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NORMAL FAMILIES OF MEROMORPHIC MAPPINGS OF SEVERAL COMPLEX VARIABLES FOR MOVING HYPERSURFACES IN A COMPLEX PROJECTIVE SPACE

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Abstract. The main aim of this article is to give sufficient conditions for a family of meromorphic mappings of a domain D in $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ to be meromorphically normal if they satisfy only some very weak conditions with respect to moving hypersurfaces in $\mathbb{P}^N(\mathbb{C})$, namely, that their intersections with these moving hypersurfaces, which moreover may depend on the meromorphic maps, are in some sense uniform. Our results generalize and complete previous results in this area, especially the works of Fujimoto, Tu, Tu-Li, Mai-Thai-Trang, and the recent work of Quang-Tan.

§1. Introduction

Classically, a family $\mathcal{F}$ of holomorphic functions on a domain $D \subset \mathbb{C}$ is said to be *(holomorphically) normal* if every sequence in $\mathcal{F}$ contains a subsequence which converges uniformly on every compact subset of D to a holomorphic map from D into P^1.

In 1957, Lehto and Virtanen [7] introduced the concept of normal meromorphic functions in connection with the study of boundary behavior of meromorphic functions of one complex variable. Since then normal families of holomorphic maps have been studied intensively, resulting in an extensive development in the one-complex-variable context and in generalizations to the several-complex-variables setting (see [1], [5], [6], [23], and references cited in [5] and [23]).

The first ideas and results on normal families of meromorphic mappings of several complex variables were introduced by Rutishauser [14] and Stoll [15].

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The notion of a meromorphically normal family into the N-dimensional complex projective space was introduced by Fujimoto [4]. (See Section 2.5 below for the definition of these concepts.) Fujimoto in [4] also gave some sufficient conditions for a family of meromorphic mappings of a domain D in $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ to be meromorphically normal. In 2002, Tu [21] considered meromorphically normal families of meromorphic mappings of a domain D in $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ for hyperplanes. Generalizing the above results of Fujimoto and Tu, in 2005 Mai, Thai, and Trang [8] gave a sufficient condition for the meromorphic normality of a family of meromorphic mappings of a domain D in $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ for fixed hypersurfaces, as follows. (See Section 2 below for the necessary definitions, in particular Section 2.3 for the definition of $D(\cdots)$.)

THEOREM A ([8, Theorem A]). *Let $\mathcal{F}$ be a family of meromorphic mappings of a domain D in $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. Suppose that for each $f \in \mathcal{F}$ there exist $q \geq 2N+1$ hypersurfaces $H_1(f), H_2(f), \ldots, H_q(f)$ in $\mathbb{P}^N(\mathbb{C})$ with*

$$\inf\{D(H_1(f), \ldots, H_q(f)), f \in \mathcal{F}\} > 0$$

and

$$f(D) \not\subset H_i(f) \quad (1 \leq i \leq N+1),$$

where q is independent of f, but the hypersurfaces $H_i(f)$ may depend on f, such that the following two conditions are satisfied.

(i) *For any fixed compact subset K of D, the $2(n-1)$-dimensional Lebesgue areas of $f^{-1}(H_i(f)) \cap K$ $(1 \leq i \leq N+1)$ with counting multiplicities are bounded above for all f in $\mathcal{F}$.*

(ii) *There exists a closed subset S of D with $\Lambda^{2n-1}(S) = 0$ such that, for any fixed compact subset K of $D - S$, the $2(n-1)$-dimensional Lebesgue areas of $f^{-1}(H_i(f)) \cap K$ $(N+2 \leq i \leq q)$ with counting multiplicities are bounded above for all f in $\mathcal{F}$.*

Then $\mathcal{F}$ is a meromorphically normal family on D.

Recently, motivated by the investigation of value distribution theory for moving hyperplanes (e.g., in [12], [13], [16]–[18]), the study of the normality of families of meromorphic mappings of a domain D in $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ for moving hyperplanes or hypersurfaces has started. While a substantial amount of information has been amassed concerning the normality of families of meromorphic mappings for fixed targets through the years, the present knowledge of this problem for moving targets has remained meagre. There

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are only a few such results in some restricted situations (see [11], [22]). For instance, we recall a recent result of Quang and Tan [11] which is the best result available at present and which generalizes [22, Theorem 2.2].

THEOREM B ([11, Theorem 1.4]). *Let $\mathcal{F}$ be a family of meromorphic mappings of a domain $D \subset \mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$, and let $Q_1, \ldots, Q_q$ $(q \geq 2N+1)$ be q moving hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ in (weakly) general position such that*

(i) *for any fixed compact subset K of D, the $2(n-1)$-dimensional Lebesgue areas of $f^{-1}(Q_j) \cap K$ $(1 \leq j \leq N+1)$ counting multiplicities are uniformly bounded above for all f in $\mathcal{F}$; and*

(ii) *there exists a thin analytic subset S of D such that for any fixed compact subset K of D, the $2(n-1)$-dimensional Lebesgue areas of $f^{-1}(Q_j) \cap (K - S)$ $(N+2 \leq j \leq q)$ regardless of multiplicities are uniformly bounded above for all f in $\mathcal{F}$.*

Then $\mathcal{F}$ is a meromorphically normal family on D.

We would like to emphasize that, in Theorem B, the q moving hypersurfaces $Q_1, \ldots, Q_q$ in $\mathbb{P}^N(\mathbb{C})$ are independent on $f \in \mathcal{F}$ (i.e., are common for all $f \in \mathcal{F}$). Thus, the following question arose naturally at this point: *Does Theorem A hold for moving hypersurfaces $H_1(f), H_2(f), \ldots, H_q(f)$ which may depend on $f \in \mathcal{F}$?* The main aim of this article is to give an affirmative answer to this question. Namely, we prove the following result, which generalizes both Theorems A and B.

THEOREM 1.1. *Let $\mathcal{F}$ be a family of meromorphic mappings of a domain D in $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. Suppose that for each $f \in \mathcal{F}$ there exist $q \geq 2N+1$ moving hypersurfaces $H_1(f), H_2(f), \ldots, H_q(f)$ in $\mathbb{P}^N(\mathbb{C})$ such that the following three conditions are satisfied.*

(i) *For each $1 \leq k \leq q$, the coefficients of the homogeneous polynomials $Q_k(f)$ which define the $H_k(f)$ are bounded above uniformly for all f in $\mathcal{F}$ on compact subsets of D, and for any sequence $\{f^{(p)}\} \subset \mathcal{F}$, there exists $z \in D$ (which may depend on the sequence) such that*

$$\inf_{p \in \mathbb{N}} \{D(Q_1(f^{(p)}), \ldots, Q_q(f^{(p)}))(z)\} > 0.$$

(ii) *For any fixed compact subset K of D, the $2(n-1)$-dimensional Lebesgue areas of $f^{-1}(H_i(f)) \cap K$ $(1 \leq i \leq N+1)$ counting multiplicities are bounded above for all f in $\mathcal{F}$ (in particular, $f(D) \not\subset H_i(f)$ $(1 \leq i \leq N+1)$).*

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(iii) *There exists a closed subset S of D with $\Lambda^{2n-1}(S)=0$ such that, for any fixed compact subset K of $D-S$, the $2(n-1)$-dimensional Lebesgue areas of $f^{-1}(H_i(f))\cap K$ $(N+2\le i\le q)$ ignoring multiplicities are bounded above for all f in $\mathcal{F}$.*

Then $\mathcal{F}$ is a meromorphically normal family on D.

In the special case of a family of holomorphic mappings, we get the following with the same proof methods.

THEOREM 1.2. *Let $\mathcal{F}$ be a family of holomorphic mappings of a domain D in $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. Suppose that for each $f\in\mathcal{F}$ there exist $q\ge 2N+1$ moving hypersurfaces $H_1(f),H_2(f),\dots,H_q(f)$ in $\mathbb{P}^N(\mathbb{C})$ such that the following three conditions are satisfied.*

(i) *For each $1\le k\le q$, the coefficients of the homogeneous polynomials $Q_k(f)$ which define the $H_k(f)$ are bounded above uniformly for all f in $\mathcal{F}$ on compact subsets of D, and for any sequence $\{f^{(p)}\}\subset\mathcal{F}$, there exists $z\in D$ (which may depend on the sequence) such that*

$$\inf_{p\in\mathbb{N}}\big\{D\big(Q_1(f^{(p)}),\dots,Q_q(f^{(p)})\big)(z)\big\}>0.$$

(ii) *Assume that $f(D)\cap H_i(f)=\emptyset$ $(1\le i\le N+1)$ for any $f\in F$.*

(iii) *There exists a closed subset S of D with $\Lambda^{2n-1}(S)=0$ such that, for any fixed compact subset K of $D-S$, the $2(n-1)$-dimensional Lebesgue areas of $f^{-1}(H_i(f))\cap K$ $(N+2\le i\le q)$ ignoring multiplicities are bounded above for all f in $\mathcal{F}$.*

Then $\mathcal{F}$ is a holomorphically normal family on D.

