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Functorial Decompositions of Looped Coassociative Co-H Spaces

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Abstract. Selick and Wu gave a functorial decomposition of $\Omega\Sigma X$ for path-connected, p-local CW-complexes X which obtained the smallest nontrivial functorial retract $A^{\min}(X)$ of $\Omega\Sigma X$. This paper uses methods developed by the second author in order to extend such functorial decompositions to the loops on coassociative co-H spaces.

1 Introduction

Let X be a path-connected, p-local CW-complex. Selick and Wu [SW1, SW2] gave a functorial decomposition $\Omega\Sigma X \simeq A^{\min}(X) \times \Omega Q^{\max}(X)$, where $A^{\min}(X)$ is the minimal functorial retract whose homology contains the homology of X. This homotopy decomposition is the geometric realization of a more general algebraic result, which obtains the minimal functorial coalgebra retract of a tensor algebra. The transition from algebra to geometry is suggested by the Bott–Samelson theorem, which gives an algebra isomorphism $H_*(\Omega\Sigma X) \cong T(\widetilde{H}_*(X))$ (homology with mod-p coefficients).

The Bott–Samelson theorem can be generalized to co-H spaces. If Y is a simply connected co-H space, then there is an algebra isomorphism

$$H_*(\Omega Y) \cong T(\Sigma^{-1}\widetilde{H}_*(Y)).$$

The question arises whether Selick and Wu's functorial decomposition of $\Omega\Sigma X$ can be generalized to the case of ΩY. A question in the same spirit was addressed in [T]. There, the functorial decomposition $\Sigma\Omega\Sigma X \simeq \bigvee_{n=1}^{\infty} \Sigma X^{(n)}$ was generalized to the case of a coassociative co-H space Y. It was shown that $\Sigma\Omega Y \simeq \bigvee_{n=1}^{\infty} M_n(Y)$ for spaces $M_n(Y)$ satisfying $\Sigma^{n-1}M_n(Y) \simeq Y^{(n)}$. The purpose of this paper is to show that the homotopy decomposition of Selick and Wu generalizes to the case of a coassociative co-H space.

Theorem 1.1 *Let Y be a simply connected, homotopy coassociative co-H space. Then there is a space $MQ^{\max}(Y)$, a homotopy fibration*

$$A^{\min}(Y) \xrightarrow{*} MQ^{\max}(Y) \longrightarrow Y,$$

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and a homotopy decomposition

$$\Omega Y \simeq A^{\min}(Y) \times \Omega M Q^{\max}(Y),$$

where $H_*(A^{\min}(Y))$ is the minimal functorial coalgebra retract of

$$H_*(\Omega Y) \cong T(\Sigma^{-1}\widetilde{H}_*(Y)).$$

This homotopy decomposition is natural for co-H maps $Y \longrightarrow Z$ between homotopy coassociative co-H spaces.

The most common example of a coassociative co-H space is a suspension. However, Berstein and Harper [BH] constructed explicit examples of coassociative co-H spaces which are not suspensions. Theorem 1.1 can then be applied so that their homotopy theory can be analyzed in exactly the same manner as suspensions. In particular, letting $V = \Sigma^{-1}\widetilde{H}_*(Y)$, the minimal functorial retract $A^{\min}(Y)$ of ΩY is the geometric realization of the minimal functorial coalgebra retract $A^{\min}(V)$ of the tensor algebra $T(V)$. That is, $H_*(A^{\min}(Y)) \cong A^{\min}(V)$. One key property of $A^{\min}(V)$ proved in [SW1] is that its primitives are concentrated in the submodules of monomials of length a power of p.

This paper is organized as follows. In Section 2 we review the work of Selick and Wu which gives functorial coalgebra decompositions of tensor algebras and loop suspensions. In Section 3 we review the constructions in [T]. Selick and Wu's result depends on particular wedge decompositions of $\Sigma X^{(n)}$. The wedge summands are obtained as telescopes of self-maps of $\Sigma X^{(n)}$ which arise from the action of the symmetric group on n letters. In Section 4 we show that such wedge decompositions can be generalized to a decomposition of $M_n(Y)$ when Y is a coassociative co-H space. Finally, in Section 5 we prove Theorem 1.1.

2 Functorial Coalgebra Decompositions of Tensor Algebras and Loop Suspensions

The material in this section comes from [SW1, SW2]. Let V be a vector space over a field $\mathbf{k}$ of characteristic p. Let $T(V)$ be the tensor algebra generated by V; this becomes a Hopf algebra by letting the elements of V be primitive. Let $L_n(V)$ be the set of homogeneous Lie elements of tensor length n in $T(V)$.

Theorem 2.1 *There are functorial submodules $Q_n^{\max}(V)$ of $L_n(V)$ such that, if we define $B^{\max}(V) = T(\bigoplus_{n=2}^{\infty} Q_n^{\max}(V))$, then:*

(a) *$B^{\max}(V)$ is a sub-Hopf algebra of $T(V)$,*
(b) *$L_n(V) \subseteq B^{\max}(V)$ if n is not a power of p,*
(c) *there is a natural coalgebra decomposition $T(V) \cong A^{\min}(V) \otimes B^{\max}(V)$, where $A^{\min}(V)$ is the smallest natural coalgebra retract of $T(V)$.* ∎

Let $B_n^{\max}(V) = B^{\max} \cap T_n(V)$. The modules $Q_n^{\max}$ and $B_n^{\max}$ are realized via idempotents on $V^{\otimes n}$. Let

$$\beta_n \colon V^{\otimes n} \longrightarrow V^{\otimes n}$$

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be defined by the iterated commutator

$$\beta_n(a_1 \otimes \cdots \otimes a_n) = [a_1, [a_2, \cdots [a_{n-1}, a_n] \cdots].$$

Let Σ_n be the symmetric group on n letters. Let $\mathbb{Z}_{(p)}$ be the p-local integers. Let $\mathbb{Z}_{(p)}[\Sigma_n]$ be the group ring. We can identify β_n with the corresponding element in $\mathbb{Z}_{(p)}[\Sigma_n]$. Then there are elements $\alpha_n^{\max}, \delta_n^{\max} \in \mathbb{Z}_{(p)}[\Sigma_n]$ such that $\lambda_n^{\max} = \beta_n \circ \alpha_n^{\max}$ and $\delta_n^{\max}$ are idempotents in $\mathbb{Z}_{(p)}[\Sigma_n]$, and

$$Q_n^{\max}(V) = \operatorname{Im}(\lambda_n^{\max} \colon V^{\otimes n} \longrightarrow V^{\otimes n})$$
$$B_n^{\max}(V) = \operatorname{Im}(\delta_n^{\max} \colon V^{\otimes n} \longrightarrow V^{\otimes n}).$$

The coalgebra decomposition in Theorem 2.1 can be realized geometrically. We use homology with mod-p coefficients throughout. Recall that if X is a path connected space then $H_*(\Omega\Sigma X) \cong T(\widetilde{H}_*(X))$.

