Problem Corner

Solutions are invited to the following problems. They should be addressed to **Chris Starr** at **Morecambe Bay Academy, Dallam Avenue, Morecambe, Lancashire LA4 5BG** (e-mail: **cstarr@morecambebayacademy.co.uk**) and should arrive not later than 10 August 2024.

Proposals for problems are equally welcome. They should also be sent to Chris Starr at the above address and should be accompanied by solutions and any relevant background information.

Change of Editor

Nick Lord writes: As was flagged in the November 2022 *Gazette*, this is the last Problem Corner that I shall edit. From now on, all solutions and problem proposals should be sent to my successor, Chris Starr, whose details appear above. Chris has been an enthusiastic supporter of Problem Corner for many years and I wish him a happy and rewarding time.

I would like to take this opportunity to thank Gerry Leversha for his unstinting support as Editor and Bill Richardson for so expertly and speedily typesetting each Problem Corner. Above all, I would like to thank readers for their solutions and problem proposals. Without these, Problem Corner simply could not exist, and I am humbled by how much I have learnt from every problem I have curated.

Submissions to Problem Corner come from all over the world, usually, but not exclusively, electronically. Indeed, I fondly recall one hand-written letter which still had its customs docket attached. Asked to describe the contents, the sender had ticked the box "Present/Gift" and that, dear reader, aptly encapsulates what the past 20 years and almost 250 problems have felt like for me.

108.A (W. C. Gosnell)

The right-angled triangle **T** with legs of length a, b and hypotenuse of length c has inradius r. These lengths satisfy the conditions:

- $r + ka = b$ for some positive rational number, k;
- $c^2 = a + b + c$.

(a) Show that **T** is similar to a triangle whose sides form a Pythagorean triple.

(b) Can all Pythagorean triples be created this way?

108.B (Luc Duc Binh and Dau Anh Hung)

Given a regular n-sided polygon $A_1A_2 \ldots A_n$ with centre O and any straight line d. Let points $B_1, \ldots, B_n$ lie on d such that $\overrightarrow{A_1B_1}, \overrightarrow{A_2B_2}, \ldots, \overrightarrow{A_nB_n}$ are parallel vectors. Show that

$$\sum_{i=1}^{n} A_iB_i^2 = \sum_{i=1}^{n} OB_i^2.$$

108.C (George Stoica)

Find all continuous functions $f : [0, \infty) \to \mathbb{R}$ with the property that, for any $a > 0$, the function $x \mapsto f(x).f(a - x)$ is constant on the interval $[0, a]$.

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108.D (Toyesh Prakash Sharma)

(a) Show that $\int_0^{\infty} \left(1 - e^{-2x}\right) \frac{\sin^2 x}{x^3} = \frac{\pi}{2}$

(b) Show that $\int_{-\infty}^{\infty} \cos^2(\tan x) \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}\left(1 + \frac{1}{e^2}\right) = \int_{-\infty}^{\infty} \sin^2(\tan x) \frac{\cos^2 x}{x^2} dx.$

Solutions and comments on **107.E**, **107.F**, **107.G**, **107.H** (July 2023).

107.E (Stan Dolan)

Let p be a prime of the form $4k + 1$ and split the numbers $1, 2, \ldots, 2k$ into pairs (x, y) such that $x < y$ and $x^2 + y^2 \equiv 0 \pmod p$. Find the sum of all such y.

[For example, when $p = 5$ there is just one pair, (1, 2), and the sum is 2. When $p = 13$ there are three pairs, (1, 5), (2, 3), (4, 6), and the sum is 14.]

Answer: The sum of the y values is $\frac{1}{12}\left(p^2 - 1\right) = \frac{2}{3}k(2k + 1)$.

The elegant solution that follows is due to the proposer, Stan Dolan.