REMARK 1.1. Several examples in [21] show that the conditions in Theorem 1.1(i)–(iii) and Theorem 1.2(i)–(iii) cannot be omitted.

We also generalize several results of Tu from [20]–[22] which allow us not to take into account at all the components of $f^{-1}(H_i(f))$ of high order, as follows.

Theorem 1.3 generalizes [22, Theorem 2.1] from the case of moving hyperplanes which are independent of f to moving hypersurfaces which may depend on f. (In fact, observe that for moving hyperplanes the condition $H_1,\dots,H_q$ in $\widetilde{\mathcal{S}}(\{T_i\}_{i=0}^N)$ is satisfied by taking $T_0,\dots,T_N$ any (fixed or moving) $N+1$ hyperplanes in general position.)

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THEOREM 1.3. *Let $\mathcal{F}$ be a family of holomorphic mappings of a domain D in $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. Let $q \geq 2N+1$ be a positive integer. Let $m_1, \ldots, m_q$ be positive integers or ∞ such that*

$$\sum_{j=1}^{q} \Big(1 - \frac{N}{m_j}\Big) > N+1.$$

Suppose that for each $f \in \mathcal{F}$ there exist $N+1$ moving hypersurfaces $T_0(f)$, $\ldots, T_N(f)$ in $\mathbb{P}^N(\mathbb{C})$ of common degree and that there exist q moving hypersurfaces $H_1(f), \ldots, H_q(f)$ in $\widetilde{\mathcal{S}}(\{T_i(f)\}_{i=0}^N)$ such that the following conditions are satisfied.

(i) *For each $0 \leq i \leq N$, the coefficients of the homogeneous polynomials $P_i(f)$ which define the $T_i(f)$ are bounded above uniformly for all f in $\mathcal{F}$ on compact subsets of D; for all $1 \leq j \leq q$, the coefficients $b_{ij}(f)$ of the linear combinations of the $P_i(f)$, $i = 0, \ldots, N$ which define the homogeneous polynomials $Q_j(f)$ which define the $H_j(f)$ are bounded above uniformly for all f in $\mathcal{F}$ on compact subsets of D; and for any fixed $z \in D$,*

$$\inf\big\{D\big(Q_1(f), \ldots, Q_q(f)\big)(z) : f \in \mathcal{F}\big\} > 0.$$

(ii) *Assume that f intersects $H_j(f)$ with multiplicity at least m_j for each $1 \leq j \leq q$. (See Section 2.6 for the necessary definitions.)*

Then $\mathcal{F}$ is a holomorphically normal family on D.

The following theorem generalizes [21, Theorem 1] from the case of fixed hyperplanes to moving hypersurfaces. (In fact, observe that for hyperplanes the condition $H_1(f), \ldots, H_q(f)$ in $\widetilde{\mathcal{S}}(\{T_i(f)\}_{i=0}^N)$ is satisfied by taking $T_0(f)$, $\ldots, T_N(f)$ any $N+1$ hyperplanes in general position.)

THEOREM 1.4. *Let $\mathcal{F}$ be a family of meromorphic mappings of a domain D in $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. Let $q \geq 2N+1$ be a positive integer. Suppose that for each $f \in \mathcal{F}$ there exist $N+1$ moving hypersurfaces $T_0(f), \ldots, T_N(f)$ in $\mathbb{P}^N(\mathbb{C})$ of common degree and that there exist q moving hypersurfaces $H_1(f), \ldots, H_q(f)$ in $\widetilde{\mathcal{S}}(\{T_i(f)\}_{i=0}^N)$ such that the following conditions are satisfied.*

(i) *For each $0 \leq i \leq N$, the coefficients of the homogeneous polynomials $P_i(f)$ which define the $T_i(f)$ are bounded above uniformly for all f in $\mathcal{F}$ on compact subsets of D; for all $1 \leq j \leq q$, the coefficients $b_{ij}(f)$ of the linear combinations of the $P_i(f)$, $i = 0, \ldots, N$ which define the homogeneous polynomials $Q_j(f)$ which define the $H_j(f)$ are bounded above uniformly for*

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all f in $\mathcal{F}$ on compact subsets of D; and for any sequence $\{f^{(p)}\} \subset \mathcal{F}$, there exists $z \in D$ (which may depend on the sequence) such that

$$\inf_{p\in\mathbb{N}} \{D(Q_1(f^{(p)}),\dots,Q_q(f^{(p)}))(z)\} > 0.$$

(ii) *For any fixed compact K of D, the $2(n-1)$-dimensional Lebesgue areas of $f^{-1}(H_k(f)) \cap K$ $(1 \le k \le N+1)$ counting multiplicities are bounded above for all $f \in \mathcal{F}$ (in particular, $f(D) \not\subset H_k(f)$ $(1 \le k \le N+1)$).*

(iii) *There exists a closed subset S of D with $\Lambda^{2n-1}(S) = 0$ such that for any fixed compact subset K of $D - S$, the $2(n-1)$-dimensional Lebesgue areas of*

$$\{z \in \operatorname{Supp}\nu(f, H_k(f)) \mid \nu(f, H_k(f))(z) < m_k\} \cap K \quad (N+2 \le k \le q)$$

ignoring multiplicities for all $f \in \mathcal{F}$ are bounded above, where $\{m_k\}_{k=N+2}^{q}$ are fixed positive integers or ∞ with

$$\sum_{k=N+2}^{q} \frac{1}{m_k} < \frac{q-(N+1)}{N}.$$

Then $\mathcal{F}$ is a meromorphically normal family on D.

The following theorem generalizes [20, Theorem 1] from the case of fixed hyperplanes to moving hypersurfaces.

THEOREM 1.5. *Let $\mathcal{F}$ be a family of holomorphic mappings of a domain D in $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. Let $q \ge 2N+1$ be a positive integer. Suppose that for each $f \in \mathcal{F}$ there exist $N+1$ moving hypersurfaces $T_0(f),\dots,T_N(f)$ in $\mathbb{P}^N(\mathbb{C})$ of common degree and that there exist q moving hypersurfaces $H_1(f),\dots,H_q(f)$ in $\widetilde{\mathcal{S}}(\{T_i(f)\}_{i=0}^{N})$ such that the following conditions are satisfied.*

(i) *For each $0 \le i \le N$, the coefficients of the homogeneous polynomials $P_i(f)$ which define the $T_i(f)$ are bounded above uniformly for all f in $\mathcal{F}$ on compact subsets of D; for all $1 \le j \le q$, the coefficients $b_{ij}(f)$ of the linear combinations of the $P_i(f)$, $i = 0,\dots,N$ which define the homogeneous polynomials $Q_j(f)$ which define the $H_j(f)$ are bounded above uniformly for all f in $\mathcal{F}$ on compact subsets of D; and for any sequence $\{f^{(p)}\} \subset \mathcal{F}$, there exists $z \in D$ (which may depend on the sequence) such that*

$$\inf_{p\in\mathbb{N}} \{D(Q_1(f^{(p)}),\dots,Q_q(f^{(p)}))(z)\} > 0.$$

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(ii) *Assume that* $f(D) \cap H_i(f) = \emptyset$ $(1 \le i \le N+1)$ *for any* $f \in \mathcal{F}$.

(iii) *There exists a closed subset* S *of* D *with* $\Lambda^{2n-1}(S) = 0$ *such that, for any fixed compact subset* K *of* $D - S$, *the* $2(n-1)$*-dimensional Lebesgue areas of*

$$\big\{z \in \operatorname{Supp} \nu\big(f, H_k(f)\big) \;\big|\; \nu\big(f, H_k(f)\big)(z) < m_k\big\} \cap K \quad (N+2 \le k \le q)$$

ignoring multiplicities for all f *in* $\mathcal{F}$ *are bounded above, where* $\{m_k\}_{k=N+2}^{q}$ *are fixed positive integers and may be* ∞ *with*

$$\sum_{k=N+2}^{q} \frac{1}{m_k} < \frac{q-(N+1)}{N}.$$

Then $\mathcal{F}$ *is a holomorphically normal family on* D.

Let us finally give some comments on our proof methods.