Theorem 2.2 *Let X be a path-connected p-local CW-complex. Then there are homotopy functors $Q_n^{\max}$ and $A^{\min}$ from path-connected p-local CW complexes to spaces such that the following properties hold:*

(a) *$Q_n^{\max}(X)$ is a functorial retract of $\Sigma X^{(n)}$;*

(b) *there is a functorial fiber sequence*

$$A^{\min}(X) \xrightarrow{\ *\ } \bigvee_{n=2}^{\infty} Q_n^{\max}(X) \xrightarrow{\pi_X} \Sigma X;$$

(c) *there is a functorial homotopy decomposition*

$$\Omega\Sigma X \simeq A^{\min}(X) \times \Omega\Big(\bigvee_{n=2}^{\infty} Q_n^{\max}(X)\Big);$$

(d) *there is a functorial coalgebra filtration on $H_*(A^{\min}(X))$ such that there is a functorial isomorphism of coalgebras*

$$Gr\, H_*(A^{\min}(X)) \cong A^{\min}(\widetilde{H}_*(X)). \qquad \blacksquare$$

To describe how the algebraic decomposition in Theorem 2.1 translates into the geometric decomposition in Theorem 2.2, observe that an element $\sigma \in \Sigma_n$ corresponds to a map $\sigma_n \colon X^{(n)} \longrightarrow X^{(n)}$ by permuting the factors in the smash product. Suspending, such maps can be added, so the idempotents $\lambda_n^{\max}, \delta_n^{\max} \in \mathbb{Z}_{(p)}[\Sigma_n]$ correspond to maps

$$\lambda_n^{\max}, \delta_n^{\max} \colon \Sigma X^{(n)} \longrightarrow \Sigma X^{(n)}.$$

In general, any self-map $f \colon Z \longrightarrow Z$ has a mapping telescope defined by

$$T = \Big(\coprod_{i=0}^{\infty} Z \times [i, i+1]\Big) / \sim$$

where $(z, j) \sim (f(z), j)$ for all $z \in Z$ and $j \in \mathbb{N}$. This is a special case of a homotopy colimit, so we write $T = \mathrm{hocolim}_f\ Z$. An introductory discussion of mapping telescopes can be found in [M] and a thorough treatment of homotopy colimits is given in [BK]. A key property of homotopy colimits is that they commute with the homology functor. In the case of a mapping telescope the directed system in homology induced by f_* has $\mathrm{Im}\, f_*$ as its direct limit. Thus $H_*(\mathrm{hocolim}_f\ Z) \cong \mathrm{Im}\, f_*$. Applying this to the idempotents $\lambda_n^{\max}$ and $\delta_n^{\max}$, let

$$Q_n^{\max}(X) = \mathrm{hocolim}_{\lambda_n^{\max}} \Sigma X^{(n)}$$
$$B_n^{\max}(X) = \mathrm{hocolim}_{\delta_n^{\max}} \Sigma X^{(n)}.$$

Then

$$H_*(Q_n^{\max}(X)) \cong Q_n^{\max}(\widetilde{H}_*(X))$$
$$H_*(B_n^{\max}(X)) \cong B_n^{\max}(\widetilde{H}_*(X)).$$

Similarly, $1 - \lambda_n^{\max}, 1 - \delta_n^{\max} \in \mathbb{Z}_{(p)}[\Sigma_n]$ are idempotents. Let

$$\overline{Q}_n^{\max}(X) = \mathrm{hocolim}_{1-\lambda_n^{\max}} \Sigma X^{(n)}$$
$$\overline{B}_n^{\max}(X) = \mathrm{hocolim}_{1-\delta_n^{\max}} \Sigma X^{(n)}.$$

Since $\lambda_n^{\max} + (1 - \lambda_n^{\max}) = 1$ while $\lambda_n^{\max} \circ (1 - \lambda_n^{\max}) = 0$, we get a sum

$$\Sigma X^{(n)} \longrightarrow Q_n^{\max}(X) \vee \overline{Q}_n^{\max}(X),$$

which is a homology isomorphism and therefore a homotopy equivalence. Similarly, there is a homotopy equivalence

$$\Sigma X^{(n)} \longrightarrow B_n^{\max}(X) \vee \overline{B}_n^{\max}(X).$$

Let $w_n\colon \Sigma X^{(n)} \longrightarrow \Sigma X$ be the n-fold iterated Whitehead product of the identity map with itself. Let $Q_n^{\max}(X) \longrightarrow \Sigma X^{(n)}$ be a right homotopy inverse for the hocolim map $\Sigma X^{(n)} \longrightarrow Q_n^{\max}(X)$. Let $\pi_{n,X}$ be the composite

$$\pi_{n,X}\colon Q_n^{\max}(X) \longrightarrow \Sigma X^{(n)} \xrightarrow{\alpha_n^{\max}} \Sigma X^{(n)} \xrightarrow{w_n} \Sigma X.$$

Let

$$Q^{\max}(X) = \bigvee_{n=2}^{\infty} Q_n^{\max}(X).$$

Let π_X be the wedge sum of the $\pi_{n,X}$'s:

$$\pi_X\colon Q^{\max}(X) \longrightarrow \Sigma X.$$

Note that

$$\begin{aligned} H_*(\Omega Q^{\max}(X)) = H_*(\Omega(\bigvee_{n=2}^{\infty} Q_n^{\max}(X))) &\cong T\Big(\bigoplus_{n=2}^{\infty} Q_n^{\max}(\widetilde{H}_*(X)\Big) \\ &\cong B^{\max}(\widetilde{H}_*(X)) \cong H_*(B^{\max}(X)). \end{aligned}$$

Thus, if $A^{\min}(X)$ is defined as the homotopy fiber of π_X, then the homotopy decomposition in Theorem 2.2 and the coalgebra decomposition in Theorem 2.1 combine to show there is a coalgebra isomorphism $H_*(A^{\min}(X)) \cong A^{\min}(\widetilde{H}_*(X))$.

3 The Construction and Properties of the Spaces $M_n(Y)$

Let Y be a homotopy coassociative co-H space. This section reviews the construction of the space $M_n(Y)$ in [T] and describes some of the properties proven there. We also take the opportunity here to prove two additional properties of $M_n(Y)$ which should have been included in [T]. These will subsequently be needed in Section 4. They are Lemmas 3.5 and 3.8.

We first record three general facts about co-H spaces (see [G]).

Lemma 3.1 *The following hold:*

(a) *A space Y is a co-H space if and only if there is a map $s\colon Y \longrightarrow \Sigma\Omega Y$ which is a right homotopy inverse of the evaluation map $\Sigma\Omega Y \longrightarrow Y$.*
(b) *A co-H space Y is homotopy coassociative if and only if the map s in part* (a) *can be chosen to be a co-H map.*
(c) *If $f\colon Y \longrightarrow Z$ is a co-H map between homotopy coassociative co-H spaces then there is a homotopy commutative diagram*

$$\begin{tikzcd} Y \arrow[r, "s_Y"] \arrow[d, "f"] & \Sigma\Omega Y \arrow[d, "\Sigma\Omega f"] \\ Z \arrow[r, "s_Z"] & \Sigma\Omega Z \end{tikzcd}$$

where s_Y and s_Z are both co-H maps. ∎

Suppose X is a connected space. One consequence of the James construction is a homotopy equivalence

$$\Sigma\Omega\Sigma X \simeq \bigvee_{n=1}^{\infty} \Sigma X^{(n)},$$

which is natural for maps $X \longrightarrow \overline{X}$. The following theorem generalizes this decomposition from suspensions to coassociative co-H spaces.