Let $U = \left\{(x, y) \mid 1 \leqslant x < y \leqslant \frac{1}{2}(p - 1), x^2 + y^2 \equiv 0 \pmod p\right\}$. It is readily shown that every integer u such that $1 \leqslant u \leqslant \frac{1}{2}(p - 1)$ is in precisely one element of U so, summing over every element of U, we have

$$\sum x + \sum y = 1 + 2 + \ldots + \tfrac{1}{2}(p - 1) = \frac{1}{2}\left(\frac{p-1}{2}\right)\left(\frac{p+1}{2}\right) = \frac{1}{8}\left(p^2 - 1\right). \quad (*)$$

Define two maps $f, g : U \to U$ as follows:

$$f(x, y) = \begin{cases} (y - x, y + x) & \text{if } y + x \leqslant \frac{1}{2}(p - 1) \\ (y - x, p - y - x) & \text{otherwise,} \end{cases}$$

$$g(x, y) = \begin{cases} \frac{1}{2}(y - x, y + x) & \text{if } y - x \text{ is even} \\ \frac{1}{2}(p - y - x, p - y + x) & \text{otherwise.} \end{cases}$$

Then f and g are well-defined and are inverses of each other. Therefore, from the definition of f, summing over every element of U gives $\sum(y - x) = \sum x$ from which $\sum x = \frac{1}{2}\sum y$. Substituting into (*) gives $\frac{3}{2}\sum y = \frac{1}{8}\left(p^2 - 1\right)$ and $\sum y = \frac{1}{12}\left(p^2 - 1\right)$.

Correct solutions were received from: M. G. Elliott, J. A. Mundie and the proposer Stan Dolan.

107.F (Michael Fox)

The point M lies in the plane of a parallelogram $KLK'L'$ but is not collinear with any pair of its vertices, nor concyclic with any triple. The circles KLM, $K'L'M$ meet again in N; circles $K'LM$, $KL'M$ meet again in N'; circles $K'LN$, $KL'N$ meet again in M'. Prove that K, L, M', N' are concyclic, as are K', L', M', N', and show that the eight named points lie on a conic.

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The neat solution that follows is due to the proposer, Michael Fox.

The five given points lie on a unique conic that has parallel chords KL, $K'L'$. Hence the line through their midpoints is a diameter. Similarly, so is the line through the midpoints of LK', $L'K$. The centre of the conic is therefore the central point of $KLK'L'$, and the curve is either an ellipse or a hyperbola. We show that each new intersection point lies on this conic.

(a) Suppose it is an ellipse. We can take axes so that it is $\mathscr{E}\left(a\frac{1-t^2}{1+t^2}, b\frac{2t}{1+t^2}\right)$. This meets the circle $x^2+y^2+2gx+2fy+c=0$ where

$$\left(a^2-2ag+c\right)t^4+4bft^3+2\left(-a^2+2b^2+c\right)t^2+4bft+\left(a^2+2ag+c\right)=0.$$

Since the coefficients of t^3 and t are identical, the roots satisfy

$$t_1t_2t_3+t_1t_2t_4+t_1t_3t_4+t_2t_3t_4 = t_1+t_2+t_3+t_4.$$

Hence (KLM) – the circle KLM – meets $\mathscr{E}$ again in the point with parameter $\frac{k+l+m-klm}{-1+lm+mk+kl}$. The parameters of diametrically opposite points are negative reciprocals, so since K and K' are opposites, the parameter of K' is $-\frac{1}{k}$. Replacing k with $-\frac{1}{k}$ gives $-\left(\frac{-1+lm+mk+kl}{k+l+m-klm}\right)$. Hence the intersection of $\mathscr{E}$ with $\left(K'LM\right)$ is opposite its intersection with (KLM). Replacing any pair of the three points with their opposites leaves the fourth intersection unchanged. Thus (KLM) and $\left(K'L'M\right)$ meet $\mathscr{E}$ in the same point, which we identify with N. Also $(K'LM)$ and $\left(KL'M\right)$ meet in the point opposite N, that is N'. Since $\left(K'LN'\right)$ meets $\mathscr{E}$ in M, we have immediately that $\left(K'LN\right)$ and $\left(KL'N\right)$ meet in M', opposite M. Finally, since $\left(K'LM\right)$ passes through N', double interchanges give immediately that $\left(KLM'\right)$ and $\left(K'L'M'\right)$ also pass through N'. It is clear that all eight points lie on $\mathscr{E}$.

