigma} \in \text{BNC}(\widehat{\chi}, \widehat{\epsilon}) \\ \widehat{\sigma} \geq_{\text{lat}} \widehat{\pi} \\ k, k+1 \text{ in separated blocks of } \widehat{\sigma}}} (-1)^{|\widehat{\pi}|-|\widehat{\sigma}|}.$$

However, the map taking $\sigma \in \text{BNC}(\chi)$ to $\widehat{\sigma} \in \text{BNC}(\widehat{\chi})$ is such that k and $k + 1$ are in separated blocks of σ if and only if k and $k + 1$ are in separated blocks of $\widehat{\sigma}$, and under these conditions $\sigma \in \text{BNC}(\chi, \epsilon)$ if and only if $\widehat{\sigma} \in \text{BNC}(\widehat{\chi}, \widehat{\epsilon})$, $\sigma \geq_{\text{lat}} \pi$ if and only if $\widehat{\sigma} \geq_{\text{lat}} \widehat{\pi}$, and $(-1)^{|\pi|-|\sigma|} = (-1)^{|\widehat{\pi}|-|\widehat{\sigma}|}$. Hence the first equation holds in this final case. ■

Proof of Proposition 4.2.1 Given π , a $\widehat{\pi}$ in $\text{BNC}(\widehat{\chi})$ where $\widehat{\chi}: \{1, \dots, n\} \rightarrow \{\ell\}$ can be constructed such that $\widehat{\pi}$ can be modified to make π via the operations in used in Lemmas 4.2.3 and 4.2.4. Since the sums are equal for $\widehat{\pi}$ by Lemma 4.2.2, and since Lemmas 4.2.3 and 4.2.4 preserve the equality of the sums, the result hold for π . ■

We apply Proposition 4.2.1 to Corollary 4.1.2 to immediately obtain the following corollary.

Corollary 4.2.5 *Let z' and z'' be a pair of two-faced families in (\mathcal{A}, ϕ) . Then z' and z'' are bi-free if and only if for every map $\alpha: \{1, \dots, n\} \rightarrow I \sqcup J$ and $\epsilon \in \{', ''\}^n$ we have*

$$\phi_\alpha(z^\epsilon) = \sum_{\pi \in \text{BNC}(\alpha)} \left[\sum_{\substack{\sigma \in \text{BNC}(\alpha) \\ \pi \leq \sigma \leq \epsilon}} \mu_{\text{BNC}}(\pi, \sigma) \right] \phi_\pi(z^\epsilon),$$

where $z^\epsilon = (z_{\alpha(1)}^{\epsilon_1}, \dots, z_{\alpha(n)}^{\epsilon_n})$.

4.3 Bi-free is Equivalent to Combinatorially-bi-free

Theorem 4.3.1 *Let $z' = ((z'_i)_{i \in I}, (z'_j)_{j \in J})$ and $z'' = ((z''_i)_{i \in I}, (z''_j)_{j \in J})$ be a pair of two-faced families in a non-commutative probability space (\mathcal{A}, ϕ) . Then z' and z'' are bi-free if and only if they are combinatorially-bi-free.*

Proof Suppose z' and z'' are bi-free, and fix a shading $\epsilon \in \{', ''\}^n$. By Corollary 4.2.5, for $\alpha: \{1, \dots, n\} \rightarrow I \sqcup J$ we have

$$\phi_\alpha(z^\epsilon) = \sum_{\pi \in \text{BNC}(\alpha)} \left(\sum_{\substack{\sigma \in \text{BNC}(\alpha) \\ \pi \leq \sigma \leq \epsilon}} \mu_{\text{BNC}}(\pi, \sigma) \right) \phi_\pi(z^\epsilon).$$

Therefore

$$\phi_\alpha(z^\epsilon) = \sum_{\substack{\sigma \in \text{BNC}(\alpha) \\ \sigma \leq \epsilon}} \kappa_\sigma(z^\epsilon)$$

by Remark 3.1.5. Using the above formula, we will proceed inductively to show that $\kappa_\sigma(z^\epsilon) = 0$ if $\sigma \in \text{BNC}(\alpha)$ and $\sigma \not\leq \epsilon$. The base case is where $n = 1$ is immediate.

For the inductive case, suppose the result holds for any $\beta: \{1, \dots, k\} \rightarrow I \sqcup J$ with $k < n$. Let $\alpha: \{1, \dots, n\} \rightarrow I \sqcup J$. Suppose ϵ is not constant (so, in particular, $1_\alpha \not\leq \epsilon$). Then

$$\sum_{\sigma \in \text{BNC}(\alpha)} \kappa_\sigma(z^\epsilon) = \phi_\alpha(z^\epsilon) = \sum_{\substack{\sigma \in \text{BNC}(\alpha) \\ \sigma \leq \epsilon}} \kappa_\sigma(z^\epsilon).$$

By induction, $\kappa_\sigma(z^\epsilon) = 0$ if $\sigma \in \text{BNC}(\alpha) \setminus \{1_\alpha\}$ and $\sigma \not\leq \epsilon$. Consequently,

$$\sum_{\sigma \in \text{BNC}(\alpha)} \kappa_\sigma(z^\epsilon) = \kappa_{1_\alpha}(z^\epsilon) + \sum_{\substack{\sigma \in \text{BNC}(\alpha) \\ \sigma \leq \epsilon}} \kappa_\sigma(z^\epsilon).$$

Combining these two equations gives $\kappa_{1_\alpha}(z^\epsilon) = 0$, completing the inductive step.

Now suppose that z' and z'' are combinatorially-bi-free. Then, for any $\epsilon \in \{', ''\}^n$ and $\alpha: \{1, \dots, n\} \rightarrow I \sqcup J$,

$$\begin{aligned} \phi_\alpha(z^\epsilon) &= \sum_{\sigma \in \text{BNC}(\alpha)} \kappa_\sigma(z^\epsilon) = \sum_{\substack{\sigma \in \text{BNC}(\alpha) \\ \sigma \leq \epsilon}} \kappa_\sigma(z^\epsilon) \\ &= \sum_{\substack{\sigma \in \text{BNC}(\alpha) \\ \sigma \leq \epsilon}} \sum_{\substack{\pi \in \text{BNC}(\alpha) \\ \pi \leq \sigma}} \phi_\pi(z^\epsilon) \mu_{\text{BNC}}(\pi, \sigma) \\ &= \sum_{\pi \in \text{BNC}(\alpha)} \left(\sum_{\substack{\sigma \in \text{BNC}(\alpha) \\ \pi \leq \sigma \leq \epsilon}} \mu_{\text{BNC}}(\pi, \sigma) \right) \phi_\pi(z^\epsilon). \end{aligned}$$

Hence, Corollary 4.2.5 implies that z' and z'' are bi-free. ■

4.4 Voiculescu’s Universal Bi-free Polynomials

Using the equivalence of bi-free independence and combinatorial-bi-free independence, we obtain explicit formulas for several universal polynomials appearing in [5].