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Theorem 3.2 *Let Y be a simply connected, homotopy coassociative co-H space. Then for each $n \geq 1$ there are spaces $M_n(Y)$ and a homotopy equivalence*

$$\Sigma\Omega Y \simeq \bigvee_{n=1}^{\infty} M_n(Y)$$

where:

(a) $\Sigma^{n-1}M_n(Y) \simeq Y^{(n)}$;
(b) *if $Y = \Sigma\overline{Y}$, then $M_n(Y) \simeq \Sigma\overline{Y}^{(n)}$;*
(c) *each $M_n(Y)$ is a homotopy coassociative co-H space;*
(d) *if Y is also homotopy cocommutative, then so is each $M_n(Y)$;*
(e) *this homotopy decomposition is natural for co-H maps $Y \longrightarrow Z$ between homotopy coassociative co-H spaces.* ■

Each $M_n(Y)$ is constructed as a telescope of an idempotent γ_n on $\Sigma(\Omega Y)^{(n)}$. The basic composite to work with is $\theta\colon \Sigma\Omega Y \xrightarrow{\text{ev}} Y \xrightarrow{s} \Sigma\Omega Y$, where ev is the evaluation map and s is the co-H map in Lemma 3.1(b). Shifting the suspension coordinate on $\Sigma(\Omega Y)^{(n)}$ to the i-th smash factor, we can do θ on the i-th smash factor and the identity map on the remaining factors. Do this once for each $1 \leq i \leq n$. The composite of all n iterations defines $\gamma_n\colon \Sigma(\Omega Y)^{(n)} \longrightarrow \Sigma(\Omega Y)^{(n)}$. Let $X = \Omega Y$. Let $M_n(Y) = \operatorname{hocolim}_{\gamma_n} \Sigma X^{(n)}$. Let $r_n\colon \Sigma X^{(n)} \longrightarrow M_n(Y)$ be the telescope map. Since γ_n is an idempotent, r_n has a right homotopy inverse $s_n\colon M_n(Y) \longrightarrow \Sigma X^{(n)}$. Note that when $n = 1$, the map s_n is just the co-H structure map $Y \xrightarrow{s} \Sigma\Omega Y = \Sigma X$. When $n > 1$, the map s_n has two properties analogous to those of $s = s_1$.

Proposition 3.3 *The map $M_n \xrightarrow{s_n} \Sigma X^{(n)}$ has the following properties:*

(a) *s_n can be chosen to be a co-H map,*
(b) *there is an isomorphism $\widetilde{H}_*(M_n(Y)) \cong \Sigma(\Sigma^{-1}\widetilde{H}_*(Y))^{\otimes n}$ and $(s_n)_*$ includes $\widetilde{H}_*(M_n(Y))$ into $\widetilde{H}_*(\Sigma X^{(n)}) \cong \Sigma\widetilde{H}_*(X)^{\otimes n}$ by the n-fold tensor inclusion.* ■

We next describe naturality. Suppose $f\colon Y \longrightarrow Z$ is a co-H map between homotopy coassociative co-H spaces. We continue to use $X = \Omega Y$. Let $\overline{X} = \Omega Z$. Let $g\colon X \longrightarrow \overline{X}$ be Ωf. Let $M_n(f)$ be the composite

$$M_n(f)\colon M_n(Y) \xrightarrow{s_n} \Sigma X^{(n)} \xrightarrow{\Sigma g^{(n)}} \Sigma\overline{X}^{(n)} \xrightarrow{r_n} M_n(Z).$$

Lemma 3.4 *The construction of $M_n(\ )$ is natural for co-H maps $Y \longrightarrow Z$ between homotopy coassociative co-H spaces. That is, there are homotopy commutative diagrams*

$$\begin{tikzcd} M_n(Y) \arrow[r, "s_n"] \arrow[d, "M_n(f)"] & \Sigma X^{(n)} \arrow[d, "\Sigma g^{(n)}"] \\ M_n(Z) \arrow[r, "s_n"] & \Sigma\overline{X}^{(n)} \end{tikzcd} \qquad \begin{tikzcd} \Sigma X^{(n)} \arrow[r, "r_n"] \arrow[d, "\Sigma g^{(n)}"] & M_n(Y) \arrow[d, "M_n(f)"] \\ \Sigma\overline{X}^{(n)} \arrow[r, "r_n"] & M_n(Z) \end{tikzcd}$$ ■

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We now prove an additional feature of the $M_n(\ )$'s. In Lemma 3.5 we will show that $M_n(f)$ is a co-H map. Note that this is not immediate from the definition of $M_n(f)$ as the composite $r_n \circ \Sigma g^{(n)} \circ s_n$. By Proposition 3.3(a), s_n is a co-H map, while $\Sigma g^{(n)}$ is a co-H map because it is a suspension. But r_n is defined in part by evaluation maps and is not co-H. So it is not true that $M_n(f)$ is co-H because it is the composite of co-H maps. Nevertheless, we have:

Lemma 3.5 *The map $M_n(Y) \xrightarrow{M_n(f)} M_n(Z)$ is a co-H map.*

Proof Argue exactly as in [T, 7.2] (which shows a certain other map is co-H, in fact, the map $M_n(\sigma)$ appearing below in Proposition 3.7(c)). ■

Corollary 3.6 *M_n defines a functor from the category of homotopy coassociative co-H spaces and co-H maps to itself.* ■

We next see how the symmetric group Σ_n acts on $M_n(Y)$. Let $\sigma \in \Sigma_n$. There is a self-map $\sigma\colon X^{(n)} \longrightarrow X^{(n)}$ given by permuting the factors in the smash product. Let $M_n(\sigma)$ be the composite

$$M_n(\sigma)\colon M_n(Y) \xrightarrow{s_n} \Sigma X^{(n)} \xrightarrow{\Sigma\sigma} \Sigma X^{(n)} \xrightarrow{r_n} M_n(Y).$$

From [T] we have:

Proposition 3.7 *The following hold:*

(a) *there is a homotopy commutative diagram*

$$\begin{tikzcd} M_n(Y) \arrow[r, "M_n(\sigma)"] \arrow[d, "s_n"] & M_n(Y) \arrow[d, "s_n"] \\ \Sigma X^{(n)} \arrow[r, "\Sigma\sigma"] & \Sigma X^{(n)}, \end{tikzcd}$$

(b) *if $\sigma_1, \sigma_2 \in \Sigma_n$, then $M_n(\sigma_1 \circ \sigma_2) \simeq M_n(\sigma_1) \circ M_n(\sigma_2)$,*

(c) *$M_n(\sigma)$ is a co-H map,*

(d) *$(M_n(\sigma))_*$ acts on $\widetilde{H}_*(M_n(Y)) \cong \Sigma(\Sigma^{-1}\widetilde{H}_*(Y))^{\otimes n}$ by permuting the tensor factors.* ■

We now show that $M_n(\sigma)$ is natural. Again, suppose $f\colon Y \longrightarrow Z$ is a co-H map between homotopy coassociative co-H spaces. Again, let $X = \Omega Y$, $\overline{X} = \Omega Z$, $g = \Omega f$, and recall the definition of the map $M_n(Y) \xrightarrow{M_n(f)} M_n(Z)$ (which precedes Lemma 3.4).

Lemma 3.8 *There is a homotopy commutative diagram*

$$\begin{tikzcd} M_n(Y) \arrow[r, "M_n(\sigma)"] \arrow[d, "M_n(f)"] & M_n(Y) \arrow[d, "M_n(f)"] \\ M_n(Z) \arrow[r, "M_n(\sigma)"] & M_n(Z). \end{tikzcd}$$

Proof To keep track of the space involved, denote the maps $M_n(Y) \xrightarrow{s_n} \Sigma X^{(n)} \xrightarrow{r_n} M_n(Y)$ by s_Y and r_Y, respectively. Recall that $r_Y \circ s_Y$ is homotopic to the identity on $M_n(Y)$. As well, denote the map $M_n(Y) \xrightarrow{M_n(\sigma)} M_n(Y)$ by $M_n^Y(\sigma)$.