(b) If the conic is a hyperbola $\mathscr{H}$, we can take it as $\left(\alpha\frac{1-\tau^2}{1+\tau^2}, \beta\frac{2\tau}{1+\tau^2}\right)$, where β and τ are pure imaginary, and reinterpret the results we found for the ellipse. Or. we can express $\mathscr{H}$ as $(a\sec\theta, b\tan\theta)$, which gives $\left(a\frac{1+t^2}{1-t^2}, b\frac{2t}{1-t^2}\right)$. This meets $x^2+y^2+2gx+2fy+c=0$ where $\left(a^2-2ag+c\right)t^4-4bft^3+2\left(a^2+2b^2-c\right)t^2+4bft+\left(a^2+2ag+c\right)=0$. We find that $t_1t_2t_3+t_1t_2t_4+t_1t_3t_4+t_2t_3t_4+t_1+t_2+t_3+t_4=0$; so $t_4=-\left(\frac{t_1+t_2+t_3+t_1t_2t_3}{1+t_2t_3+t_3t_1+t_1t_2}\right)$. In this case the parameters of diametrically opposite points are reciprocals, and if we replace t_1, say, by its reciprocal, the expression for t_4 is also replaced by its reciprocal. The argument used for the ellipse is now easily adapted.

Michael Fox further comments that the centres of the circles in **107.F** lie on a rectangular hyperbola. Also, that if $KLK'L'$ is a general quadrilateral, the circles meet as in the problem, although the eight points do not lie on a conic.

Correct solutions were received from: M. G. Elliott, J. A. Mundie, L. Wimmer and the proposer Michael Fox.

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107.G (Mihály Bencze)

Let M be an interior point of a triangle ABC and let R_a, R_b, R_c be the circumradii of triangles MBC, MCA, MAB respectively. Prove that:

(a) $\sum \frac{R_a}{\sqrt{MB.MC}} \geqslant 3$,

(b) $3 + \sum \frac{R_a}{\sqrt{MB.MC}} \geqslant \sum \frac{1}{\sin\left(\frac{1}{4}B\hat{M}C\right)}$.

(a) Several solvers, including the proposer Mihály Bencze, used the following approach to part (a).

Let $\angle BMC = 2\alpha$, $\angle AMC = 2\beta$, $\angle AMB = 2\gamma$ with $0 < \alpha, \beta, \gamma < \frac{\pi}{2}$ and $\alpha + \beta + \gamma = \pi$. Then, by the cosine rule and the AM-GM inequality,

$$BC^2 = MB^2 + MC^2 - 2MB.MC \cos 2\alpha$$

$$\geqslant 2MB.MC\left(1 - \cos 2\alpha\right) = 4MB.MC \sin^2 \alpha.$$

But $BC = 2R_a \sin 2\alpha$, so $2R_a \sin 2\alpha \geqslant 2\sqrt{MB.MC} \sin \alpha$ and

$$\sum \frac{R_a}{\sqrt{MB.MC}} \geqslant \frac{1}{2} \sum \sec \alpha \geqslant \frac{1}{2} \cdot 3 \sec\left(\frac{\alpha + \beta + \gamma}{3}\right) = 3,$$

by Jensen's inequality since $x \mapsto \sec x$ is convex on $\left(0, \frac{\pi}{2}\right)$.

(b) Here, we note that (b) is a stronger inquality than (a) since, by Jensen's inequality, $\sum \frac{1}{\sin\left(\frac{1}{4}\angle BMC\right)} \geqslant \frac{3}{\sin\left(\frac{1}{6}(\alpha + \beta + \gamma)\right)} = 6$, which means that (b) $\Rightarrow$ (a).