Proposition 4.4.1 *Let $\alpha: \{1, \dots, n\} \rightarrow I \sqcup J$. For each shading $\epsilon \in \{', ''\}^n$ we define a polynomial $P_{\alpha, \epsilon}$ on indeterminates X'_K and X''_K indexed by non-empty subsets $K \subset \{1, \dots, n\}$ by the formula*

$$P_{\alpha, \epsilon} := \sum_{\pi \in \text{BNC}(\alpha, \epsilon)} \left[\sum_{\substack{\sigma \in \text{BNC}(\alpha) \\ \pi \leq \sigma \leq \epsilon}} \mu_{\text{BNC}}(\pi, \sigma) \right] \prod_{V \in \pi} X_V^{\epsilon(V)}.$$

Then for z' and z'' a bi-free pair of two-faced families in (\mathcal{A}, ϕ) we have

$$\phi_\alpha(z^\epsilon) = P_{\alpha, \epsilon}(z', z''),$$

where $P_{\alpha, \epsilon}(z', z'')$ is given by evaluating $P_{\alpha, \epsilon}$ at $X_{\{k_1 < \dots < k_r\}}^\delta = \phi(z_{\alpha(k_1)}^\delta \cdots z_{\alpha(k_r)}^\delta)$, $\delta \in \{', ''\}$.

Furthermore, if we define Q_α as the sum of the $P_{\alpha, \epsilon}$ over all possible shadings then

$$Q_\alpha = X'_{\{1, \dots, n\}} + X''_{\{1, \dots, n\}} + \sum P_{\alpha, \epsilon},$$

where the summation is over non-constant shadings ϵ , and

$$\phi_\alpha(z' + z'') = Q_\alpha(z', z''),$$

where $Q_\alpha(z', z'')$ is Q_α evaluated at the same point as the $P_{\alpha, \epsilon}$ above.

Proof The first part of this corollary is immediate from Corollary 4.2.5. The assertion regarding $Q_\alpha(z', z'')$ is also immediate when expanding the product in the left-hand side. All that remains to show is

$$Q_\alpha = X'_{\{1, \dots, n\}} + X''_{\{1, \dots, n\}} + \sum P_{\alpha, \epsilon},$$

which is equivalent to saying $P_{\alpha,\epsilon} = X_{\{1,\dots,n\}}^\delta$ when ϵ is the constant shading $\epsilon = (\delta, \dots, \delta)$, $\delta \in \{', ''\}$. Such a shading induces the full partition 1_α , and hence

$$\sum_{\substack{\sigma \in \text{BNC}(\alpha) \\ \pi \leq \sigma \leq \epsilon}} \mu_{\text{BNC}}(\pi, \sigma) = \sum_{\substack{\sigma \in \text{BNC}(\alpha) \\ \pi \leq \sigma \leq 1_\alpha}} \mu_{\text{BNC}}(\pi, \sigma) = \delta_{\text{BNC}}(\pi, 1_\alpha).$$

Then the only term in $P_{\alpha,\epsilon}$ with a non-zero coefficient is the one corresponding to $\pi = 1_\alpha$. ■

Proposition 4.4.2 For any $\alpha: \{1, \dots, n\} \rightarrow I \sqcup J$, recursively define polynomials R_α on indeterminates X_K indexed by non-empty subsets $K \subseteq \{1, \dots, n\}$ by the formula

$$R_\alpha = \sum_{\pi \in \text{BNC}(\alpha)} \mu_{\text{BNC}}(\pi, 1_\alpha) \prod_{V \in \pi} X_V.$$

If X_K is given degree $|K|$, then R_α is homogeneous with degree n .

For z a two-faced family in (\mathcal{A}, ϕ) , if $R_\alpha(z)$ denotes R_α evaluated at the point

$$X_{\{k_1 < \dots < k_r\}} = \phi(z_{\alpha(k_1)} \cdots z_{\alpha(k_r)})$$

then $R_\alpha(z) = \kappa_\alpha(z)$. Moreover, if z' and z'' are bi-free in (\mathcal{A}, ϕ) , then $R_\alpha(z' + z'') = R_\alpha(z') + R_\alpha(z'')$; that is, R_α has the cumulant property.

Proof We see that $R_\alpha(z)$ and $\kappa_\alpha(z)$ are equal by Remark 3.1.5. Then R_α has the cumulant property simply because κ_α does. ■

Remark 4.4.3 The polynomials $P_{\alpha,\epsilon}$, Q_α , and R_α are precisely the universal polynomials from [5], Propositions 2.18, 5.6, and Theorem 5.7, respectively.

5 A Multiplicative Bi-free Convolution

5.1 Kreweras Complement on Bi-non-crossing Partitions

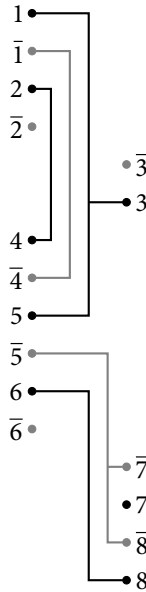
In [3], the Kreweras complement K_{NC} on the non-crossing partitions was used to simplify the convolution of multiplicative functions. In particular, we have the following extension to BNC.

Definition 5.1.1 For any $\chi: \{1, \dots, n\} \rightarrow I \sqcup J$ and $\pi \in \text{BNC}(\chi)$, the Kreweras complement of π , denoted $K_{\text{BNC}}(\pi)$, is the element of $\text{BNC}(\chi)$ obtained by applying s_χ to the Kreweras complement in $\text{NC}(n)$ of $s_\chi^{-1} \cdot \pi$; explicitly,

$$K_{\text{BNC}}(\pi) = s_\chi \cdot K_{\text{NC}}(s_\chi^{-1} \cdot \pi).$$

Remark 5.1.2 Note that $K_{\text{BNC}}(\pi)$ can be obtained by taking the diagram corresponding to π , placing a node beneath each left node and above each right node of π , and drawing the largest bi-non-crossing diagram on the new nodes.

Example 5.1.3 In the following diagram, if π is the bi-non-crossing partition drawn in black, $K_{\text{BNC}}(\pi)$ is the bi-non-crossing partition in grey.



Remark 5.1.4 Since K_{BNC} is an order reversing and s_χ is order preserving, K_{BNC} is an order reversing bijection. Thus $[\pi, 1_\alpha] \simeq [K_{\text{BNC}}(1_\alpha), K_{\text{BNC}}(\pi)] = [0_\alpha, K_{\text{BNC}}(\pi)]$ for all $\pi \in \text{BNC}(\alpha)$. Hence, if $f, g \in \text{IA}(\text{BNC})$ are multiplicative functions, then

$$(f * g)(0_\alpha, 1_\alpha) = \sum_{\pi \in \text{BNC}(\alpha)} f(0_\alpha, \pi)g(0_\alpha, K_{\text{BNC}}(\pi)) = (g * f)(0_\alpha, 1_\alpha),$$

and thus $f * g = g * f$.

5.2 Computing Cumulants of a Multiplicative Bi-free Convolution

Taking inspiration from [3], we use the Kreweras complement to examine the bi-free cumulants of a two-faced family generated by products of a bi-free pair of two-faced families.