Consider the following diagram

$$\begin{tikzcd} & M_n(Y) \arrow[dr, "s_Y"] & & \\ M_n(Y) \arrow[ur, "M_n^Y(\sigma)"] \arrow[r, "s_Y"] \arrow[d, "M_n(f)"] & \Sigma X^{(n)} \arrow[r, "\Sigma\sigma"] \arrow[d, "\Sigma g^{(n)}"] & \Sigma X^{(n)} \arrow[d, "\Sigma g^{(n)}"] & \\ M_n(Z) \arrow[r, "s_Z"] \arrow[dr, "M_n^Z(\sigma)"'] & \Sigma \overline{X}^{(n)} \arrow[r, "\Sigma\sigma"] & \Sigma \overline{X}^{(n)} \arrow[r, "r_Z"] & M_n(Z). \\ & M_n(Z) \arrow[ur, "s_Z"'] & & \end{tikzcd}$$

The top and bottom triangles homotopy commute by Proposition 3.7(a). The left inner square homotopy commutes by the naturality in Lemma 3.4 while the right inner square homotopy commutes by the naturality of $\Sigma\sigma$. Thus the entire diagram homotopy commutes. By definition, $M_n(f) = r_Z \circ \Sigma g^{(n)} \circ s_Y$ so the upper direction around the diagram is homotopic to $M_n(f) \circ M_n^Y(\sigma)$. Since $r_Z \circ s_Z$ is homotopic to the identity on $M_n(Z)$, the lower direction around the diagram is homotopic to $M_n^Z(\sigma) \circ M_n(f)$. Thus $M_n(f) \circ M_n^Y(\sigma) \simeq M_n^Z(\sigma) \circ M_n(f)$, proving the lemma. ■

Finally, we end this section with more review material from [T]. One application of the spaces $M_n(Y)$ is to construct generalizations of Whitehead products. For a space X, recall that

$$w_n \colon \Sigma X^{(n)} \longrightarrow \Sigma X$$

is the n-fold iterated Whitehead product of the identity map with itself. Consider the special case when $X = \Omega Y$. Let $\mathrm{ev} \colon \Sigma X = \Sigma\Omega Y \longrightarrow Y$ be the evaluation map. Let $s \colon Y \longrightarrow \Sigma\Omega Y = \Sigma X$ be the co-H map in Lemma 3.1(b). Define a generalized Whitehead product on Y by the composite

$$\overline{w}_n \colon M_n(Y) \xrightarrow{s_n} \Sigma X^{(n)} \xrightarrow{w_n} \Sigma X \xrightarrow{ev} Y.$$

This generalized Whitehead product $\overline{w}_n$ is compatible with w_n:

Lemma 3.9 *There is a homotopy commutative diagram*

$$\begin{tikzcd} M_n(Y) \arrow[r, "\overline{w}_n"] \arrow[d, "s_n"] & Y \arrow[d, "s"] \\ \Sigma X^{(n)} \arrow[r, "w_n"] & \Sigma X. \end{tikzcd}$$

Further, the map $\overline{w}_n$ is natural for co-H maps $Y \longrightarrow Z$ between homotopy coassociative co-H spaces. ∎

4 Idempotent Decompositions of $\Sigma X^{(n)}$ and $M_n(Y)$

In this section we consider wedge decompositions of $\Sigma X^{(n)}$ which arise from the action of the symmetric group Σ_n, and show that analogous wedge decompositions exist for $M_n(Y)$. Such decompositions were considered in [SW3] for the case of $X^{(n)}$ when X is a suspension and $n \geq 2$. We begin with a definition.

Definition 4.1 Maps $f_1, \dots, f_k \in \mathbb{Z}_{(p)}[\Sigma_n]$ give an *orthogonal decomposition of the identity* if:

(1) $\sum_{i=1}^{k} f_i = 1$,
(2) $f_i \circ f_i = f_i$ for $1 \leq i \leq k$,
(3) $f_i \circ f_j = 0$ whenever $i \neq j$.

To each $\sigma \in \Sigma_n$ there corresponds a map

$$\sigma \colon X^{(n)} \longrightarrow X^{(n)}$$

given by permuting the factors in the smash product. In order to add such maps we need to suspend. However, while $[\Sigma X^{(n)}, \Sigma X^{(n)}]$ is a group, it is not necessarily commutative (it will be, for example, if X is a suspension). So in general, there is no representation $\mathbb{Z}_{(p)}[\Sigma_n] \longrightarrow [\Sigma X^{(n)}, \Sigma X^{(n)}]$. Nevertheless, given $\alpha = a_1\sigma_1 + \dots + a_{n!}\sigma_{n!} \in \mathbb{Z}_{(p)}[\Sigma_n]$, we can still define a map $\alpha \colon \Sigma X^{(n)} \longrightarrow \Sigma X^{(n)}$. It only has to be remembered that the homotopy class of α depends on the order of $\sigma_1, \dots, \sigma_{n!}$. Once we take homology, however, the non-commutativity problem goes away. There is a representation $\mathbb{Z}_{(p)}[\Sigma_n] \longrightarrow \mathrm{Hom}(H_*(\Sigma X^{(n)}), H_*(\Sigma X^{(n)}))$. In particular, suppose $f_1, \dots, f_k \in \mathbb{Z}_{(p)}[\Sigma_n]$ is an orthogonal decomposition of the identity. Then the maps

$$f_i \colon \Sigma X^{(n)} \longrightarrow \Sigma X^{(n)}$$

have the property that $(f_1)_*, \dots, (f_k)_*$ give an orthogonal decomposition of the identity in $\mathrm{Hom}(H_*(\Sigma X^{(n)}), H_*(\Sigma X^{(n)}))$. Let

$$Q_{n,i}(X) = \mathrm{hocolim}_{f_i}\ \Sigma X^{(n)}.$$

Then $H_*(Q_{n,i}(X))$ is isomorphic to the image of $(f_i)_*$. Thus the sum $\Sigma X^{(n)} \longrightarrow \bigvee_{i=1}^k Q_{n,i}(X)$ is a homology isomorphism and therefore a homotopy equivalence.