The proofs of Theorems 1.1 and 1.2 are obtained by generalizing ideas, which have been used by Mai, Thai, and Trang in [8] to prove Theorem A, to moving targets, which presents several highly nontrivial technical difficulties. Among others, for a sequence of moving targets $H(f^{(p)})$ which at the same time may depend of the meromorphic maps $f^{(p)} : D \to \mathbb{P}^N(\mathbb{C})$, obtaining a subsequence which converges locally uniformly on D is much more difficult than for fixed targets. (Among other difficulties, we cannot normalize the coefficients to have norm equal to 1 everywhere as we can for fixed targets.) This is obtained in Lemma 3.6, after having proved in Lemma 3.5 that the condition $D(Q_1, \dots, Q_q) > \delta > 0$ forces a uniform bound, only in terms of δ, on the degrees of the Q_i, $1 \le i \le q$. (In fact the latter result fixes also a gap in [8] even for the case of fixed targets.)

The proofs of Theorems 1.3, 1.4, and 1.5 are obtained by combining methods used by Tu in [20] and [21] and by Tu and Li in [22] with the methods which we developed to prove our first two theorems. However, in order to apply the techniques used by Tu and by Tu and Li for the case of hyperplanes, we still need that, for every meromorphic map $f^{(p)} : D \to \mathbb{P}^N(\mathbb{C})$, the $Q_1(f^{(p)}), \dots, Q_q(f^{(p)})$ are still in a linear system given by $N+1$ such maps $P_0(f^{(p)}), \dots, P_N(f^{(p)})$. Lemmas 3.11–3.14 adapt our techniques to this situation. (For example, Lemma 3.14 is an adaptation of Lemma 3.6.)

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§2. Basic notions

2.1. Meromorphic mappings

Let A be a nonempty open subset of a domain D in $\mathbb{C}^n$ such that $S = D - A$ is an analytic set in D. Let $f : A \to \mathbb{P}^N(\mathbb{C})$ be a holomorphic mapping. Let U be a nonempty connected open subset of D. A holomorphic mapping $\tilde{f} \not\equiv 0$ from U into $\mathbb{C}^{N+1}$ is said to be a *representation of* f *on* U if $f(z) = \rho(\tilde{f}(z))$ for all $z \in U \cap A - \tilde{f}^{-1}(0)$, where $\rho : \mathbb{C}^{N+1} - \{0\} \to \mathbb{P}^N(\mathbb{C})$ is the canonical projection. A holomorphic mapping $f : A \to \mathbb{P}^N(\mathbb{C})$ is said to be a *meromorphic mapping from* D *into* $\mathbb{P}^N(\mathbb{C})$ if for each $z \in D$ there exists a representation of f on some neighborhood of z in D.

2.2. Admissible representations

Let f be a meromorphic mapping of a domain D in $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. Then for any $a \in D$, f always has an admissible representation $\tilde{f}(z) = (f_0(z), f_1(z), \ldots, f_N(z))$ on some neighborhood U of a in D, which means that each $f_i(z)$ is a holomorphic function on U and that $f(z) = (f_0(z) : f_1(z) : \cdots : f_N(z))$ outside the analytic set $I(f) := \{z \in U : f_0(z) = f_1(z) = \cdots = f_N(z) = 0\}$ of codimension ≥ 2.

2.3. Moving hypersurfaces in general position

Let D be a domain in $\mathbb{C}^n$. Denote by $\mathcal{H}_D$ the ring of all holomorphic functions on D, and denote by $\widetilde{\mathcal{H}}_D[\omega_0, \ldots, \omega_N]$ the set of all homogeneous polynomials $Q \in \mathcal{H}_D[\omega_0, \ldots, \omega_N]$ such that the coefficients of Q are not all identically zero. Each element of $\widetilde{\mathcal{H}}_D[\omega_0, \ldots, \omega_N]$ is said to be a *moving hypersurface* in $\mathbb{P}^N(\mathbb{C})$.

Let Q be a moving hypersurface of degree $d \geq 1$. Denote by $Q(z)$ the homogeneous polynomial over $\mathbb{C}^{N+1}$ obtained by evaluating the coefficients of Q in a specific point $z \in D$. We remark that for generic $z \in D$ this is a nonzero homogeneous polynomial with coefficients in $\mathbb{C}$. The hypersurface H given by $H(z) := \{w \in \mathbb{C}^{N+1} : Q(z)(w) = 0\}$ (for generic $z \in D$) is also called a *moving hypersurface* in $\mathbb{P}^N(\mathbb{C})$, which is defined by Q. In this article, we identify Q with H if no confusion arises.

We say that moving hypersurfaces $\{Q_j\}_{j=1}^q$ of degree d_j $(q \geq N+1)$ in $\mathbb{P}^N(\mathbb{C})$ are located in *(weakly) general position* if there exists $z \in D$ such that, for any $1 \leq j_0 < \cdots < j_N \leq q$, the system of equations

$$\begin{cases} Q_{j_i}(z)(\omega_0, \ldots, \omega_N) = 0, \\ 0 \leq i \leq N \end{cases}$$

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has only the trivial solution $\omega = (0, \dots, 0)$ in $\mathbb{C}^{N+1}$. This is equivalent to

$$D(Q_1, \dots, Q_q)(z) := \prod_{1 \le j_0 < \cdots < j_N \le q} \inf_{\|\omega\|=1} \left(\left|Q_{j_0}(z)(\omega)\right|^2 + \cdots + \left|Q_{j_N}(z)(\omega)\right|^2 \right) > 0,$$

where $Q_j(z)(\omega) = \sum_{|I|=d_j} a_{jI}(z) \cdot \omega^I$ and $\|\omega\| = (\sum |\omega_j|^2)^{1/2}$.

2.4. Divisors

Let D be a domain in $\mathbb{C}^n$, and let f be a nonidentically zero holomorphic function on D. For a point $a = (a_1, a_2, \dots, a_n) \in D$ we expand f as a compactly convergent series

$$f(u_1 + a_1, \dots, u_n + a_n) = \sum_{m=0}^{\infty} P_m(u_1, \dots, u_n)$$

on a neighborhood of a, where P_m is either identically zero or a homogeneous polynomial of degree m. The number

$$\nu_f(a) := \min\{m; P_m(u) \not\equiv 0\}$$

is said to be the *zero multiplicity* of f at a. By definition, a divisor on D is an integer-valued function ν on D such that for every $a \in D$ there are holomorphic functions $g(z)$ $(\not\equiv 0)$ and $h(z)$ $(\not\equiv 0)$ on a neighborhood U of a with $\nu(z) = \nu_g(z) - \nu_h(z)$ on U. We define the support of the divisor ν on D by

$$\operatorname{Supp} \nu := \overline{\{z \in D : \nu(z) \neq 0\}}.$$

We denote $\mathcal{D}^+(D) = \{\nu\text{: a nonnegative divisor on } D\}$.

Let f be a meromorphic mapping from a domain D into $\mathbb{P}^N(\mathbb{C})$. For each homogeneous polynomial $Q \in \widetilde{\mathcal{H}}_D[\omega_0, \dots, \omega_N]$, we define the divisor $\nu(f, Q)$ on D as follows. For each $a \in D$, let $\tilde{f} = (f_0, \dots, f_N)$ be an admissible representation of f in a neighborhood U of a. Then we put

$$\nu(f, Q)(a) := \nu_{Q(\tilde{f})}(a),$$

where $Q(\tilde{f}) := Q(f_0, \dots, f_N)$.

Let H be a moving hypersurface which is defined by the homogeneous polynomial $Q \in \widetilde{\mathcal{H}}_D[\omega_0, \dots, \omega_N]$, and let f be a meromorphic mapping of D into $\mathbb{P}^N(\mathbb{C})$. As above, we define the divisor $\nu(f, H)(z) := \nu(f, Q)(z)$. Obvi-

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ously, $\operatorname{Supp}\nu(f,H)$ is either an empty set or a pure $(n-1)$-dimensional analytic set in D if $f(D)\not\subset H$ (i.e., $Q(\tilde{f})\not\equiv 0$ on U). We define $\nu(f,H)=\infty$ on D and $\operatorname{Supp}\nu(f,H)=D$ if $f(D)\subset H$. Sometimes we identify $f^{-1}(H)$ with the divisor $\nu(f,H)$ on D. We can rewrite $\nu(f,H)$ as the formal sum $\nu(f,H)=\sum_{i\in I}n_iX_i$, where X_i are the irreducible components of $\operatorname{Supp}\nu(f,H)$ and n_i are the constants $\nu(f,H)(z)$ on $X_i\cap\operatorname{Reg}(\operatorname{Supp}\nu(f,H))$, where $\operatorname{Reg}(\ )$ denotes the set of all the regular points.