Theorem 5.2.1 Let $z' = (\{z'_\ell\}, \{z'_r\})$ and $z'' = (\{z''_\ell\}, \{z''_r\})$ be a bi-free family of pairs of faces and let $z = (\{z'_\ell z''_\ell\}, \{z'_r z''_r\})$. Then

$$\kappa_\chi(z) = \sum_{\pi \in \text{BNC}(\chi)} \kappa_\pi(z') \kappa_{K_{\text{BNC}}(\pi)}(z'')$$

for all $\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}$.

Proof Recall from Remark 3.1.5 the definition of the moment and bi-free cumulant functions m_x and κ_x , and, moreover, that these are uniquely determined by the moments and cumulants of the family (x) , respectively. Since the bi-free cumulant functions are multiplicative and by the structure of the convolution of multiplicative

functions given in Remark 5.1.4, it suffices to show $\kappa_z = \kappa_{z'} * \kappa_{z''}$. Using the relations $m_z * \mu_{\text{BNC}} = \kappa_z$ and $\kappa_z * \zeta_{\text{BNC}} = m_z$, it suffices to show $m_z = \kappa_{z'} * m_{z''}$.

Suppose $\chi: \{1, \dots, n\} \rightarrow \{\ell, r\}$. Let $\beta: \{1, \dots, 2n\} \rightarrow \{\ell, r\}$ be given by $\beta(2k-1) = \beta(2k) = \chi(k)$. Take $\epsilon \in \{', ''\}^{2n}$ so that $\epsilon_{2k-1} = '$ and $\epsilon_{2k} = ''$ if $k \in \chi^{-1}(\ell)$, and the opposite if $k \in \chi^{-1}(r)$. Then

$$\begin{aligned} m_z(0_\chi, 1_\chi) &= \phi_\chi(z) = \phi(z_{\chi(1)}^{\epsilon_1} z_{\chi(1)}^{\epsilon_2} \cdots z_{\chi(n)}^{\epsilon_{2n-1}} z_{\chi(n)}^{\epsilon_{2n}}) \\ &= \phi(z_{\beta(1)}^{\epsilon_1} z_{\beta(2)}^{\epsilon_2} \cdots z_{\beta(2n-1)}^{\epsilon_{2n-1}} z_{\beta(2n)}^{\epsilon_{2n}}) \\ &= \sum_{\pi \in \text{BNC}(\beta, \epsilon)} \kappa_\pi(z^\epsilon) \\ &= \sum_{\pi_1 \in \text{BNC}(\chi)} \kappa_{\pi_1}(z') \sum_{\substack{\pi_2 \in \text{BNC}(\chi) \\ \pi_2 \leq K_{\text{BNC}}(\pi_1)}} \kappa_{\pi_2}(z'') \\ &= \sum_{\pi_1 \in \text{BNC}(\chi)} \kappa_{\pi_1}(z') \phi_{K_{\text{BNC}}(\pi_1)}(z'') \\ &= (\kappa_{z'} * m_{z''})(0_\chi, 1_\chi) \end{aligned}$$

Hence, as m_z and $\kappa_{z'} * m_{z''}$ are multiplicative functions that agree on full lattices in BNC, and consequently on all intervals by bi-multiplicativity. ■

Remark 5.2.2 Note that the above generalizes the formula for the free cumulants of the multiplicative convolution of freely independent random variables in terms of their individual cumulants (cf. [3, Section 3.5]). This seems to suggest that when defining the multiplicative convolution of a bi-free pair of two-faced families, one should multiply the right faces as if in the opposite algebra.

Remark 5.2.3 Since convolution is abelian on multiplicative functions, we obtain that $(\{z'_\ell z''_\ell\}, \{z'_r z''_r\})$ and $(\{z''_\ell z'_\ell\}, \{z''_r z'_r\})$ have the same joint distributions.

6 An Operator Model for Pairs of Faces

In this section we will construct an operator model for a two-faced family in a non-commutative probability space. This model will generalize the operator model usually considered in free probability introduced by Nica in [2].

In [2, Definition 3.2.1], Nica’s operator model is constructed via unbounded operators on a Fock space making use of the left creation and annihilation operators where each product of creation operators is weighted by a free cumulant of the random variables. The operator model for a pair of faces in a non-commutative probability space will be constructed in Theorem 6.4.1, with terms similarly weighted by the corresponding bi-free cumulants. We use left annihilation operators in the same way as Nica’s model, though we must use more complex operators than simply left and right creation, essentially to account for the fact that the order in which variables are annihilated does not correspond to the order in which they were added as strongly as in the free case. Our model reduces to Nica’s model when all variables are left (or right) variables. Moreover, a model using only left and right creation and annihilation operators is unlikely, by discussions in [1].

Nica’s operator model also gives a direct analogue to the R -series of a collection of random variables in a non-commutative probability space. We introduce an analogous operator Θ_z in Theorem 6.4.1, which serves as the R -series of the two-faced family $z = (\{z_i\}_{i \in I}, \{z_j\}_{j \in J})$. In particular, if

$$z' = (\{z'_i\}_{i \in I}, \{z'_j\}_{j \in J}) \quad \text{and} \quad z'' = (\{z''_i\}_{i \in I}, \{z''_j\}_{j \in J})$$

is a bi-free pair of two-faced families, we can consider the single family

$$z = (\{z'_i + z''_i\}_{i \in I}, \{z'_j + z''_j\}_{j \in J})$$

and construct the corresponding operator Θ_z . It will follow that

$$\Theta_z - I = (\Theta_{z'} - I) + (\Theta_{z''} - I).$$

Hence, the operator Θ_z from Theorem 6.4.1 behaves like an R -series.

6.1 Nica’s Operator Model

We will take a moment to recall Nica’s operator model from [1] to demystify our construction. Given an index set I , let $\mathcal{F}(\mathbb{C}^I)$ be the Fock space generated by I ; that is,

$$\mathcal{F}(\mathbb{C}^I) := \mathbb{C}\Omega \oplus \left(\bigoplus_{\substack{k \geq 1 \\ i_1, \dots, i_k \in I}} \mathbb{C}(e_{i_1} \otimes \dots \otimes e_{i_k}) \right),$$

where $\{e_i\}_{i \in I}$ is an orthonormal basis of \mathbb{C}^I . Recall that Ω is called the vacuum vector of $\mathcal{F}(\mathbb{C}^I)$, and is thought of as a “length zero tensor”; that is, for example, $\xi_1 \otimes \dots \otimes \xi_k = \Omega$ if $k = 0$. To simplify notation, we will sometimes join impure tensors of varying lengths together with a tensor product symbol, which should be distributed across sums. For example, $(\xi_1 + \xi_2 \otimes \xi_3) \otimes \xi_4 = \xi_1 \otimes \xi_4 + \xi_2 \otimes \xi_3 \otimes \xi_4$, while $\xi \otimes \Omega = \xi = \Omega \otimes \xi$.