We now wish to reproduce such wedge decompositions in the case of $M_n(Y)$, where Y is a homotopy coassociative co-H space. Let $\sigma \in \Sigma_n$. As in Section 3, let $M_n(\sigma)$ be the composite

$$M_n(\sigma)\colon M_n(Y) \xrightarrow{s_n} \Sigma X^{(n)} \xrightarrow{\Sigma\sigma} \Sigma X^{(n)} \xrightarrow{r_n} M_n(Y).$$

Suppose $f_1, \dots, f_k \in \mathbb{Z}_{(p)}[\Sigma_n]$ is an orthogonal decomposition of the identity. Suppose $f_i = \Sigma_{j=1}^{n!} a_j\sigma_j$, where the σ_j's are distinct elements of Σ_n and each $a_j \in \mathbb{Z}_{(p)}$. Let

$$M_n(f_i)\colon M_n(Y) \longrightarrow M_n(Y)$$

be the sum $M_n(f_i) = \Sigma_{j=1}^{n!} a_j M_n(\sigma_j)$. Since the inclusion $M_n(Y) \xrightarrow{s_n} \Sigma X^{(n)}$ is a co-H map, Lemma 3.7(a) implies:

Lemma 4.2 *There is a homotopy commutative diagram*

$$\begin{tikzcd} M_n(Y) \arrow[r, "M_n(f_i)"] \arrow[d, "s_n"] & M_n(Y) \arrow[d, "s_n"] \\ \Sigma X^{(n)} \arrow[r, "f_i"] & \Sigma X^{(n)}. \end{tikzcd}$$

■

Let

$$N_i(Y) = \operatorname{hocolim}_{M_n(f_i)} M_n(Y).$$

By Lemma 4.2 there is a homotopy commutative diagram of telescopes

$$\begin{tikzcd} M_n(Y) \arrow[r] \arrow[d, "s_n"] & N_{n,i}(Y) \arrow[d] \\ \Sigma X^{(n)} \arrow[r] & Q_{n,i}(Y). \end{tikzcd}$$

As $M_n(Y) \xrightarrow{s_n} \Sigma X^{(n)}$ is a co-H map, we can add over i to get the homotopy commutativity of the diagram in the following proposition.

Proposition 4.3 *There is a homotopy commutative diagram of equivalences*

$$\begin{tikzcd} M_n(Y) \arrow[r, "\simeq"] \arrow[d, "s_n"] & \bigvee_{i=1}^k N_{n,i}(Y) \arrow[d] \\ \Sigma X^{(n)} \arrow[r, "\simeq"] & \bigvee_{i=1}^k Q_{n,i}(X). \end{tikzcd}$$

Proof It remains to show that the map $e\colon M_n(Y) \longrightarrow \bigvee_{i=1}^{k} N_{n,i}(Y)$ along the top row is a homotopy equivalence. By definition, $M_n(f_i) = \Sigma_{j=1}^{n!} a_j M_n(\sigma_j)$. By Lemma 3.7(d), $(M_n(\sigma))_*$ acts on $\widetilde{H}_*(M_n(Y)) \cong \Sigma(\Sigma^{-1}\widetilde{H}_*(Y))^{\otimes n}$ by permuting tensor factors. That is, $(M_n(\sigma))_* = \Sigma\sigma$. Thus $(M_n(f_i))_* = \Sigma f_i$. The telescope $N_{n,i}(Y)$ of the map $M_n(f_i)$ therefore has its homology isomorphic to $\operatorname{Im}(\Sigma f_i)$. Adding over i, we see that e_* has image isomorphic to that of $\operatorname{Im}\big(\Sigma_{j=1}^{n!}(\Sigma f_i)\big)$. But $f_1, \dots, f_k$ is an orthogonal decomposition of the identity so the latter image is isomorphic to that of the identity map. Thus e_* is an isomorphism and hence e is a homotopy equivalence. ■

Next consider the naturality of the homotopy decomposition

$$M_n(Y) \xrightarrow{\simeq} \bigvee_{i=1}^{k} N_{n,i}(Y).$$

As in Section 3, let $f\colon Y \longrightarrow Z$ be a co-H map between homotopy coassociative co-H spaces. Let $X = \Omega Y$, $\overline{X} = \Omega Z$, $g = \Omega f$, and recall the definition of the map $M_n(Y) \xrightarrow{M_n(f)} M_n(Z)$ (preceeding Lemma 3.4).

Lemma 4.4 *There is a homotopy commutative diagram*

$$\begin{tikzcd} M_n(Y) \arrow[r, "M_n(f_i)"] \arrow[d, "M_n(f)"] & M_n(Y) \arrow[d, "M_n(f)"] \\ M_n(Z) \arrow[r, "M_n(f_i)"] & M_n(Z). \end{tikzcd}$$

Proof To keep track of the space involved, denote the telescope maps $M_n(Y) \xrightarrow{s_n} \Sigma X^{(n)} \xrightarrow{r_n} M_n(Y)$ by s_Y and r_Y, respectively. Recall that $r_Y \circ s_Y$ is homotopic to the identity map on $M_n(Y)$. Denote the map $M_n(Y) \xrightarrow{M_n(\sigma)} M_n(Y)$ by $M_n^Y(\sigma)$. Also write $M_n^Y(f_i)$ for $M_n(Y) \xrightarrow{M_n(f_i)} M_n(Y)$.

Consider the sequence:

(1) $$M_n^Z(f_i) \circ M_n(f) = \big(\Sigma_{j=1}^{n!} a_j M_n^Z(\sigma_j)\big) \circ M_n(f)$$

(2) $$\simeq \Sigma_{j=1}^{n!} \big(a_j M_n^Z(\sigma_j) \circ M_n(f)\big)$$

(3) $$\simeq \Sigma_{j=1}^{n!} \big(a_j M_n(f) \circ M_n^Y(\sigma_j)\big)$$

(4) $$\simeq \Sigma_{j=1}^{n!} \big(M_n(f) \circ a_j M_n^Y(\sigma_j)\big)$$

(5) $$\simeq M_n(f) \circ \big(\Sigma_{j=1}^{n!} a_j M_n^Y(\sigma_j)\big)$$

(6) $$= M_n(f) \circ M_n^Y(f_i).$$

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The definition of $M_n^Z(f_i)$ gives the equality in line (1). By Lemma 3.5, $M_n(f)$ is a co-H map, so it will distribute on the right, giving the homotopy in line (2). The homotopy in line (3) follows from Lemma 3.8. The commutation of a_j and $M_n(f)$ in line (4) follows from the fact that $M_n(f)$ is co-H. Any map distributes on the left when a sum is taken via a co-H structure, giving line (5). Finally, the equality in line (6) comes from the definition of $M_n^Y(f_i)$. This sequence of equalities and homotopies proves the lemma. ■

Taking horizontal telescopes in Lemma 4.4 gives a homotopy commutative diagram

$$\begin{array}{ccc} M_n(Y) & \longrightarrow & N_{n,i}(Y) \\ \Big\downarrow{\scriptstyle M_n(f)} & & \Big\downarrow \\ M_n(Z) & \longrightarrow & N_{n,i}(Z). \end{array}$$

Adding over i gives:

Proposition 4.5 *There is a homotopy commutative diagram of equivalences*

$$\begin{array}{ccc} M_n(Y) & \xrightarrow{\simeq} & \bigvee_{i=1}^k N_{n,i}(Y) \\ \Big\downarrow{\scriptstyle M_n(f)} & & \Big\downarrow \\ M_N(Z) & \xrightarrow{\simeq} & \bigvee N_{n,i}(Z). \end{array}$$

■

The following theorem summarizes Propositions 4.3 and 4.5.