We say that the *meromorphic mapping f intersects H with multiplicity at least m on D* if $\nu(f,H)(z)\geq m$ for all $z\in\operatorname{Supp}\nu(f,H)$ and, in particular, that *f intersects H with multiplicity ∞ on D* if $f(D)\subset H$ or $f(D)\cap H=\emptyset$.

2.5. Meromorphically normal families

Let D be a domain in $\mathbb{C}^n$.

(i) (See [1].) Let $\mathcal{F}$ be a family of holomorphic mappings of D into a compact complex manifold M. This family $\mathcal{F}$ is said to be *a (holomorphically) normal family on D* if any sequence in $\mathcal{F}$ contains a subsequence which converges uniformly on compact subsets of D to a holomorphic mapping of D into M.

(ii) (See [4].) A sequence $\{f^{(p)}\}$ of meromorphic mappings from D into $\mathbb{P}^N(\mathbb{C})$ is said *to converge meromorphically on D to a meromorphic mapping f* if and only if, for any $z\in D$, each $f^{(p)}$ has an admissible representation

$$\tilde{f}^{(p)}=(f_0^{(p)}:f_1^{(p)}:\cdots:f_N^{(p)})$$

on some fixed neighborhood U of z such that $\{f_i^{(p)}\}_{p=1}^{\infty}$ converges uniformly on compact subsets of U to a holomorphic function f_i $(0\leq i\leq N)$ on U with the property that $\tilde{f}=(f_0:f_1:\cdots:f_N)$ is a representation of f on U (not necessarily an admissible one!).

(iii) (See [4].) Let $\mathcal{F}$ be a family of meromorphic mappings of D into $\mathbb{P}^N(\mathbb{C})$. One may call $\mathcal{F}$ *a meromorphically normal family on D* if any sequence in $\mathcal{F}$ has a meromorphically convergent subsequence on D.

(iv) (See [15].) Let $\{\nu_i\}$ be a sequence of nonnegative divisors on D. It is said to converge to a nonnegative divisor ν on D if and only if any $a\in D$ has a neighborhood U such that there exist holomorphic functions h_i $(\not\equiv 0)$ and h $(\not\equiv 0)$ on U such that $\nu_i=\nu_{h_i}$, $\nu=\nu_h$, and $\{h_i\}$ converges compactly to h on U.

(v) (See [15].) Let $\{A_i\}$ be a sequence of closed subsets of D. It is said to converge to a closed subset A of D if and only if A coincides with the set of all z such that every neighborhood U of z intersects A_i for all but

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finitely many i and, simultaneously, with the set of all z such that every U intersects A_i for infinitely many i.

2.6. Other notation

Let $P_0,\dots,P_N$ be $N+1$ homogeneous polynomials of common degree in $\mathbb{C}[\omega_0,\dots,\omega_N]$. Denote by $\mathcal{S}(\{P_i\}_{i=0}^N)$ the set of all homogeneous polynomials $Q=\sum_{i=0}^N b_iP_i$ $(b_i\in\mathbb{C})$.

Let $\{Q_j:=\sum_{i=0}^N b_{ji}P_i\}_{j=1}^q$ be q $(q\ge N+1)$ homogeneous polynomials in $\mathcal{S}(\{P_i\}_{i=0}^N)$. We say that $\{Q_j\}_{j=1}^q$ are *located in general position in* $\mathcal{S}(\{P_i\}_{i=0}^N)$ if

$$\forall 1\le j_0<\cdots<j_N\le q,\quad \det(b_{j_ki})_{0\le k,i\le N}\ne 0.$$

Let $T_0,\dots,T_N$ be hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ of common degree, where T_i is defined by the (not zero) polynomial P_i $(0\le i\le N)$. Denote by $\widetilde{\mathcal{S}}(\{T_i\}_{i=0}^N)$ the set of all hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ which are defined by $Q\in\mathcal{S}(\{P_i\}_{i=0}^N)$ with Q not zero.

Let $P_0,\dots,P_N$ be $N+1$ homogeneous polynomials of common degree in $\widetilde{\mathcal{H}}_D[\omega_0,\dots,\omega_N]$. Denote by $\widetilde{\mathcal{S}}(\{P_i\}_{i=0}^N)$ the set of all homogeneous not identically zero polynomials $Q=\sum_{i=0}^N b_iP_i$ $(b_i\in\mathcal{H}_D)$.

Let $T_0,\dots,T_N$ be moving hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ of common degree, where T_i is defined by the (not identically zero) polynomial P_i $(0\le i\le N)$. Denote by $\widetilde{\mathcal{S}}(\{T_i\}_{i=0}^N)$ the set of all moving hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ which are defined by $Q\in\widetilde{\mathcal{S}}(\{P_i\}_{i=0}^N)$.

Denote by $\mathrm{Hol}(X,Y)$ the set of all holomorphic mappings from a complex space X to a complex space Y.

For each $x\in\mathbb{C}^n$ and $R>0$, we set $B(x,R)=\{z\in\mathbb{C}^n:\|z-x\|<R\}$ and $B(R)=B(0,R)$.

Let $\Lambda^d(S)$ denote the real d-dimensional Hausdorff measure of $S\subset\mathbb{C}^n$. For a formal $\mathbb{Z}$-linear combination $X=\sum_{i\in I}n_iX_i$ of analytic subsets $X_i\subset\mathbb{C}^n$ and for a subset $E\subset\mathbb{C}^n$, we call $\sum_{i\in I}\Lambda^d(X_i\cap E)$ (resp., $\sum_{i\in I}n_i\Lambda^d(X_i\cap E)$) the *$d$-dimensional Lebesgue area* of $X\cap E$, ignoring multiplicities (resp., with counting multiplicities).

Finally, we list some facts on Hausdorff measures for later use which can be found, for example, in [2].

LEMMA 2.1 ([2, pp. 351–352 and 299–300]). *Let $B:=B(x,R)\subset\mathbb{C}^n$, and let $S\subset B$ be a closed subset with $\Lambda^{2n-1}(S)=0$. Let $h:B\to\mathbb{C}$ be a holomorphic function, let $h\not\equiv 0$, and let $S_1:=S\cup\{h=0\}$. Let $z_0\in B$. Then we have the following.*

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(a) *For almost every complex line* $l := \{z_0 + z \cdot u, z \in \mathbb{C}\}$ *passing through* z_0, *we have* $\Lambda^1(S_1 \cap l) = 0$.

(b) *Let* $r_0 := \operatorname{dist}(z_0, \partial B)$, *and for every* $r \in]0, r_0[$, *let* $\mathcal{C}_r := \{z_0 + r \cdot e^{i\theta} \cdot u : \theta \in [0, 2\pi]\} \subset l$. *If* $\Lambda^1(S_1 \cap l) = 0$, *then the subset of the* r *such that* $\mathcal{C}_r \cap S_1 \neq \emptyset$ *is nowhere dense in the interval* $]0, r_0[$.

COROLLARY 2.2. *With the notations of Lemma 2.1, the set* $B \setminus S_1$ *is pathwise connected.*

Proof. This is an immediate consequence of Lemma 2.1 and of the fact that $B \setminus S_1 \subset B$ is an open subset. □

§3. Lemmas

LEMMA 3.1 ([15, Theorem 2.24]). *A sequence* $\{\nu_i\}$ *of nonnegative divisors on a domain* D *in* $\mathbb{C}^n$ *is normal in the sense of the convergence of divisors on* D *if and only if the* $2(n-1)$*-dimensional Lebesgue areas of* $\nu_i \cap E$ $(i \geq 1)$ *with counting multiplicities are bounded above for any fixed compact set* E *of* D.

LEMMA 3.2 ([15, Theorem 4.10]). *If a sequence* $\{\nu_i\}$ *converges to* ν *in* $\mathcal{D}^+(B(R))$, *then* $\{\operatorname{supp} \nu_i\}$ *converges to* $\operatorname{supp} \nu$ *(in the sense of closed subsets of* D*).*

LEMMA 3.3 ([15, Proposition 4.12]). *Let* $\{N_i\}$ *be a sequence of pure* $(n-1)$*-dimensional analytic subsets of a domain* D *in* $\mathbb{C}^n$. *If the* $2(n-1)$*-dimensional Lebesgue areas of* $N_i \cap K$ *ignoring multiplicities* $(i = 1, 2, \ldots)$ *are bounded above for any fixed compact subset* K *of* D, *then* $\{N_i\}$ *is normal in the sense of the convergence of closed subsets in* D.