Recall next that the left creation operator corresponding to e_i , denoted L_i , is defined by $L_i(\xi_1 \otimes \dots \otimes \xi_k) = e_i \otimes \xi_1 \otimes \dots \otimes \xi_k$, while its adjoint L_i^* is called the left annihilation operator corresponding to e_i . We take $\omega: \mathcal{L}(\mathcal{F}(\mathbb{C}^I)) \rightarrow \mathbb{C}$ to be the vector state corresponding to Ω , so $\omega(T) := \langle T\Omega, \Omega \rangle$.

Suppose that $X = (X_i)_{i \in I}$ is a collection of random variables in a non-commutative probability space (\mathcal{A}, ϕ) , and consider the (unbounded) operator

$$\Theta_X := I_{\mathcal{F}(\mathbb{C}^I)} + \sum_{k \geq 1} \sum_{i_1, \dots, i_k \in I} \kappa(X_{i_1}, \dots, X_{i_k}) L_{i_k} \cdots L_{i_1},$$

where κ is the free cumulant functional. Next, set

$$Z_i := L_i^* \Theta_X = L_i^* + \sum_{k \geq 0} \sum_{i_1, \dots, i_k \in I} \kappa(X_{i_1}, \dots, X_{i_k}, X_i) L_{i_k} \cdots L_{i_1}.$$

Then the joint distribution of $(Z_i)_{i \in I}$ with respect to ω is the same as that of $(X_i)_{i \in I}$ with respect to ϕ . Observe that given i_1, \dots, i_n , there is a bijection between the non-crossing partitions $\text{NC}(n)$ and the terms in $Z_{i_1} \cdots Z_{i_n}$ of non-zero trace: one takes the finest partition such that if the annihilation operator of Z_{i_t} cancels a creation operator added by a term from the variable Z_{i_s} , then t and s lie in the same block.

Example 6.1.1 Consider the product $T_1 T_2 T_3 T_4 T_5 = Z_1 Z_2 Z_3 Z_3 Z_1$, which contains the term of non-zero trace

$$\kappa(X_2, X_3) \kappa(X_1, X_3, X_1) L_1^* L_2^* (L_3^* L_3 L_2) L_3^* (L_1^* L_1 L_3 L_1).$$

This corresponds to the non-crossing partition $\{\{1, 4, 5\}\{2, 3\}\}$, since the term $\kappa(X_2, X_3) L_3^* L_3 L_2$ was selected from T_3 , while the surviving L_2 was annihilated by T_2 ; this accounts for the block $\{2, 3\}$. Similarly, $\kappa(X_1, X_3, X_1) L_1^* L_1 L_3 L_1$ was added by T_5 , and its remaining pieces were annihilated by T_4 and T_1 , which gives us the block $\{1, 4, 5\}$.

On the other hand, to find the term corresponding to the non-crossing partition $\{\{1, 5\}, \{2, 4\}, \{3\}\}$, we note that T_5 must introduce an operator annihilated by T_1 , T_4 must for T_2 , and T_3 annihilate any term it adds. That is, we have the term

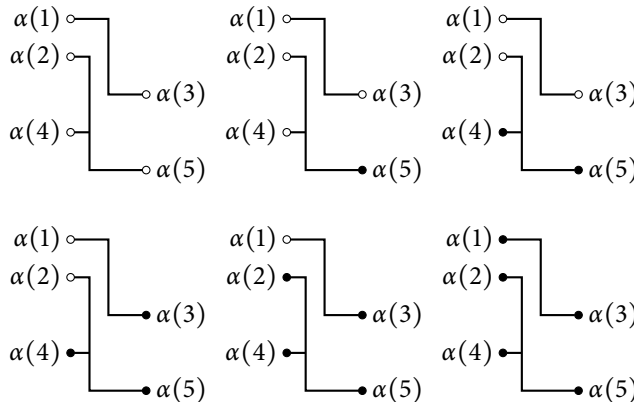
$$L_1^* L_2^* (\kappa(X_3) L_3^* L_3) (\kappa(X_2, X_3) L_3^* L_3 L_2) (\kappa(X_1, X_1) L_1^* L_1 L_1).$$

6.2 Skeletons Corresponding to Bi-non-crossing Partitions

The operator model from [2] can be thought of as a systematic way of constructing all non-crossing partitions weighted by products of free cumulants. Recall that non-crossing partitions can be viewed as bi-non-crossing partitions where all nodes are on the left-hand side.

Definition 6.2.1 Let $\alpha: \{1, \dots, n\} \rightarrow I \sqcup J$. For a bi-non-crossing partition $\pi \in \text{BNC}(\alpha)$, a *skeleton* on π is a bi-non-crossing diagram of π (as in Subsection 2.4), labelled by α , with a choice of each node being either closed or open subject to the constraint that any node below a closed node is also closed.

Example 6.2.2 If α and π are as in Example 2.4.1, the skeletons corresponding to π are the following diagrams.



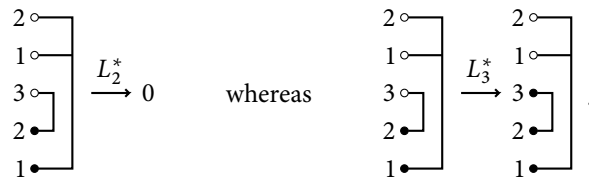
Definition 6.2.3 We will refer to a skeleton where all nodes are closed circles as the *completed skeleton*. For a skeleton on $1_\alpha \in \text{BNC}(\alpha)$, the skeleton where all nodes are open will be referred to as the *empty skeleton* corresponding to α , while the skeleton where all but the bottom node is open will be referred to as the *starter skeleton*

corresponding to α . Any skeleton that is not empty will be referred to as a *partially completed skeleton*.

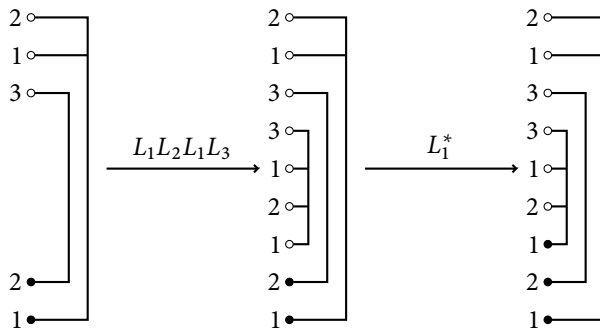
Remark 6.2.4 We will examine Nica’s model in the language of skeletons, which we will think of as a bi-free situation where all variables come from the left face. Let $\{X_i\}_{i \in I}$ be a family of non-commutative random variables, $\{Z_i\}_{i \in I}$ the corresponding operator model, and fix some $i_1, \dots, i_n \in I$ and consider a product $(L_{i_1}^* T_1) \cdots (L_{i_n}^* T_n)$ where

$$T_k \in \{I\} \cup \{\kappa(X_{i'_1}, \dots, X_{i'_m})L_{i'_m} \cdots L_{i'_1} \mid m \geq 1, i'_1, \dots, i'_m \in I\}.$$