Theorem 4.6 *Let $\Sigma_{i=1}^k f_i = 1$ be an orthogonal decomposition of the identity in $\mathbb{Z}_{(p)}[\Sigma_n]$. To this there corresponds a homotopy equivalence*

$$M_n(Y) \xrightarrow{\simeq} \bigvee_{i=1}^k N_{n,i}(Y)$$

which is natural for co-H maps $Y \longrightarrow Z$ between coassociative co-H spaces. ■

Next, we examine the telescope maps $N_{n,i}(Y) \longrightarrow Q_{n,i}(X)$ in Proposition 4.3 and show they have natural left homotopy inverses. First, taking homotopy inverses in Proposition 4.3 and restricting to the i-th wedge summand gives:

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Corollary 4.7 *For each $1 \leq i \leq k$, there is a homotopy commutative diagram*

$$\begin{tikzcd} N_{n,i}(Y) \arrow[r] \arrow[d] & \bigvee_{i=1}^{k} N_{n,i} \arrow[r, "\simeq"] \arrow[d] & M_n(Y) \arrow[d, "s_n"] \\ Q_{n,i}(X) \arrow[r] & \bigvee_{i=1}^{k} Q_{n,i}(X) \arrow[r, "\simeq"] & \Sigma X^{(n)}. \end{tikzcd}$$

where the left square is the inclusion of the i-th summand into the wedge. ∎

Corollary 4.8 *For each $1 \leq i \leq k$, the telescope map $N_{n,i}(Y) \longrightarrow Q_{n,i}(X)$ has a left homotopy inverse.*

Proof Since $M_n(Y) \xrightarrow{s_n} \Sigma X^{(n)}$ and $N_{n,i}(Y) \longrightarrow M_n(Y)$ each have left homotopy inverses, using Corollary 4.7 we obtain a homotopy commutative diagram

$$\begin{tikzcd} N_{n,i}(Y) \arrow[r] \arrow[d] & M_n(Y) \arrow[d, "s_n"] \arrow[dr, equal] & & \\ Q_{n,i}(X) \arrow[r] & \Sigma X^{(n)} \arrow[r, "r_n"] & M_n(Y) \arrow[r] & N_{n,i}(Y) \end{tikzcd}$$

in which the upper direction is homotopic to the identity map on $N_{n,i}(Y)$. ∎

Define $\psi_{n,i} \colon Q_{n,i}(X) \longrightarrow N_{n,i}(Y)$ by the composite

$$\psi_{n,i} \colon Q_{n,i}(X) \hookrightarrow \bigvee_{i=1}^{k} Q_{n,i}(X) \xrightarrow{\simeq} \Sigma X^{(n)} \xrightarrow{r_n} M_n(Y) \longrightarrow N_{n,i}(Y).$$

Each of the four maps in this composite is natural. The wedge inclusion and the homotopy equivalence are natural with respect to maps $X \longrightarrow \overline{X}$. By Lemma 3.4 and Proposition 4.5, respectively, the maps r_n and $M_n(Y) \longrightarrow N_{n,i}(Y)$ are natural for co-H maps $Y \longrightarrow Z$ between coassociative co-H spaces. Thus:

Lemma 4.9 *The telescope map $N_{n,i}(Y) \longrightarrow Q_{n,i}(X)$ has a left homotopy inverse*

$$Q_{n,i}(X) \xrightarrow{\psi_{n,i}} N_{n,i}(Y)$$

which is natural with respect to co-H maps $Y \longrightarrow Z$ between homotopy coassociative co-H spaces. That is, there is a homotopy commutative diagram

$$\begin{tikzcd} Q_{n,i}(X) \arrow[r, "\psi_{n,i}"] \arrow[d] & N_{n,i}(Y) \arrow[d] \\ Q_{n,i}(\overline{X}) \arrow[r, "\psi_{n,i}"] & N_{n,i}(Z). \end{tikzcd}$$

∎

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5 The Construction of $A^{\min}$ for Coassociative Co-H Spaces

Recall from Section 2 the idempotents $\lambda_n^{\max}, \delta_n^{\max} \in \mathbb{Z}_{(p)}[\Sigma_n]$. These give orthogonal decompositions of the identity, $1 = \lambda_n^{\max} + (1 - \lambda_n^{\max})$ and $1 = \delta_n^{\max} + (1 - \delta_n^{\max})$. Recall that $Q_n^{\max}(X) = \mathrm{hocolim}_{\lambda_n^{\max}} \Sigma X^{(n)}$, $\overline{Q}_n^{\max}(X) = \mathrm{hocolim}_{1-\lambda_n^{\max}} \Sigma X^{(n)}$, and $B_n^{\max}(X), \overline{B}_n^{\max}(X)$ were defined similarly with respect to $\delta_n^{\max}$. Let Y be a homotopy coassociative co-H space. As in Section 4, the element $\lambda_n^{\max} \in \mathbb{Z}_{(p)}[\Sigma_n]$ corresponds to a map $M_n(\lambda_n^{\max})\colon M_n(Y) \longrightarrow M_n(Y)$, and similarly for $1-\lambda_n^{\max}$, $\delta_n^{\max}$, and $1-\delta_n^{\max}$. Let

$$MQ_n^{\max}(Y) = \mathrm{hocolim}_{M_n(\lambda_n^{\max})} M_n(Y),$$
$$M\overline{Q}_n^{\max}(Y) = \mathrm{hocolim}_{M_n(1-\lambda_n^{\max})} M_n(Y),$$
$$MB_n^{\max}(Y) = \mathrm{hocolim}_{M_n(\delta_n^{\max})} M_n(Y),$$
$$M\overline{B}_n^{\max}(Y) = \mathrm{hocolim}_{M_n(1-\delta_n^{\max})} M_n(Y).$$

Suppose $Y \longrightarrow Z$ is a co-H map between homotopy coassociative co-H spaces. As before, let $X = \Omega Y$ and $\overline{X} = \Omega Z$. By Theorem 4.6 and Lemma 4.9 we have:

Lemma 5.1 *The following hold:*

(a) *There is a homotopy equivalence $MQ_n^{\max}(Y) \vee M\overline{Q}_n^{\max}(Y) \longrightarrow M_n(Y)$ which is natural for co-H maps $Y \longrightarrow Z$ between coassociative co-H spaces.*

(b) *There is a homotopy commutative diagram of equivalences*

$$\begin{tikzcd} MQ_n^{\max}(Y) \vee M\overline{Q}_n^{\max}(Y) \arrow[r, "\simeq"] \arrow[d] & M_n(Y) \arrow[d, "s_n"] \\ Q_n^{\max}(X) \vee \overline{Q}_n^{\max}(X) \arrow[r, "\simeq"] & \Sigma X^{(n)}. \end{tikzcd}$$

(c) *The map $MQ_n^{\max}(Y) \longrightarrow Q_n^{\max}(X)$ in part* (b) *has a left homotopy inverse which is natural for co-H maps $Y \longrightarrow Z$ between homotopy coassociative co-H spaces. That is, there is a homotopy commutative diagram*

$$\begin{tikzcd} MQ_n^{\max}(Y) \arrow[r] \arrow[d] & MQ_n^{\max}(Z) \arrow[d] \\ Q_n^{\max}(X) \arrow[r] & Q_n^{\max}(\overline{X}). \end{tikzcd}$$

The same statements hold when $Q_n^{\max}$ is replaced by $B_n^{\max}$. ■

Now consider what happens in homology. Recall from Section 2 that

$$H_*(Q_n^{\max}(X)) \cong Q_n^{\max}(\widetilde{H}_*(X)),$$

where $Q_n^{\max}(\widetilde{H}_*(X))$ is the image of the idempotent $\lambda_n^{\max}\colon \widetilde{H}_*(X)^{\otimes n} \longrightarrow \widetilde{H}_*(X)^{\otimes n}$. By Proposition 3.7(d), the inclusion $M_n(Y) \xrightarrow{s_n} \Sigma X^{(n)}$ becomes an n-fold tensor inclusion

$$\widetilde{H}_*(M_n(Y)) \cong \Sigma(\Sigma^{-1}\widetilde{H}_*(Y))^{\otimes n} \xrightarrow{(s_n)_*} \widetilde{H}_*(\Sigma X^{(n)}) \cong \Sigma\widetilde{H}_*(X)^{\otimes n}.$$