LEMMA 3.4 ([15, Proposition 4.11]). *Let* $\{N_i\}$ *be a sequence of pure* $(n-1)$*-dimensional analytic subsets of a domain* D *in* $\mathbb{C}^n$. *Assume that the* $2(n-1)$*-dimensional Lebesgue areas of* $N_i \cap K$ *ignoring multiplicities* $(i = 1, 2, \ldots)$ *are bounded above for any fixed compact subset* K *of* D *and that* $\{N_i\}$ *converges to* N *as a sequence of closed subsets of* D. *Then* N *is either empty or a pure* $(n-1)$*-dimensional analytic subset of* D.

LEMMA 3.5. *Let natural numbers* N *and* $q \geq N + 1$ *be fixed. Then for each* $\delta > 0$, *there exists* $M(\delta) = M(\delta, N, q) > 0$ *such that the following is satisfied.*

For any homogeneous polynomials $Q_1, \ldots, Q_q$ *on* $\mathbb{C}^{N+1}$ *with complex coefficients with norms bounded above by* 1 *such that* $D(Q_1, \ldots, Q_q) > \delta$, *we have* $\max\{\deg Q_1, \ldots, \deg Q_q\} < M(\delta)$.

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Proof. First of all, we make the following three remarks.

(i) Let $Q(\omega)$ be a homogeneous polynomial on $\mathbb{C}^{N+1}$ such that

$$Q(\omega) = \sum_{|\alpha|=d} a_\alpha \omega^\alpha,$$

where $|a_\alpha| \le 1$. Then

$$\big|Q(\omega)\big| \le \sum_{|\alpha|=d} |\omega_0|^{\alpha_0} \cdots |\omega_N|^{\alpha_N} \le (d+1)^{N+1} r^d,$$

when $|\omega_k| \le r \ \forall 0 \le k \le N$.

We set

$$M_0 = \sup_{d\in\mathbb{Z}^+} (d+1)^{N+1} \Big(\frac{1}{\sqrt{N+1}}\Big)^d.$$

Since $\lim_{d\longrightarrow+\infty}(d+1)^{N+1}(1/\sqrt{N+1})^d = 0$, it implies that $M_0 < +\infty$.

(ii) Let $Q_0, \ldots, Q_N$ be homogeneous polynomials on $\mathbb{C}^{N+1}$ such that the norms of their complex coefficients are bounded above by 1 and $D(Q_0, \ldots, Q_N) > 0$. We choose $\omega^{(0)} = (1/\sqrt{N+1}, \ldots, 1/\sqrt{N+1}) \in \mathbb{C}^{N+1}$. Then $\|\omega^{(0)}\| = 1$. By (i), we have

$$D(Q_0, \ldots, Q_N) \le (N+1) M_0^2 < +\infty.$$

(iii) Since $\lim_{x\to\infty}(1-(1/x))^x = 1/e$, we have $(e(1-(1/x))^x)/2 < 1$ for x big enough. Therefore,

$$\lim_{x\to\infty} (x^2+1)^{N+1} \Big(1-\frac{1}{x}\Big)^{x^2} = \lim_{x\to\infty} \Big(\frac{e(1-\frac{1}{x})^x}{2}\Big)^x \frac{(x^2+1)^{N+1}}{(\frac{e}{2})^x} = 0.$$

We now come back to the proof of Lemma 3.5, and we consider the following two cases.

Case 1: $q = N+1$.

Assume that such a constant $M(\delta) = M(\delta, N, N+1)$ does not exist. Then there exist homogeneous polynomials $Q_0^{(j)}, \ldots, Q_N^{(j)}$ $(j \ge 1)$ with coefficients being bounded above by 1 such that

$$\inf\big\{D(Q_0^{(j)}, \ldots, Q_N^{(j)}) : j \ge 1\big\} > \delta > 0,$$

$$\lim_{j\to\infty} \big(\max\{\deg Q_0^{(j)}, \ldots, \deg Q_N^{(j)}\}\big) = \infty.$$

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Without loss of generality, we may assume that

$$\deg Q_i^{(j)} = d_i \quad \forall\ 0 \le i \le k, \forall j \ge 1, \qquad \text{and}$$

$$\deg Q_i^{(j)} = d_i^{(j)} \to \infty \text{ as } j \to \infty \quad \text{for each } k+1 \le i \le N,$$

where k is some integer such that $0 \le k \le N-1$.

Since $\deg Q_i^{(j)} = d_i$ $\forall 0 \le i \le k$, $\forall j \ge 1$, we may assume that, for each $0 \le i \le k$, $\{Q_i^{(j)}\}_{j\ge 1}$ converges uniformly on compact subsets of $\mathbb{C}^{N+1}$ either to a homogeneous polynomial Q_i of degree d_i with coefficients being bounded above by 1 or to the zero polynomial. Since $0 \le k \le N-1$, we have

$$\bigcap_{i=0}^{k} \big\{ H_i := \text{Zero}(Q_i) \big\} \ne \emptyset.$$

Hence, there exists $\omega^{(0)} \in \bigcap_{i=0}^{k} H_i$ with $\|\omega^{(0)}\| = 1$. We now consider two subcases.

Subcase 1.1. Assume that $r = \max\{|\omega_0^{(0)}|, \dots, |\omega_N^{(0)}|\} < 1$.

- If $0 \le i \le k$, then
$$\lim_{j\to\infty} Q_i^{(j)}(\omega^{(0)}) = 0.$$
- If $k+1 \le i \le N$, then, by (i), we have
$$\big|Q_i^{(j)}(\omega^{(0)})\big| \le (d_i^{(j)}+1)^{N+1} r^{d_i^{(j)}}.$$

Since $\lim_{j\to\infty} d_i^{(j)} = \infty$ and $r < 1$, it implies that

$$\lim_{j\to\infty} Q_i^{(j)}(\omega^{(0)}) = 0 \quad \text{for } k+1 \le i \le N.$$

Therefore, we get

$$\lim_{j\to\infty} D(Q_0^{(j)}, \dots, Q_N^{(j)}) \le \lim_{j\to\infty} \sum_{i=0}^{N} \big|Q_i^{(j)}(\omega^{(0)})\big|^2 = 0.$$

This is a contradiction.

Subcase 1.2. Assume that $\max\{|\omega_0^{(0)}|, \dots, |\omega_N^{(0)}|\} = 1$.

We may assume that $\omega^{(0)} = (1, 0, \dots, 0)$. Set $\{\omega^{(j)}\}_{j\ge 1}$ such that

$$\omega_0^{(j)} = 1 - \frac{1}{\sqrt{d^{(j)}}}, \qquad \omega_1^{(j)} = \cdots = \omega_N^{(j)} = \frac{1}{\sqrt{N}} \sqrt{\frac{2}{\sqrt{d^{(j)}}} - \frac{1}{d^{(j)}}},$$

where $d^{(j)} = \min_{k+1\le i\le N} d_i^{(j)}$.

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- If $0 \le i \le k$, then

$$\lim_{j\to\infty} Q_i^{(j)}(\omega^{(j)}) = Q_i(\omega^{(0)}) = 0.$$

- Suppose that $k+1 \le i \le N$.

Since $\lim_{j\to\infty} d^{(j)} = \infty$, there exists j_0 such that

$$\max\{|\omega_0^{(j)}|, \dots, |\omega_N^{(j)}|\} = |\omega_0^{(j)}| = 1 - \frac{1}{\sqrt{d^{(j)}}} = r_j \quad \text{for any } j > j_0.$$

By (i) and (iii), for each $k+1 \le i \le N$, we have

$$\begin{aligned} |Q_i^{(j)}(\omega^{(j)})| &\le (d_i^{(j)}+1)^{N+1}\Big(1 - \frac{1}{\sqrt{d^{(j)}}}\Big)^{d_i^{(j)}} \\ &\le (d_i^{(j)}+1)^{N+1}\Big(1 - \frac{1}{\sqrt{d_i^{(j)}}}\Big)^{d_i^{(j)}} \to 0 \quad \text{as } j \to \infty. \end{aligned}$$

This is a contradiction by the same argument as above.

Case 2: $q > N+1$.

By (ii) we have

$$\delta < D(Q_1, \dots, Q_q) = \prod_{1\le j_0<\cdots<j_N\le q} D(Q_{j_0}, \dots, Q_{j_N}) \le C D(Q_{j_0}, \dots, Q_{j_N})$$

for any set $\{j_0, \dots, j_N\} \subset \{1, \dots, q\}$, where C is a constant which depends only on N and q.