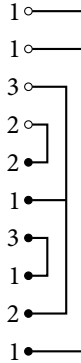
Note that $L_{i_k}^* T_k = 0$ unless the T_k chosen is either I or begins with L_{i_k} . For $t \leq n$, we think of $(L_{i_t}^* T_t) \cdots (L_{i_n}^* T_n)\Omega$ as a partially completed skeleton weighted by a scalar which is a product of free cumulants. There is no bijection between partially completed skeletons and basis vectors of our Fock space as the partially completed skeleton will retain the information of how the vector was created. Each annihilation operator acts on the skeleton by filling in the lowest open node if it is labelled appropriately (to make the node closed in the new skeleton), and otherwise weights the skeleton by zero (which removes the skeleton from consideration). Note, then, that the closed nodes correspond to variables in the block which have been encountered, and the requirement that they be filled from bottom to top ensures that the ordering of variables matches the cumulant. For example,



Each product of creation operators $\kappa(X_{i'_1}, \dots, X_{i'_m})L_{i'_m} \cdots L_{i'_1}$ adds an empty skeleton (corresponding to the creation operators chosen) to the skeleton under consideration directly above the highest closed node, and is weighted by the appropriate cumulant. The lowest node of the new block is immediately filled by the following $L_{i_k}^*$. For example,



For a product $(L_{i_1}^* T_1) \cdots (L_{i_n}^* T_n) \Omega$, we will get precisely one partially completed skeleton. For example,



corresponds to the product

$$\begin{aligned} & (\kappa(X_2, X_2)L_2)L_1^*L_3^* (\kappa(X_3, X_1)L_1^*L_1L_3) \\ & (\kappa(X_3, X_1, X_2)L_2^*L_2L_1L_3) (\kappa(X_1, X_1, X_1)L_1^*L_1L_1L_1) \Omega. \end{aligned}$$

Notice that when the above operators are applied to Ω in the order listed, we obtain the vector $\lambda e_2 \otimes e_3 \otimes e_1 \otimes e_1$, where λ is a product of cumulants. The indices of the tensor can be seen in the partially completed skeleton by reading the open nodes from bottom to top. In this manner, vectors $(L_{i_1}^* T_1) \cdots (L_{i_n}^* T_n) \Omega$ correspond to partially completed skeletons and the only products such that

$$\langle (L_{i_1}^* T_1) \cdots (L_{i_n}^* T_n) \Omega, \Omega \rangle \neq 0$$

arise from completed skeletons. It is easy to see that a completed skeleton corresponds to an element of $\pi \in \text{NC}(n)$. These completed skeletons are weighted by the correct product of cumulants so that when we sum over all completed skeletons, we get

$$\langle Z_{i_1} \cdots Z_{i_n} \Omega, \Omega \rangle = \sum_{\pi \in \text{NC}(n)} \kappa_{\pi}(X_{i_1}, \dots, X_{i_n}) = \phi(X_{i_1} \cdots X_{i_n}),$$

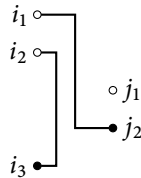
as desired.

6.3 A Construction

We will now construct our operator model for pairs of faces, motivated by our realization of Nica’s operator model. Above, the model constructed all weighted non-crossing partitions by using creation operators to glue in full non-crossing blocks and annihilation operators to approve or reject non-crossing diagrams. As the combinatorics of pairs of faces is dictated by bi-non-crossing partitions, we must construct the appropriate creation operators to glue together bi-non-crossing partitions. However, unlike with non-crossing partitions where there is only one way to glue in a full block at any given point, there may be multiple or no ways to glue one bi-non-crossing skeleton into another. As such, the description of the appropriate creation operators is more complicated.

Let $z = ((z_i)_{i \in I}, (z_j)_{j \in J})$ be a two-faced family in (\mathcal{A}, ϕ) . As before, consider the Fock space $\mathcal{H} := \mathcal{F}(\mathbb{C}^{I \sqcup J})$ with $\{e_k\}_{k \in I \sqcup J}$ an orthonormal basis.

For $\alpha: \{1, \dots, n\} \rightarrow I \sqcup J$, we will define operators $T_\alpha \in \mathcal{L}(\mathcal{H})$ that should be thought of as playing the same roll as the operators T_k in our discussion of Nica's model; that is, each adds an appropriate empty skeleton. Though we will often speak of actions of these operators in terms of their actions on skeletons, one can return to the context of \mathcal{H} by letting a partially completed skeleton correspond to the vector formed by taking the tensor product of the basis elements matching the labels of its open nodes, from bottom to top, and weighting it based on which cumulants have been chosen. For example, the skeleton

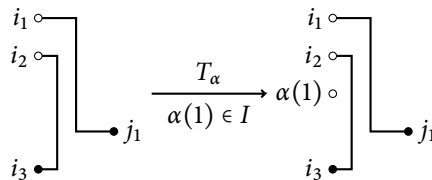


corresponds to the vector $e_{j_1} \otimes e_{i_2} \otimes e_{i_1}$ and will be weighted by

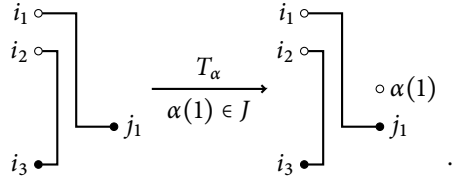
$$\kappa(z_{i_2}, z_{i_3})\kappa(z_{i_1}, z_{j_2})\kappa(z_{j_1}).$$

The key point here is that the only choices of future Z_k that yield a non-zero Ω component when applied to such a vector have annihilation operators in the correct order. In the above example, in order for this skeleton to make a contribution to the final term, we must act on it by Z_{j_1} , Z_{i_2} , and Z_{i_1} , in that order (though other variables may occur between them). Since the closed nodes of the skeleton only effect the resulting quantity in terms of its weight and cannot affect the action of future operators (as indeed they must not, for the vector has forgotten them), we will sometimes truncate diagrams of skeletons to show only the open nodes. It is implied that there may be significantly more nodes and blocks below the bottom of the diagrams that follow, but their representation is eschewed. Likewise, in order to ensure that T_α is well defined, we cannot have behaviour depending on which partial skeletons have been chosen, but only the choice of side and of labels of the open nodes.

For $n = 1$, we define $T_\alpha := L_{\alpha(1)}$. In this setting, one may think of T_α as adding an empty skeleton in the lowest possible position with a single open node on the left or on the right depending on whether $\alpha(1)$ is in I or J . For example,



and



Observe that T_α adds an open node in the lowest valid location (*i.e.*, immediately above all closed nodes); this behaviour will be mimicked by the other T_α as well. That is, the lowest open node added will always be added directly above the highest closed node.