There were several detailed and clearly argued solutions to (b), but the proposer Mihály Bencze gave a succinct solution by invoking *Popoviciu's inequality* which states that, for convex $f(x)$,

$$f(\alpha) + f(\beta) + f(\gamma) + 3f\left(\tfrac{1}{3}(\alpha + \beta + \gamma)\right) \geqslant 2 \sum f\left(\tfrac{1}{2}(\alpha + \beta)\right).$$

Applying this to $f(x) = \sec x$ on $\left(0, \frac{1}{2}\pi\right)$ with notation as in part (a),

$$3 + \sum \frac{R_a}{\sqrt{MB.MC}} \geqslant \tfrac{1}{2} \sum \sec \alpha + 3$$

$$= \tfrac{1}{2}\left[f(\alpha) + f(\beta) + f(\gamma) + 3f\left(\tfrac{1}{3}(\alpha + \beta + \gamma)\right)\right]$$

$$\geqslant \sum f\left(\tfrac{1}{2}(\alpha + \beta)\right) = \sum \frac{1}{\cos\left(\frac{1}{2}(\alpha + \beta)\right)} = \sum \frac{1}{\sin \frac{1}{2}\gamma},$$

as required.

Correct solutions were received from: M. G. Elliott, P. F. Johnson, J. A. Mundie, I. D. Sfikas, L. Wimmer and the proposer Mihály Bencze.

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107.H (Isaac Sofair)

The probability that a biased coin turns up heads when tossed is p. Derive closed form expressions for the following probabilities:

(a) The probability of getting no consecutive heads in n tosses of the coin.

(b) The probability that, if n people sitting around a circular table each flip the coin, no two adjacent people flip heads.

Answer: (a) The probability, p_n, of getting no consecutive heads in n tosses is

$$p_n = \left(\frac{1}{2} + \frac{1+p}{2\sqrt{(1-p)(1+3p)}}\right)\alpha^n + \left(\frac{1}{2} - \frac{1+p}{2\sqrt{(1-p)(1+3p)}}\right)\beta^n$$

where $\alpha = \frac{1}{2}\left(1 - p + \sqrt{(1-p)(1+3p)}\right)$ and $\beta = \frac{1}{2}\left(1 - p - \sqrt{(1-p)(1+3p)}\right)$.

(b) The probability, q_n, of getting no adjacent heads from n flips is $q_n = \alpha^n + \beta^n, n \geqslant 2$.

The following solution pools ideas from solvers, but most closely follows M. G. Elliott's argument.

(a) Clearly $p_0 = 1$, $p_1 = 1$, $p_2 = 1 - p(HH) = 1 - p^2$ and $p_{n+2} = p(T)p_{n+1} + p(HT)p_n = (1-p)p_{n+1} + p(1-p)p_n$ (*). This second-order recurrence relation has auxiliary quadratic $\lambda^2 - (1-p)\lambda - p(1-p) = 0$ with roots α, β as above and general solution $p_n = A\alpha^n + B\beta^n$. Fitting the initial conditions then gives the solution quoted above.

(b) All solvers connected part (b) with part (a).

Clearly $q_1 = 1, q_2 = 1 - p^2$ and

$$q_3 = p(TTT, HTT, THT, TTH) = (1-p)^3 + 3(1-p)^2 p = (1-p)^2(1+2p).$$

Splitting the circle at some point and considering the sequence of n flips as we go round the circle, we see that for $n \geqslant 4$,

$$\begin{aligned} q_n &= p_n - p(HT)p_{n-4}p(TH) \\ &= p_n - p^2(1-p)^2 p_{n-4} \\ &= p_n - p(1-p)\left[p_{n-2} - (1-p)p_{n-3}\right] \\ &= (1-p)p_{n-1} + p(1-p)^2 p_{n-3}, \text{ on using (*) twice.} \end{aligned}$$

Substituting for p_n from (a) and simplifying we obtain $q_n = \alpha^n + \beta^n$, which is readily checked to work for all $n \geqslant 2$.

The proposer, Isaac Sofair, observed that, if the coin is fair, then the probability in (b) becomes $q_n = \left(\frac{\phi}{2}\right)^n + \left(\frac{-1}{2\phi}\right)^n$ where $\phi = \frac{1}{2}\left(\sqrt{5} + 1\right)$ is the golden ratio.

Correct solutions were received from: P. F. Johnson, J. A. Mundie, S. Riccarelli, I. D. Sfikas, C. Starr and the proposer Isaac Sofair.

N.J.L.

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