By Case 1, we have

$$\max\{\deg Q_{j_0}, \dots, \deg Q_{j_N}\} < M(\delta/C, N, N+1)$$

for any set $\{j_0, \dots, j_N\} \subset \{1, \dots, q\}$. So if we define

$$M(\delta, N, q) := M(\delta/C, N, N+1)$$

(which is well defined since C only depends on N and q), then we have

$$\max\{\deg Q_1, \dots, \deg Q_q\} < M(\delta, N, q). \qquad \square$$

LEMMA 3.6. *Let natural numbers N and $q \ge N+1$ be fixed. Let $H_k^{(p)}$ $(1 \le k \le q,\ p \ge 1)$ be moving hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ such that the following conditions are satisfied.*

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(i) *For each* $1 \le k \le q$, $p \ge 1$, *the coefficients of the homogeneous polynomials* $Q_k^{(p)}$ *which define the* $H_k^{(p)}$ *are bounded above uniformly for all* $p \ge 1$ *on compact subsets of* D.

(ii) *There exists* $z_0 \in D$ *such that*

$$\inf_{p\in\mathbb{N}} \{D(Q_1^{(p)},\dots,Q_q^{(p)})(z_0)\} > \delta > 0.$$

Then, we have the following.

(a) *There exists a subsequence* $\{j_p\} \subset \mathbb{N}$ *such that for* $1 \le k \le q$, $Q_k^{(j_p)}$ *converge uniformly on compact subsets of* D *to not identically zero homogeneous polynomials* Q_k *(meaning that the* $Q_k^{(j_p)}$ *and* Q_k *are homogeneous polynomials in* $\widetilde{\mathcal{H}}_D[\omega_0,\dots,\omega_N]$ *of the same degree, and all their coefficients converge uniformly on compact subsets of* D*). Moreover, we have that* $D(Q_1,\dots,Q_q)(z_0) > \delta > 0$, *the hypersurfaces* $Q_1(z_0),\dots,Q_q(z_0)$ *are located in general position, and the moving hypersurfaces* $Q_1(z),\dots,Q_q(z)$ *are located in (weakly) general position.*

(b) *There exist a subsequence* $\{j_p\} \subset \mathbb{N}$ *and* $r = r(\delta) > 0$ *such that*

$$\inf\{D(Q_1^{(j_p)},\dots,Q_q^{(j_p)})(z) \mid p \ge 1\} > \frac{\delta}{4}, \quad \forall z \in B(z_0,r).$$

Proof. Let $d_k^{(p)} = \deg Q_k^{(p)}$ be the degree of the nonidentically vanishing homogeneous polynomial $Q_k^{(p)}$ $(1 \le k \le q,\ p \ge 1)$. Then we have

$$Q_k^{(p)}(z)(\omega) = \sum_{|I|=d_k^{(p)}} a_{kpI}(z)\cdot\omega^I,$$

where $I = (i_1,\dots,i_{N+1})$, $|I| = i_1 + \cdots + i_{N+1}$, and $a_{kpI}(z)$ are holomorphic functions which are bounded above uniformly for all $p \ge 1$ on compact subsets of D. Since the coefficients of the polynomials $Q_k^{(j_p)}$ are bounded above uniformly for all $p \ge 1$ on compact subsets of D, there exists $c > 0$ such that $|a_{kpI}(z_0)| \le c$ for all k, p, I. Define homogeneous polynomials

$$\tilde{Q}_k^{(p)}(z)(\omega) := \frac{1}{c} Q_k^{(p)}(z)(\omega).$$

Then the $\tilde{Q}_k^{(p)}(z)(\omega)$ satisfy the condition

$$\inf_{p\in\mathbb{N}} \{D(\tilde{Q}_1^{(p)},\dots,\tilde{Q}_q^{(p)})(z_0)\} > \tilde{\delta} > 0, \tag{3.1}$$

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with $\tilde{\delta} := (1/c)^{2\binom{q}{N+1}}\delta$. By Lemma 3.5, we have

$$\max\{\deg \tilde{Q}_1^{(p)}(z_0), \ldots, \deg \tilde{Q}_q^{(p)}(z_0)\} < M(\tilde{\delta}).$$

Since by (3.1) none of the homogeneous polynomials $Q_k^{(p)}(z_0)$ $(1 \le k \le q,$ $p \ge 1)$ can be the zero polynomial, we get that

$$\max\{\deg \tilde{Q}_1^{(p)}(z), \ldots, \deg \tilde{Q}_q^{(p)}(z)\} < M(\tilde{\delta})$$

for all $z \in D$. So if again

$$\tilde{Q}_k^{(p)}(z)(\omega) = \sum_{|I|=d_k^{(p)}} \tilde{a}_{kpI}(z) \cdot \omega^I,$$

after passing to a subsequence $\{j_p\} \subset \mathbb{N}$ (which we denote for simplicity again by $\{p\} \subset \mathbb{N}$), we can assume that $d_k^{(p)} = d_k$ for $1 \le k \le q$. So if we still multiply by c, we get

$$Q_k^{(p)}(z)(\omega) = \sum_{|I|=d_k} a_{kpI}(z) \cdot \omega^I.$$

Now, since the $a_{kpI}(z)$ are locally bounded uniformly for all $p \ge 1$ on D, by using Montel's theorem and a standard diagonal argument with respect to an exhaustion of D with compact subsets, after passing to a subsequence $\{j_p\} \subset \mathbb{N}$ (which we denote for simplicity again by $\{p\} \subset \mathbb{N}$), we also can assume that $\{a_{kpI}(z)\}_{p=1}^{\infty}$ converges uniformly on compact subsets of D to a_{kI} for each k, I. Denote

$$Q_k(z)(\omega) = \sum_{|I|=d_k} a_{kI}(z) \cdot \omega^I.$$

Then

$$D(Q_1, \ldots, Q_q)(z_0) \ge \liminf_{p \longrightarrow \infty} D(Q_1^{(p)}, \ldots, Q_q^{(p)})(z_0) > \delta > 0; \tag{3.2}$$

hence, the hypersurfaces $Q_1(z_0), \ldots, Q_q(z_0)$ are located in general position, and so the moving hypersurfaces $Q_1(z), \ldots, Q_q(z)$ are located (weakly) general position (and, in particular, all the $Q_1(z), \ldots, Q_q(z)$ are not identically zero), which proves (a).

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Moreover, by (3.2), there exists $r = r(\delta)$ such that

$$D(Q_1, \dots, Q_q)(z) > \frac{\delta}{2}, \quad \forall z \in B(z_0, r).$$

Since $\{Q_k^{(p)}\}$ converges uniformly on compact subsets of D to Q_k, after shrinking r a bit if necessary, there exists M such that

$$D(Q_1^{(p)}, \dots, Q_q^{(p)})(z) > \frac{\delta}{4}, \quad \forall z \in B(z_0, r), p > M,$$

which proves (b). □

LEMMA 3.7. *Let $\{f^{(p)}\}$ be a sequence of meromorphic mappings of a domain D in $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$, and let S be a closed subset of D with $\Lambda^{2n-1}(S) = 0$. Suppose that $\{f^{(p)}\}$ meromorphically converges on $D - S$ to a meromorphic mapping f of $D - S$ into $\mathbb{P}^N(\mathbb{C})$. Suppose that, for each $f^{(p)}$, there exist $N+1$ moving hypersurfaces $H_1(f^{(p)}), \dots, H_{N+1}(f^{(p)})$ in $\mathbb{P}^N(\mathbb{C})$, where the moving hypersurfaces $H_i(f^{(p)})$ may depend on $f^{(p)}$, such that the following three conditions are satisfied.*

(i) *For each $1 \le k \le N+1$, the coefficients of homogeneous polynomial $Q_k(f^{(p)})$ which define $H_k(f^{(p)})$ for all $f^{(p)}$ are bounded above uniformly for all $p \ge 1$ on compact subsets of D.*

(ii) *There exists $z_0 \in D$ such that*

$$\inf\{D(Q_1(f^{(p)}), \dots, Q_{N+1}(f^{(p)}))(z_0) \mid p \ge 1\} > 0.$$

(iii) *The $2(n-1)$-dimensional Lebesgue areas of $(f^{(p)})^{-1}(H_k(f^{(p)})) \cap K$ $(1 \le k \le N+1,\ p \ge 1)$ counting multiplicities are bounded above for any fixed compact subset K of D.*

Then we have the following:

(a) *$\{f^{(p)}\}$ has a meromorphically convergent subsequence on D, and*

(b) *if, moreover, $\{f^{(p)}\}$ is a sequence of holomorphic mappings of a domain D in $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ and condition (iii) is sharpened to*

$$f^{(p)}(D) \cap H_k(f^{(p)}) = \emptyset \quad (1 \le k \le N+1, p \ge 1),$$

then $\{f^{(p)}\}$ has a subsequence which converges uniformly on compact subsets of D to a holomorphic mapping of D to $\mathbb{P}^N(\mathbb{C})$.