In general, if $W \subseteq V$ are vector spaces, then the idempotent $\lambda_n^{\max}(V)\colon V^{\otimes n} \longrightarrow V^{\otimes n}$ restricts to the corresponding idempotent on $W^{\otimes n}$, that is, $\lambda_n^{\max}(V)_{|W} = \lambda_n^{\max}(W)$. So the image $Q_n^{\max}(V)$ of $\lambda_n^{\max}(V)$ restricts to the image $Q_n^{\max}(W)$ of $\lambda_n^{\max}(W)$. A similar argument holds for the idempotent $\delta_n^{\max}(V)$ and its image $B_n^{\max}(V)$. Applied to the case $W = \Sigma^{-1}\widetilde{H}_*(Y)$ and $V = \widetilde{H}_*(X)$ we have:

Lemma 5.2 *There are isomorphisms*

$$H_*(MQ_n^{\max}(Y)) \cong Q_n^{\max}(\Sigma^{-1}\widetilde{H}_*(Y))$$
$$H_*(MB_n^{\max}(Y)) \cong B_n^{\max}(\Sigma^{-1}\widetilde{H}_*(Y)). \qquad \blacksquare$$

We now begin the construction of $A^{\min}(Y)$ for ΩY. This is done with the help of the known construction of $A^{\min}(X)$ for $\Omega\Sigma X$ in the case when $X = \Omega Y$. The two are linked by the co-H structure map $Y \xrightarrow{s} \Sigma\Omega Y = \Sigma X$ in Lemma 3.1(b). Recall from Section 2 there is a homotopy fibration

$$A^{\min}(X) \xrightarrow{*} Q^{\max}(X) \xrightarrow{\pi_X} \Sigma X,$$

where $Q^{\max}(X) = \bigvee_{n=2}^{\infty} Q_n^{\max}(X)$, and π_X is the wedge sum of the composites

$$\pi_{n,X}\colon Q_n^{\max}(X) \longrightarrow \Sigma X^{(n)} \xrightarrow{w_n} \Sigma X.$$

Similarly, let

$$MQ^{\max}(Y) = \bigvee_{n=2}^{\infty} MQ_n^{\max}(Y)$$

and define

$$M\pi_Y\colon MQ^{\max}(Y) \longrightarrow Y$$

by adding each of the composites $M\pi_{n,Y}\colon MQ_n^{\max}(Y) \longrightarrow M_n(Y) \xrightarrow{\overline{w}_n} Y$. Define $A^{\min}(Y)$ by the homotopy fibration

$$A^{\min}(Y) \longrightarrow MQ^{\max}(Y) \xrightarrow{M\pi_Y} Y.$$

Now we put together the two fibrations defining $A^{\min}(Y)$ and $A^{\min}(X)$. Consider the diagram

$$\begin{tikzcd} MQ_n^{\max}(Y) \arrow[r] \arrow[d] & M_n(Y) \arrow[r, "\overline{w}_n"] \arrow[d, "s_n"] & Y \arrow[d, "s"] \\ Q_n^{\max}(X) \arrow[r] & \Sigma X^{(n)} \arrow[r, "w_n"] & \Sigma X. \end{tikzcd}$$

The left square homotopy commutes by restricting the diagram in Lemma 5.1(b) to the left wedge summand. The right square homotopy commutes by Lemma 3.9. Adding over n then gives a homotopy commutative diagram

$$\begin{tikzcd} MQ^{\max}(Y) \arrow[r, "M\pi_Y"] \arrow[d] & Y \arrow[d, "s"] \\ Q^{\max}(X) \arrow[r, "\pi_X"] & \Sigma X. \end{tikzcd}$$

From this we obtain a homotopy fibration diagram

$$\begin{tikzcd} \Omega MQ^{\max}(Y) \arrow[r, "\Omega M\pi_Y"] \arrow[d] & \Omega Y \arrow[r] \arrow[d, "\Omega s"] & A^{\min}(Y) \arrow[r] \arrow[d] & MQ^{\max}(Y) \arrow[r, "M\pi_Y"] \arrow[d] & Y \arrow[d, "s"] \\ \Omega Q^{\max}(X) \arrow[r, "\Omega\pi_X"] & \Omega\Sigma X \arrow[r] & A^{\min}(X) \arrow[r, "*"] & Q^{\max}(X) \arrow[r, "\pi_X"] & \Sigma X. \end{tikzcd}$$

Note that along the bottom row we have $\Omega\Sigma X \simeq A^{\min}(X) \times \Omega Q^{\max}(X)$. By Theorem 2.2, $\Omega\pi_X$ has a left homotopy inverse

$$\theta_X \colon \Omega\Sigma X \longrightarrow \Omega Q^{\max}(X)$$

which is natural for maps $X \longrightarrow \overline{X}$. By Lemma 5.1(c) the map $MQ_n^{\max}(Y) \longrightarrow Q_n^{\max}(X)$ has a left homotopy inverse $\psi_{n,Y} \colon Q_n^{\max}(X) \longrightarrow MQ_n^{\max}(Y)$ which is natural for co-H maps $Y \longrightarrow Z$ between coassociative co-H spaces. Adding over n we obtain a left homotopy inverse

$$\psi_Y \colon Q^{\max}(X) \longrightarrow MQ^{\max}(Y)$$

of $MQ^{\max}(Y) \longrightarrow Q^{\max}(X)$. Let $M\theta_Y$ be the composite

$$M\theta_Y \colon \Omega Y \xrightarrow{\Omega s} \Omega\Sigma X \xrightarrow{\theta_X} \Omega Q^{\max}(X) \xrightarrow{\Omega\psi_Y} \Omega MQ^{\max}(Y).$$

Proposition 5.3 *$M\theta_Y$ is a left homotopy inverse of $\Omega M\pi_Y$. Thus*

$$\Omega Y \simeq A^{\min}(Y) \times \Omega MQ^{\max}(Y).$$

Proof We have a homotopy commutative diagram

$$\begin{tikzcd} \Omega MQ^{\max}(Y) \arrow[r] \arrow[d, "\Omega M\pi_Y"] & \Omega Q^{\max}(X) \arrow[d, "\Omega\pi_X"] \arrow[dr, equal] & & \\ \Omega Y \arrow[r, "\Omega s"] & \Omega\Sigma X \arrow[r, "\theta_X"] & \Omega Q^{\max}(X) \arrow[r, "\Omega\psi_Y"] & \Omega MQ^{\max}(Y) \end{tikzcd}$$

in which the lower row is the definition of $M\theta_Y$ and the upper direction is the identity map on $\Omega MQ^{\max}(Y)$. ∎

Remark 5.4 The left homotopy inverse for $\Omega\pi_X$ is constructed explicitly in [SW1, SW2] by using combinatorial James–Hopf invariants. It may be possible to reproduce combinatorial James–Hopf maps for ΩY using the methods in Section 3, but it is certainly more efficient to proceed as above by the timely use of existing retractions.

The next proposition shows that $A^{\min}(Y)$ is the minimal functorial coalgebra retract of ΩY.

Proposition 5.5 $H_*(A^{\min}(Y)) \cong A^{\min}(\Sigma^{-1}\widetilde{H}_*(Y))$.