Let $\Sigma: \mathcal{H} \oplus \mathcal{H} \rightarrow \mathcal{H}$ be defined by

$$\Sigma(f_1 \otimes \dots \otimes f_n, f_{n+1} \otimes \dots \otimes f_{n+m}) := \sum_{\sigma} f_{\sigma(1)} \otimes \dots \otimes f_{\sigma(n+m)},$$

where the sum is over all permutations $\sigma \in S_{n+m}$ so that $\sigma|_{[1,n]}$ and $\sigma|_{[n+1,n+m]}$ are increasing; that is, σ interleaves the sets $\{1, \dots, n\}$ and $\{n+1, \dots, n+m\}$. Note that $\Sigma(\xi, \Omega) = \xi = \Sigma(\Omega, \xi)$. As an example,

$$\begin{aligned} \Sigma(e_1 \otimes e_2, e_3 \otimes e_4) &= e_1 \otimes e_2 \otimes e_3 \otimes e_4 + e_1 \otimes e_3 \otimes e_2 \otimes e_4 + e_3 \otimes e_1 \otimes e_2 \otimes e_4 \\ &\quad + e_1 \otimes e_3 \otimes e_4 \otimes e_2 + e_3 \otimes e_1 \otimes e_4 \otimes e_2 + e_3 \otimes e_4 \otimes e_1 \otimes e_2. \end{aligned}$$

We will use Σ to account for the fact that nodes on the right may be added with any order to nodes on the left to obtain a valid skeleton.

For $\alpha: \{1, \dots, n\} \rightarrow I \sqcup J$, we define

$$T_\alpha(\Omega) := L_{\alpha(n)} L_{\alpha(n-1)} \dots L_{\alpha(1)}(\Omega) = e_{\alpha(1)} \otimes \dots \otimes e_{\alpha(n)}.$$

Note that this corresponds to taking a completed skeleton (possibly with no nodes), and adding the empty skeleton corresponding to α above it.

We will now define T_α for $n \geq 2$ on tensor products of basis elements, and extend it by linearity to their span (which is dense \mathcal{H}). We consider only the case $\alpha(n) \in I$, as the case when $\alpha(n) \in J$ will be similar. Let $\eta = e_{\beta(m)} \otimes \dots \otimes e_{\beta(1)} \in \mathcal{H}$, where $\beta: \{1, \dots, m\} \rightarrow I \sqcup J$.

If $\beta^{-1}(I) = \emptyset$, we define $T_\alpha(\eta)$ as follows. Let $k = \max(\{0\} \cup \alpha^{-1}(J))$, so that k is the lowest of the nodes to be added by which falls on the right. We define $T_\alpha(\eta)$ as follows:

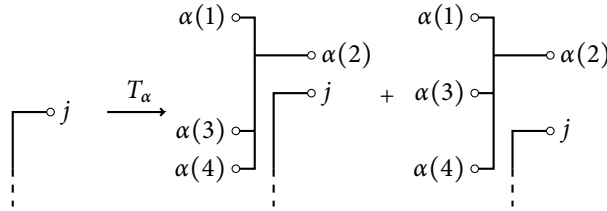
$$T_\alpha(\eta) := e_{\alpha(n)} \otimes \Sigma(e_{\alpha(n-1)} \otimes \dots \otimes e_{\alpha(k+1)}, e_{\beta(m)} \otimes \dots \otimes e_{\beta(1)}) \otimes e_{\alpha(k)} \otimes \dots \otimes e_{\alpha(1)}.$$

This is mimicking the action of adding a new skeleton to the existing skeleton. In order to ensure that no crossings are introduced, all new nodes on the right must be placed above all existing nodes on the right. Before any new right nodes are added, though, nodes on the left can be added freely. One should think of this as the sum of all valid partially completed skeletons where the old skeleton is below and to the right of starter skeleton corresponding to α , with the node corresponding to $\alpha(n)$ in the lowest possible position.

Example 6.3.1 If $\alpha: \{1, 2, 3, 4\} \rightarrow I \sqcup J$ satisfies $\alpha^{-1}(I) = \{1, 3, 4\}$ and $\alpha^{-1}(J) = \{2\}$, and $j \in J$, then

$$T_\alpha(e_j) = e_{\alpha(4)} \otimes e_{\alpha(3)} \otimes e_j \otimes e_{\alpha(2)} \otimes e_{\alpha(1)} + e_{\alpha(4)} \otimes e_j \otimes e_{\alpha(3)} \otimes e_{\alpha(2)} \otimes e_{\alpha(1)}.$$

This action corresponds to the following diagram:

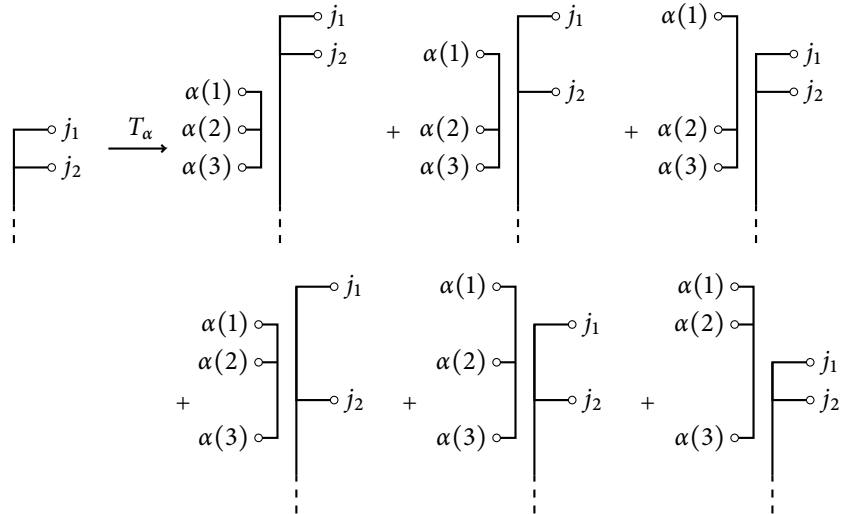


The purpose of allowing multiple diagrams is that the cumulant corresponding to a bi-non-crossing diagram for a sequence of operators is equal to the same cumulant for the sequence of operators obtained by interchanging the k -th and $(k+1)$ -th operators and the k -th and $(k+1)$ -th nodes in the bi-non-crossing diagram, provided k and $k+1$ are in different blocks and on different sides of the diagram. In the end, a sequence of annihilation operators can complete at most one skeleton and will produce the correct completed skeleton for a given sequence of operators.

As a further example, suppose $\alpha: \{1, 2, 3\} \rightarrow I$ and $j_1, j_2 \in J$. Then

$$\begin{aligned} T_\alpha(e_{j_2} \otimes e_{j_1}) &= e_{\alpha(3)} \otimes e_{\alpha(2)} \otimes e_{\alpha(1)} \otimes e_{j_2} \otimes e_{j_1} + e_{\alpha(3)} \otimes e_{\alpha(2)} \otimes e_{j_2} \otimes e_{\alpha(1)} \otimes e_{j_1} \\ &\quad + e_{\alpha(3)} \otimes e_{\alpha(2)} \otimes e_{j_2} \otimes e_{j_1} \otimes e_{\alpha(1)} + e_{\alpha(3)} \otimes e_{j_2} \otimes e_{\alpha(2)} \otimes e_{\alpha(1)} \otimes e_{j_1} \\ &\quad + e_{\alpha(3)} \otimes e_{j_2} \otimes e_{\alpha(2)} \otimes e_{j_1} \otimes e_{\alpha(1)} + e_{\alpha(3)} \otimes e_{j_2} \otimes e_{j_1} \otimes e_{\alpha(2)} \otimes e_{\alpha(1)}. \end{aligned}$$

This action corresponds to the following diagram:



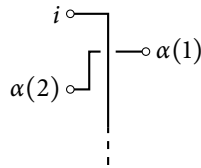
Now, suppose that $\beta^{-1}(I) \neq \emptyset$, and let $k = \max(\beta^{-1}(I))$. This corresponds to a partially completed skeleton with open nodes on both the left and right, where the

lowest open node on the left is the k -th from the top. We set $T_\alpha(\eta) = 0$ if $\alpha(t) \in J$ for some t , since the partially completed skeleton has open nodes on the left and right, we cannot add the empty skeleton of α without introducing a crossing, since the lowest node of α is on the left. Otherwise, $\alpha(t) \in I$ for all $t \in \{1, \dots, n\}$, and we set

$$T_\alpha(\eta) := e_{\alpha(n)} \otimes \Sigma(e_{\alpha(n-1)} \otimes \dots \otimes e_{\alpha(1)}, e_{\beta(m)} \otimes \dots \otimes e_{\beta(k+1)}) \otimes e_{\beta(k)} \otimes \dots \otimes e_{\beta(1)}.$$

One can think of this as the sum of all valid partially completed skeletons where the empty skeleton of α sits below the lowest open node on the left of the old skeleton.

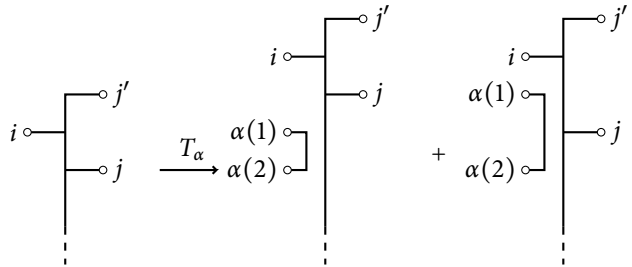
Example 6.3.2 If $\alpha: \{1, 2\} \rightarrow I \sqcup J$ has $\alpha(2) \in I$, $\alpha(1) \in J$ and $i \in I$, then $T_\alpha(e_i) = 0$. This is because there is no way to glue the empty skeleton corresponding to α into the partially completed skeleton without introducing a crossing while placing the lowest node of α at the bottom of the diagram (directly above the highest closed node):



If $\alpha: \{1, 2\} \rightarrow I$ and $i \in I, j, j' \in J$, then

$$T_\alpha(e_j \otimes e_i \otimes e_{j'}) = e_{\alpha(2)} \otimes e_{\alpha(1)} \otimes e_j \otimes e_i \otimes e_{j'} + e_{\alpha(2)} \otimes e_j \otimes e_{\alpha(1)} \otimes e_i \otimes e_{j'}.$$

This action corresponds to the following diagram:



As T_α has been defined on an orthonormal basis, we may extend it by linearity to obtain a densely defined operator on \mathcal{H} ; note that T_α may not be bounded due to the action of Σ . On the other hand, if $\alpha: \{1, \dots, n\} \rightarrow I$, then T_α acts on the Fock subspace generated by $\{e_i\}_{i \in I}$ as $L_{\alpha(n)} \cdots L_{\alpha(1)}$. Thus, if one considers only left variables, the resulting operators are precisely those of Nica’s model.

We define T_α in a similar manner when $\alpha(n) \in J$.

6.4 The Operator Model for Pairs of Faces

With the above construction, the operator model for a pair of faces is at hand.

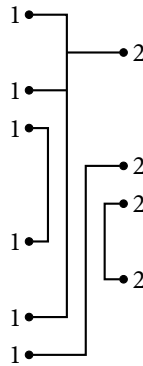
Theorem 6.4.1 Let $z = (\{z_i\}_{i \in I}, \{z_j\}_{j \in J})$ be a pair of faces in a non-commutative probability space (\mathcal{A}, ϕ) . With notation as in Construction 6.3, consider the (unbounded) operator

$$\Theta_z := I + \sum_{n \geq 1} \sum_{\alpha: \{1, \dots, n\} \rightarrow I \sqcup J} \kappa_\alpha(z) T_\alpha,$$

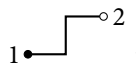
and for $k \in I \sqcup J$, set $Z_k := L_k^* \Theta_z$. If $T \in \text{alg}(\{Z_k\}_{k \in I \sqcup J})$, then $\langle T\Omega, \Omega \rangle$ is well defined. Moreover, if $\omega(T) = \langle T\Omega, \Omega \rangle$, the joint distribution of $\{Z_k\}_{k \in I \sqcup J}$ with respect to ω is the same as the joint distribution of z with respect to ϕ .

Before we begin the proof, we give the following example.

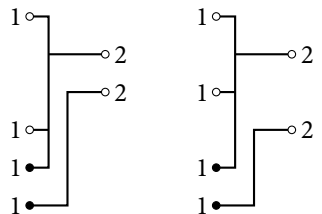
Example 6.4.2 In this example, let $I = \{1\}$ and $J = \{2\}$. We will examine how the completed skeleton below is constructed for $Z_1 Z_2 Z_1 Z_1 Z_2 Z_2 Z_1 Z_2 Z_1 Z_1$.



First $\kappa_{(21)}(z)L_1^* T_{(21)}$ is applied to get the partially completed skeleton

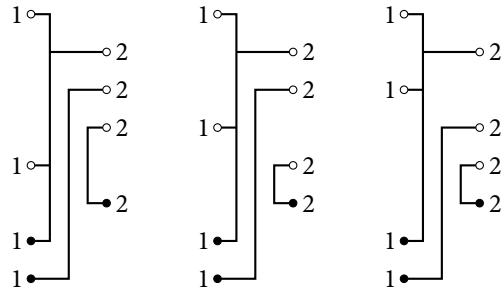


Then $\kappa_{(1211)}(z)L_1^* T_{(1211)}$ is applied to obtain the following collection of partially completed skeletons

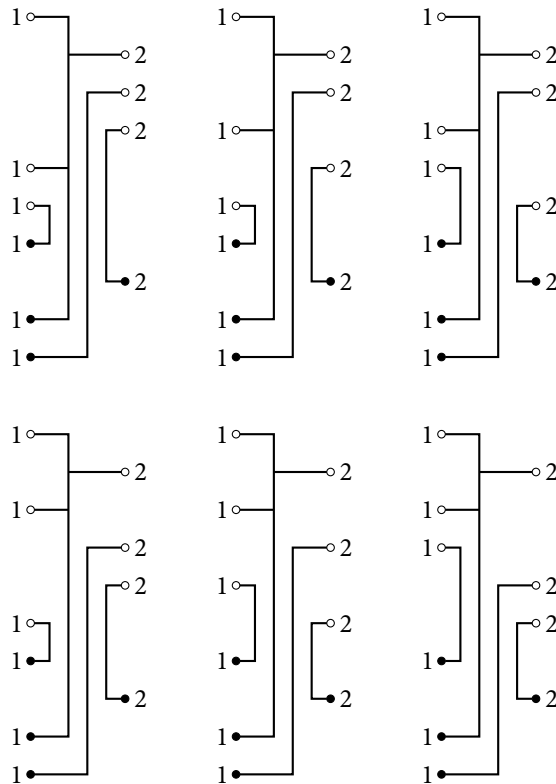


Applying $\kappa_{(22)}L_2^* T_{(22)}$ then gives the following collection of partially completed skeletons (where the first two below are from the first above and the third below is from