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Proof. By Lemma 3.6 and conditions (i) and (ii), after passing to a subsequence we may assume that for $1 \le k \le N+1$, $Q_k(f^{(p)})$ converge uniformly on compact subsets of D to Q_k; in particular, they have common degree d_k. Moreover, $Q_1, \dots, Q_{N+1}$ are located in (weakly) general position. Denote by $H_1, \dots, H_{N+1}$ the corresponding moving hypersurfaces.

By Lemma 3.1 and condition (iii), after passing to a subsequence, we may assume that for every $1 \le k \le N+1$, the divisors

$$\big\{\nu\big(f^{(p)}, H_k(f^{(p)})\big)\big\} = (f^{(p)})^{-1}\big(H_k(f^{(p)})\big) \quad (p \ge 1)$$

are convergent (in the sense of convergence of divisors in D).

By a standard diagonal argument, we may assume that $D = B(R)$ and that $\{f^{(p)}\}$ meromorphically converges on $B(R) - S$ to a meromorphic mapping $f : B(R) - S \to \mathbb{P}^N(\mathbb{C})$.

We prove that there exists $k_0 \in \{1, \dots, N+1\}$ such that $f(D-S) \not\subset H_{k_0}$; more precisely, we prove that for any representation $f = (f_0 : \cdots : f_N)$ of $f : D - S \to \mathbb{P}^N(\mathbb{C})$ (admissible or not), we have that $Q_{k_0}(f_0, \dots, f_N) \not\equiv 0$: $E = \{z \in D : f_0(z) = f_1(z) = \cdots = f_N(z) = 0\}$ is a proper analytic subset. Since $Q_1, \dots, Q_{N+1}$ are located in (weakly) general position, there exists $z \in D$ such that the system of equations

$$\begin{cases} Q_k(z)(\omega_0, \dots, \omega_N) = 0, \\ 1 \le k \le N+1 \end{cases}$$

has only the trivial solution $\omega = (0, \dots, 0)$ in $\mathbb{C}^{N+1}$. But since then the same is true for the generic point $z \in D$, it is true in particular for the generic point $z \in D - E$. So for such point z there exists some $k \in \{1, \dots, N+1\}$ such that $Q_k(z)(f_0(z), \dots, f_N(z)) \ne 0$. In order to simplify notations, from now on we put

$$Q^{(p)} := Q_{k_0}(f^{(p)}), \qquad Q := Q_{k_0},$$
$$H^{(p)} := H_{k_0}(f^{(p)}), \qquad H := H_{k_0}, \qquad d := d_{k_0}.$$

Let z_1 be any point of S. By [15, Theorem 3.6], for any r $(0 < r < \tilde{R} = R - \|z_1\|)$, we can choose holomorphic functions $h^{(p)} \not\equiv 0$ and $h \not\equiv 0$ on $B(z_1, r)$ such that $\nu(f^{(p)}, H^{(p)}) = \nu_{h^{(p)}}$, $\nu = \nu_h$ for the limit ν of $\{\nu(f^{(p)}, H^{(p)})\}$, and $\{h^{(p)}\}$ converges uniformly on compact subsets of $B(z_1, r)$ to h. Moreover, each $f^{(p)}$ has an admissible representation on $B(z_1, r)$

$$f^{(p)} = (f_0^{(p)} : f_1^{(p)} : \cdots : f_N^{(p)})$$

with suitable holomorphic functions $f_i^{(p)}$ $(0 \le i \le N)$ on $B(z_1, r)$.

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Let z be a point in $B(z_1,r)-(S\cup\{h=0\})$. Choose a simply connected relatively compact neighborhood W_z of z in $B(z_1,r)-(S\cup\{h=0\})$ such that there exists a sequence $\{u_z^{(p)}\}$ of nonvanishing holomorphic functions on W_z such that $\{u_z^{(p)}f_i^{(p)}\}\to f_i^z$ $(0\le i\le N)$ on W_z and $f=(f_0^z:f_1^z:\cdots:f_N^z)$ on W_z. It may be assumed that $h^{(p)}$ $(p\ge 1)$ has no zero on W_z. We have $Q^{(p)}(f_0^{(p)},\dots,f_N^{(p)})=v^{(p)}h^{(p)}$, where $v^{(p)}$ is a nonvanishing holomorphic function on $B(z_1,r)$. This implies that $Q^{(p)}(u_z^{(p)}f_0^{(p)},\dots,u_z^{(p)}f_N^{(p)})\ne 0$ on W_z, since $Q^{(p)}$ is a homogeneous polynomial, and we have

$$Q^{(p)}(u_z^{(p)}f_0^{(p)},\dots,u_z^{(p)}f_N^{(p)})\to Q(f_0^z,\dots,f_N^z)$$

on W_z since $Q^{(p)}$ converge uniformly on compact subsets of D to Q. Since $f(D-S)\not\subset H$, it implies that $Q(f_0^z,\dots,f_N^z)\not\equiv 0$ on W_z, and hence $Q(f_0^z,\dots,f_N^z)\ne 0$ on W_z.

We recall that the $Q^{(p)}$, $p\ge 1$, and Q have common degree d. Since

$$Q^{(p)}(u_z^{(p)}f_0^{(p)},\dots,u_z^{(p)}f_N^{(p)}) \text{ tends to } Q(f_0^z,\dots,f_N^z) \text{ on } W_z \qquad \text{and}$$
$$Q^{(p)}(u_z^{(p)}f_0^{(p)},\dots,u_z^{(p)}f_N^{(p)})=(u_z^{(p)})^d\cdot v^{(p)}\cdot h^{(p)},$$

it follows that $(u_z^{(p)})^d\cdot v^{(p)}\cdot h^{(p)}$ tends to $Q(f_0^z,\dots,f_N^z)$ on W_z. Since $v^{(p)}\ne 0$ on $B(z_1,r)$, $v^{(p)}=(k^{(p)})^d$, where $k^{(p)}$ is a nonvanishing holomorphic function on $B(z_1,r)$. We have

$$(u_z^{(p)})^d\cdot(k^{(p)})^d=(u_z^{(p)}\cdot k^{(p)})^d\to\frac{Q(f_0^z,\dots,f_N^z)}{h}\quad\text{on } W_z.$$

Define F^z such that

$$(F^z)^d:=\frac{Q(f_0^z,\dots,f_N^z)}{h}\quad\text{on } W_z.$$

We can do this because $(Q(f_0^z,\dots,f_N^z))/h\ne 0$ on W_z. So $(u_z^{(p)}\cdot k^{(p)})^d\to(F^z)^d$ on W_z; hence, $((u_z^{(p)}\cdot k^{(p)})/F^z)^d$ tends to 1 on W_z. Therefore, there exist infinite (or empty) subsets $\{N_j^z\}_{j=0}^{d-1}$ of $\mathbb{N}$ such that $\mathbb{N}$ is a disjoint union of sets N_j^z and

$$\left\{\frac{u_z^{(p)}\cdot k^{(p)}}{F^z}\right\}_{p\in N_j^z}\to\theta_j=e^{i\cdot(2\pi j/d)}\quad\text{for each } 0\le j\le d-1.$$

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This implies that $\{f_i^{(p)}/k^{(p)}\}_{p\in N_j^z} \to F_i^z/\theta_j$ on W_z, where $F_i^z = f_i^z/F^z$ on W_z.

Take $a \in B(z_1, r) - (S \cup \{h = 0\})$. Then $\{f_i^{(p)}/k^{(p)}\}_{p\in N_j^a} \to F_i^a/\theta_j$ on W_a for each $0 \le j \le d-1$.

Take $b \in B(z_1, r) - (S \cup \{h = 0\})$ such that $W_a \cap W_b \neq \emptyset$. We will prove that $\{f_i^{(p)}/k^{(p)}\}_{p\in N_j^a} \to F_i^b/\theta_j \cdot c$ for each $0 \le j \le d-1$. Indeed, without loss of generality, we may assume that $f_0^a \not\equiv 0$ on W_a. Then $f_0^x \not\equiv 0$ on W_x for each $x \in B(z_1, r) - (S \cup \{h = 0\})$. Hence, $F_0^x \not\equiv 0$ on W_x for each $x \in B(z_1, r) - (S \cup \{h = 0\})$.

Consider $|N_j^a| = \infty$, where $|\cdot|$ denotes the cardinality of a set.

