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A NOTE ON COMPLETE HYPERSURFACES OF NON-POSITIVE RICCI CURVATURE

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In this note we point out that a recent result of Leung concerning hypersurfaces of a Euclidean space has a simple generalisation to hypersurfaces of complete simply-connected Riemannian manifolds of non-positive constant sectional curvature.

The purpose of this note is to establish the following results.

THEOREM. *Let* $\bar{M}$ *be a complete simply-connected* $n+1$ *dimensional Riemannian manifold of constant curvature* $C \le 0$. *Let* M *be a complete hypersurface in* $\bar{M}$ *such that all sectional curvatures on* M *are bounded away from* $-\infty$. *If* M *is contained in a ball of radius* R *then*

$$\limsup_{\substack{p \in M \\ X \in M_p \\ \|X\|=1}} \mathrm{Ric}(X, X) \ge \frac{n-1}{R^2}\left(1+R^2 C\right) .$$

COROLLARY. *Let* $\bar{M}$ *be as in the above theorem. Let* M *be a complete hypersurface in* $\bar{M}$ *such that all sectional curvatures are bounded away from* $-\infty$. *If the Ricci curvature of* M *satisfies* $\mathrm{Ric}(X, X) \le C\|X\|^2$ *then* M *is unbounded.*

Proof. The case $C = 0$ has been proved by Leung [1]. We sketch the

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proof for the case $C = -1/\rho^2 < 0$. We choose an arbitrary point $0 \in \overline{M}$, $0 \notin M$. There exists a coordinate system $r, \theta_1, \theta_2, \ldots, \theta_n$ on $\overline{M} - \{0\}$ and 1 - forms

$$\omega^1 = \rho dr$$

$$\omega^2 = \rho \sinh r d\theta_1$$

$$\omega^3 = \rho \sinh r \cos \theta_1 d\theta_2$$

$$\cdots\cdot$$

$$\omega^{n+1} = \rho \sinh r \cos \theta_1 \cos \theta_2 \ldots \cos \theta_{n-1} d\theta_n$$

such that the metric on $\overline{M} - \{0\}$ is given by

$$g = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \ldots + \omega^{n+1} \otimes \omega^{n+1} .$$

Let $e_1, \ldots, e_{n+1}$ be dual to $\omega^1, \ldots, \omega^{n+1}$. For convenience in notation we put $\theta_0 \equiv 0$. The connection 1 - forms for the above metric are (for $2 \le k \le n+1$, $2 \le j < k$)

$$\omega_1^k = \cosh r \cos \theta_0 \cos \theta_1 \ldots \cos \theta_{k-2} d\theta_{n-1} ,$$

$$\omega_j^k = -\sin \theta_{j-1} \cos \theta_j \ldots \cos \theta_{k-2} d\theta_{k-1} .$$

Define a vector field X on $\overline{M} - \{0\}$ by

$$X_p = \rho(re_1)_p .$$

Let

$$V = a_1 e_1 + \ldots + a_{n+1} e_{n+1}$$

be a unit vector field.

An elementary calculation shows that

$$\overline{D}_V X = a_1 e_1 + \sum_{i,k} a_i r \omega_1^k (e_i) e_k$$
$$= a_1 e_1 + r \cosh r (a_2 e_2 + \ldots + a_{n+1} e_{n+1}) .$$

Consequently,

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$$\langle \bar{D}_V X, X \rangle = a_1 r = \langle V, X \rangle$$

and

$$\langle V, \bar{D}_V X \rangle = a_1^2 + r \cosh r \left(a_2^2 + \ldots + a_{n+1}^2\right)$$
$$\geq 1 .$$

Now suppose M is bounded, so that M lies inside a ball of radius R, say. We define a function f on M by $f(p) = \langle X_p, X_p \rangle$ for $p \in M$. Clearly $f(p) \leq R^2$ for all $p \in M$ and so is bounded. We have, for a unit vector $V \in M_p$,

$$\begin{aligned} \text{Hess } f(V, V) &= VVf - D_V V(f) \\ &= 2V\langle V, X \rangle - 2\langle D_V V, X \rangle \\ &= 2\langle \bar{D}_V V, X \rangle + 2\langle V, D_V X \rangle - 2\langle D_V V, X \rangle \\ &\geq 2\langle B(V, V), X \rangle + 2 , \end{aligned}$$

where B is the second fundamental form on M. Following Leung's proof we find that this inequality implies

$$\|B(V, V)\| > \frac{1}{R}\left(1 - \frac{1}{m}\right)$$

for all positive integers m. Consequently $B(V, V) \neq 0$.

We now take V_0 so that $\|B(V_0, V_0)\|^2$ is the minimum of $\|B(V, V)\|^2$ for all unit vectors $V \in M_p$.

As shown by Leung there exists an orthonormal basis $V_0, Y_1, \ldots, Y_{n-1}$ for M_p with $Y_1, \ldots, Y_{n-1}$ being an orthonormal basis for $\ker B(V_0, \cdot)$. Hence by the Gauss equation for hypersurfaces we have

$$\begin{aligned}\operatorname{Ric}(V_0, V_0) &= \sum_{i=1}^{n-1} R(V_0, Y_i, V_0, Y_i) \\ &= \frac{n-1}{\rho^2} + \sum_{i=1}^{n-1} \langle B(Y_i, Y_i), B(V_0, V_0)\rangle \\ &\geq \frac{1-n}{\rho^2} + \sum_{i=1}^{n-1} \|B(V_0, V_0)\|^2 \\ &> \frac{1-n}{\rho^2} + \frac{n-1}{R^2}\left(1 - \frac{1}{m}\right)^2 .\end{aligned}$$

Hence letting $m \to \infty$ we obtain the theorem above, from which the corollary readily follows.

Reference

[1] Pui-Fai Leung, "Complete hypersurface of non-positive Ricci curvature", *Bull. Austral. Math. Soc.* **27** (1983), 215-219.

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