Proof By Theorem 2.1 there is a coalgebra decomposition

$$H_*(\Omega Y) \cong T(\Sigma^{-1}\widetilde{H}_*(Y)) \cong A^{\min}(\Sigma^{-1}\widetilde{H}_*(Y)) \otimes B^{\max}(\Sigma^{-1}\widetilde{H}_*(Y)).$$

The homotopy decomposition in Proposition 5.3 implies that the statement we are trying to prove is equivalent to showing that $H_*(\Omega MQ^{\max}(Y)) \cong B^{\max}(\Sigma^{-1}\widetilde{H}_*(Y))$. But by definition, $MQ^{\max}(Y) = \bigvee_{n=2}^{\infty} MQ_n^{\max}(Y)$, so by Lemma 5.2 we have

$$\begin{aligned} H_*(\Omega MQ^{\max}(Y)) = H_*(\Omega(\bigvee_{n=2}^{\infty} MQ_n^{\max}(Y))) &\cong T\Big(\bigoplus_{n=2}^{\infty} Q_n^{\max}(\Sigma^{-1}\widetilde{H}_*(Y))\Big) \\ &\cong B^{\max}(\Sigma^{-1}\widetilde{H}_*(Y)). \end{aligned}$$

∎

It remains to show that the decomposition in Proposition 5.3 is natural. Suppose $Y \longrightarrow Z$ is a co-H map between coassociative co-H spaces. The naturality in Lemmas 5.1(a) and 3.9 give, for each $n \geq 2$, a homotopy commutative diagram

$$\begin{tikzcd} MQ_n^{\max}(Y) \arrow[r] \arrow[d] & M_n(Y) \arrow[r, "\overline{w}_n"] \arrow[d] & Y \arrow[d] \\ MQ_n^{\max}(Z) \arrow[r] & M_n(Z) \arrow[r, "\overline{w}_n"] & Z. \end{tikzcd}$$

Adding over n gives a homotopy commutative diagram

$$\begin{tikzcd} MQ^{\max}(Y) \arrow[r, "M\pi_Y"] \arrow[d] & Y \arrow[d] \\ MQ^{\max}(Z) \arrow[r, "M\pi_Z"] & Z. \end{tikzcd}$$

From this we obtain a homotopy fibration diagram

$$\begin{tikzcd}
\Omega MQ^{\max}(Y) \arrow[r, "\Omega M\pi_Y"] \arrow[d] & \Omega Y \arrow[r] \arrow[d] & A^{\min}(Y) \arrow[r, "*"] \arrow[d] & MQ^{\max}(Y) \arrow[r, "M\pi_Y"] \arrow[d] & Y \arrow[d] \\
\Omega MQ^{\max}(Z) \arrow[r, "\Omega M\pi_Y"] & \Omega Z \arrow[r] & A^{\min}(Z) \arrow[r, "*"] & MQ^{\max}(Z) \arrow[r, "M\pi_Z"] & Z.
\end{tikzcd}$$

First observe:

Lemma 5.6 *The left homotopy inverse $M\theta_Y$ of $\Omega M\pi_Y$ in Proposition 5.3 is natural with respect to co-H maps $Y \longrightarrow Z$ between homotopy coassociative co-H spaces. That is, there is a homotopy commutative diagram*

$$\begin{tikzcd}
\Omega Y \arrow[r, "M\theta_Y"] \arrow[d] & \Omega MQ^{\max}(Y) \arrow[d] \\
\Omega Z \arrow[r, "M\theta_Z"] & \Omega MQ^{\max}(Z).
\end{tikzcd}$$

Proof As usual, we have $X = \Omega Y$, $\overline{X} = \Omega Z$, and $g\colon X \longrightarrow \overline{X}$ is Ωf. By Lemma 3.1(c) there is a homotopy commutative diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "f"] & \Sigma X \arrow[d, "\Sigma g"] \\
Z \arrow[r] & \Sigma \overline{X}
\end{tikzcd}$$

in which all maps are co-H maps. Consider the diagram

$$\begin{tikzcd}
\Omega Y \arrow[r] \arrow[d, "\Omega f"] & \Omega\Sigma X \arrow[r, "\theta_X"] \arrow[d, "\Omega\Sigma g"] & \Omega Q^{\max}(X) \arrow[r, "\Omega\psi_Y"] \arrow[d] & \Omega MQ^{\max}(Y) \arrow[d] \\
\Omega Z \arrow[r] & \Omega\Sigma\overline{X} \arrow[r, "\theta_Z"] & \Omega Q^{\max}(\overline{X}) \arrow[r, "\Omega\psi_Z"] & \Omega MQ^{\max}(Z).
\end{tikzcd}$$

The left square homotopy commutes by the previous diagram. The middle square homotopy commutes by the naturality of Theorem 2.2. For the right square, by definition (preceeding Proposition 5.3), $\psi_Y = \bigvee_{n=1}^{\infty} \psi_{n,Y}$, where each $\psi_{n,Y}$ is natural for co-H maps $Y \longrightarrow Z$ between coassociative co-H spaces. Thus the right square homotopy commutes as well and hence the entire rectangle homotopy commutes. But the top row of the diagram is the definition of $M\theta_Y$ while the bottom row is the definition of $M\theta_Z$. ∎

Proposition 5.7 *The homotopy decomposition $\Omega Y \simeq A^{\min}(Y) \times \Omega M Q^{\max}(Y)$ of Proposition* 5.3 *is natural with respect to co-H maps $Y \longrightarrow Z$ between homotopy coassociative co-H spaces.*

Proof The homotopy fibration diagram preceeding Lemma 5.6 gives a homotopy commutative square

$$\begin{array}{ccc} \Omega Y & \longrightarrow & A^{\min}(Y) \\ \downarrow & & \downarrow \\ \Omega Z & \longrightarrow & A^{\min}(Z). \end{array}$$

Combining this with the diagram in Lemma 5.6 gives a homotopy commutative diagram of equivalences

$$\begin{array}{ccc} \Omega Y & \xrightarrow{\simeq} & A^{\min}(Y) \times \Omega M Q^{\max}(Y) \\ \downarrow & & \downarrow \\ \Omega Z & \xrightarrow{\simeq} & A^{\min}(Z) \times \Omega M Q^{\max}(Z), \end{array}$$

which proves the Proposition. ∎

Finally, observe that Theorem 1.1 is the combination of Propositions 5.3, 5.5, and 5.7.

References

[BH] I. Berstein and J. R. Harper, *Cogroups which are not suspensions.* In: Algebraic Topology, Lecture Notes in Math. 1370, 1989, pp. 63–86.

[BK] A. K. Bousfield and D. M. Kan, *Homotopy Limits, Completions and Localizations.* Lecture Notes in Mathematics 304, Springer-Verlag, Berlin, 1972.

[G] T. Ganea, *Cogroups and suspensions.* Invent. Math. **9**(1970), 185–197.

[M] J. P. May, *A Concise Course in Algebraic Topology.* Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1999.

[SW1] P. Selick and J. Wu, *On Natural Coalgebra Decompositions of Tensor Algebras and Loop Suspensions.* Mem. Amer. Math. Soc. 148, No. 701, 2000.

[SW2] ———, *The functor A^{min} on p-local spaces.* Math. Z., to appear.

[SW3] ———, *On functorial decompositions of self-smash products.* Manuscripta Math. **111**(2003), no. 4, 435–457.
[T] S. D. Theriault, *Homotopy decompositions involving the loops of coassociative co-H spaces.* Canad. J. Math. **55**(2003), no. 1, 181–203.

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