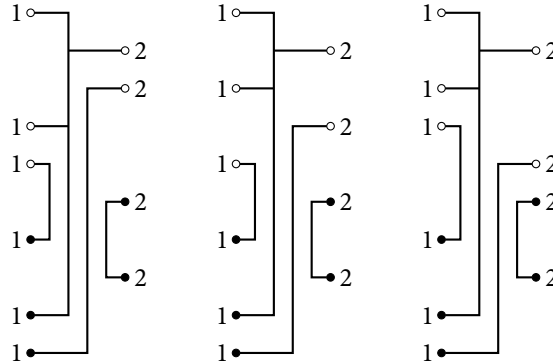
the second above)



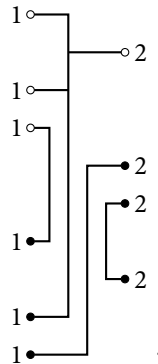
and applying $\kappa_{(11)}L_1^*T_{(11)}$ then gives the following collection of partially completed skeletons (where the first below is from the first above, the second and third are from the second above, and the last three are from the third above).



Applying L_2^* then gives the following collection of partially completed skeletons (where the first, second, and fourth diagrams above were destroyed)



and applying L_2^* removes all but the last diagram to give



Applying $L_1^* L_2^* L_1^* L_1^*$ then gives us the desired diagram. We also see the diagram was weighted by

$$\kappa_{(21)}(z)\kappa_{(1211)}(z)\kappa_{(22)}\kappa_{(11)}(z)$$

which is the correct product of bi-free cumulants for this bi-non-crossing partition.

Proof of Theorem 6.4.1 Let $\alpha: \{1, \dots, n\} \rightarrow I \sqcup J$. To see that

$$\omega(Z_{\alpha(1)} \cdots Z_{\alpha(n)}) = \phi(z_{\alpha(1)} \cdots z_{\alpha(n)}),$$

we must demonstrate that the sum of over all

$$A_k \in \{L_{\alpha(k)}^*\} \cup \{\kappa_\beta(z)L_{\alpha(k)}^*T_\beta \mid \beta: \{1, \dots, m\} \rightarrow I \sqcup J\}$$

of $\langle A_1 \cdots A_n \Omega, \Omega \rangle$ is precisely $\phi(z_{\alpha(1)} \cdots z_{\alpha(n)})$. (Note that $L_{\alpha(k)}^*T_\beta = 0$ unless $\beta(m) = \alpha(k)$.) This suffices, as these are precisely the terms that appear in expanding the product $Z_{\alpha(1)} \cdots Z_{\alpha(n)}$. By construction, $A_1 \cdots A_n$ acting on Ω corresponds to creating a (sequence of) partially completed skeletons and $\langle A_1 \cdots A_n \Omega, \Omega \rangle$ will be the weight of the skeleton if the skeleton is complete and otherwise will be zero. Since

$$\phi(z_{\alpha(1)} \cdots z_{\alpha(n)}) = \sum_{\pi \in \text{BNC}(\alpha)} \kappa_\pi(z),$$

it suffices to show that there is a bijection between completed skeletons and elements π of $\text{BNC}(\alpha)$, and that the weight of the skeleton is the corresponding cumulant.

Observe that after A_k is applied, the bottom $n - k + 1$ nodes of the partially completed skeleton will be closed, as A_k itself either closed an open node that was already present or added a new block containing one closed node and zero or more open nodes. In particular, the $(n - k + 1)$ -th node from the bottom must be on the side corresponding to $\alpha(k)$, since it was closed by $L_{\alpha(k)}^*$. Thus when we have applied $A_1 \cdots A_n$, any skeleton surviving has precisely n nodes and structure arising from α .

From a bi-non-crossing partition $\pi \in \text{BNC}(\alpha)$, we can recover the choice of A_1, \dots, A_n that produces it. To do so, for each block $V = \{k_1 < \dots < k_t\}$, we let $A_{k_i} = L_{k_i}^*$ for $i \neq t$, and if $\beta(i) = \alpha(k_i)$, we set $A_{k_t} = \kappa_\beta(z)L_{k_t}^*T_\beta$. Indeed, the partially created skeletons created by $A_k \cdots A_n$ agree with π on the bottom $n - k + 1$ nodes. Moreover, given any other product $A'_1 \cdots A'_n$ that differs from $A_1 \cdots A_n$, consider the greatest index k so that $A'_k \neq A_k$. Then all partially completed skeletons in $A'_k \cdots A'_n$ and $A_k \cdots A_n$ agree in structure for their bottom $n - k$ nodes, while the next either starts a new block in one case but not the other or starts new blocks of different shapes. Finally, note that if β corresponds to the block $V \in \pi$ as above, then $\kappa_\beta(z) = \kappa_{\pi|_V}(z)$ and so the total weight on the skeleton is precisely $\kappa_\pi(z)$. ■

Remark 6.4.3 In [5, Theorem 7.4], an operator model for the bi-free central limit distributions was given as sums of creation and annihilation operators on a Fock space. It is interesting that the operator model from Theorem 6.4.1 uses different operators. Indeed, for $i, i' \in I$ and $j \in J$, one can check that

$$T_{(i,i')} = \sum_{n \geq 0} \sum_{\alpha: \{1, \dots, n\} \rightarrow J} L_{i'} L_{\alpha(1)} \cdots L_{\alpha(n)} L_i L_{\alpha(n)}^* \cdots L_{\alpha(1)}^* \quad \text{and} \quad T_{(j,i')} = L_{i'} R_j P,$$

where P is the projection onto the Fock subspace of \mathcal{H} generated by $\{e_j\}_{j \in J}$ and R_j is the right creation operator corresponding to e_j . Therefore, if $c_{k_1, k_2} = \phi(z_{k_1} z_{k_2})$ for $k_1, k_2 \in I \sqcup J$ with z a bi-free central limit distribution, Theorem 6.4.1 produces the operators

$$Z_k = L_k^* + \sum_{k' \in I \sqcup J} c_{k', k} L_k^* T_{(k', k)},$$

which are very different from $L_k + L_k^*$ (if $k \in I$) and $R_k + R_k^*$ (if $k \in J$) proposed in [5]. The main issues with the model involving $\{L_i, L_i^*, R_j, R_j^* \mid i \in I, j \in J\}$ is that the vectors obtained by applying the algebra generated by these operators to Ω do not generate the full Fock space; indeed, they only generate vectors of the form

$$e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes e_{j_m} \otimes \cdots \otimes e_{j_1}$$

where $n, m \geq 0$, $i_1, \dots, i_n \in I$, and $j_1, \dots, j_m \in J$. It is not difficult to see that the vectors obtained by the algebra generated $\{L_i^*, L_j^*, T_{(i,i)}, T_{(j,j)} \mid i \in I, j \in J\}$ applied to Ω generate the full Fock space.

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