Assume that there exist N_1^b, N_2^b such that for $\tilde{N} := N_j^a \cap N_1^b$ and $\tilde{\tilde{N}} := N_j^a \cap N_2^b$ we have $|\tilde{N}| = |\tilde{\tilde{N}}| = \infty$. Since $\{f_0^{(p)}/k^{(p)}\}_{p\in\tilde{N}\subset N_1^b} \to F_0^b/\theta_1$ on W_b and $\{f_0^{(p)}/k^{(p)}\}_{p\in\tilde{N}\subset N_j^a} \to F_0^a/\theta_j$ on W_a, we have $F_0^b/\theta_1 = F_0^a/\theta_j$ on $W_a \cap W_b$. Similarly, $F_0^b/\theta_2 = F_0^a/\theta_j$ on $W_a \cap W_b$. This is a contradiction. Thus, every infinite subset N_j^a intersects and intersects infinitely only with the subset $N_{\alpha(j)}^b$. Moreover, $|N_j^a \Delta N_{\alpha(j)}^b| < \infty$, where $A\Delta B = (A - B) \cup (B - A)$ for arbitrary sets A, B.

From this it follows that there exists a bijection $\alpha : \{0, 1, \ldots, d-1\} \to \{0, 1, \ldots, d-1\}$ such that

$$N_j^a = \emptyset \qquad \text{if and only if } N_{\alpha(j)}^b = \emptyset;$$

$$\text{if } |N_j^a| = \infty \quad \text{then } |N_j^a \Delta N_{\alpha(j)}^b| < \infty.$$

On the other hand, since $\{f_0^{(p)}/k^{(p)}\}_{p\in N_j^a\cap N_{\alpha(j)}^b} \to F_0^a/\theta_j$ on W_a and $\{f_0^{(p)}/k^{(p)}\}_{p\in N_j^a\cap N_{\alpha(j)}^b} \to F_0^b/\theta_{\alpha(j)}$ on W_b, we have $F_0^a/\theta_j = F_0^b/\theta_{\alpha(j)}$ on $W_a \cap W_b$. This means that $F_0^a = F_0^b \cdot \theta_j/\theta_{\alpha(j)}$ on $W_a \cap W_b$ for each $0 \le j \le d-1$, and hence $c_b := \theta_j/\theta_{\alpha(j)}$ is a constant independent of j, $0 \le j \le d-1$. This implies that $\{f_i^{(p)}/k^{(p)}\}_{p\in N_j^a\cap N_{\alpha(j)}^b} \to F_i^b/\theta_{\alpha(j)} = F_i^b/\theta_j \cdot c_b$ on W_b, and hence

$$\left\{\frac{f_i^{(p)}}{k^{(p)}}\right\}_{p\in N_j^a} \to \frac{F_i^b}{\theta_j} \cdot c_b \quad \text{on } W_b.$$

Applying this procedure a finite number of times, we have

$$\left\{\frac{f_i^{(p)}}{k^{(p)}}\right\}_{p\in N_j^a} \to \frac{F_i^x}{\theta_j} \cdot c_x$$

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on W_x for each $x \in B(z_1,r) - (S \cup \{h=0\})$ and for each $0 \le j \le d-1$. Indeed, by the assumption on the Hausdorff dimension of S and by Corollary 2.2, the open set $B(z_1,r) - (S \cup \{h=0\})$ is pathwise connected, and such a path between a and x, which is compact as the image of a closed interval under a continuous map, can be covered by a finite number of such neighborhoods W_y with $y \in B(z_1,r) - (S \cup \{h=0\})$. And since the limit is unique if it exists, it does not depend on the choice of the path. For $p \in N_j^a$, put $\tilde{f}_i^{(p)} = f_i^{(p)} \cdot (\theta_j / k^{(p)})$ $(0 \le i \le N)$. Then $f^{(p)} = (\tilde{f}_0^{(p)}, \ldots, \tilde{f}_N^{(p)})$ for all $p \in N_j^a$ and $0 \le j \le d-1$, and $\{\tilde{f}_i^{(p)}\}_{p=1}^{\infty} \to F_i^x \cdot c_x$ on W_x for each $0 \le i \le N$. Note that if $W_x \cap W_y \ne \emptyset$ $(x, y \in B(z_1,r) - (S \cup \{h=0\}))$, then $F_i^x \cdot c_x = F_i^y \cdot c_y$ for each $0 \le i \le N$.

Define the function $F_i : B(z_1,r) - (S \cup \{h=0\}) \to \mathbb{C}$ given by $F_i|_{W_z} = F_i^z \cdot c_z$. Then $\{\tilde{f}_i^{(p)}\}_{p=1}^{\infty} \to F_i$ on $B(z_1,r) - (S \cup \{h=0\})$ for each $0 \le i \le N$.

We now prove that the sequence $\{f^{(p)}\}_{p=1}^{\infty}$ meromorphically converges on $B(z_1,r)$ to some meromorphic mapping $\tilde{F} = (\tilde{F}_0, \ldots, \tilde{F}_N)$. Indeed, let $z^{(0)}$ be any point of $S_1 = S \cup \{h=0\}$. Since $\Lambda^{2n-1}(S_1) = 0$, by Lemma 2.1(a) there exists a complex line $l_{z^{(0)}}$ passing through $z^{(0)}$ such that $\Lambda^1(S_1 \cap l_{z^{(0)}}) = 0$. Put $l_{z^{(0)}} = \{z^{(0)} + z \cdot u : z \in \mathbb{C}\}$. Then by Lemma 2.1(b) there exists $R > 0$ such that

$$\mathcal{C}_0 = \{z^{(0)} + R \cdot e^{i\theta} \cdot u : \theta \in [0, 2\pi]\},$$

satisfying $\mathcal{C}_0 \subset B(z_1,r)$ and $\mathcal{C}_0 \cap S_1 = \emptyset$. By the maximum principle, it implies that the sequence $\{\tilde{f}_i^{(p)}(z^{(0)})\}$ converges. Put $\lim_{p\to\infty} \tilde{f}_i^{(p)}(z^{(0)}) = \tilde{F}_i(z^{(0)})$. This means that the mapping F_i extends over $B(z_1,r)$ to the mapping $\tilde{F}_i$.

We now prove that the sequence $\{\tilde{f}_i^{(p)}(z)\}_{p=1}^{\infty}$ converges uniformly on compact subsets of $B(z_1,r)$ to $\tilde{F}_i(z)$. Indeed, assume that $\{z^{(j)}\} \subset B(z_1,r)$ converges to $z^{(0)} \in B(z_1,r)$. As above, there exists a circle $\mathcal{C}_0 = \{z^{(0)} + R \cdot e^{i\theta} \cdot u : \theta \in [0,2\pi]\} \subset B(z_1,r)$ such that $\mathcal{C}_0 \cap S_1 = \emptyset$. Since $\mathcal{C}_0$ is a compact subset of $B(z_1,r) - S_1$, there exists $\epsilon_0 > 0$ such that

$$V(\mathcal{C}_0, \epsilon_0) = \{z \in \mathbb{C}^n : \operatorname{dist}(z, \mathcal{C}_0) < \epsilon_0\} \Subset B(z_1,r) - S_1.$$

Consider the circles $\mathcal{C}_j = \{z^{(j)} + R \cdot e^{i\theta} \cdot u : \theta \in [0,2\pi]\}$. It is easy to see that $\operatorname{dist}(\mathcal{C}_0, \mathcal{C}_j) = \|z^{(j)} - z^{(0)}\| \to 0$ as $j \to \infty$. Thus, without loss of generality, we may assume that $\mathcal{C}_j \subset V(\mathcal{C}_0, \epsilon_0) \Subset B(z_1,r) - S_1$. By the hypothesis, $\forall \epsilon > 0$, $\exists N = N(\epsilon)$ such that

$$\sup\{\|\tilde{f}_i^{(p)}(z) - F_i(z)\| : z \in V(\mathcal{C}_0, \epsilon_0), p \ge N\} < \epsilon.$$

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By the maximum principle, we have $\limsup_{j\to\infty} \|\tilde{f}_i^{(j)}(z^{(j)}) - F_i(z^{(j)})\| = 0$. This implies that the sequence $\{\tilde{f}_i^{(p)}\}_{p=1}^{\infty}$ converges uniformly on compact subsets of $B(z_1, r)$ to $\tilde{F}_i$. This finishes the proof of part (a) of the lemma.

In order to prove part (b), we first remark that it suffices to prove that $\{f^{(p)}\}$ has a subsequence which converges locally uniformly on D to a holomorphic mapping f of D to $\mathbb{P}^N(\mathbb{C})$, which means that after passing to a subsequence we have the following.

Let z_1 be any point of D. Then there exists $r > 0$ and, for each $f^{(p)}$, a holomorphic representation on $B(z_1, r)$

$$f^{(p)} = (f_0^{(p)} : f_1^{(p)} : \cdots : f_N^{(p)})$$

with suitable holomorphic functions $f_i^{(p)}$ $(0 \le i \le N)$ without common zeros on $B(z_1, r)$, such that $\{f_i^{(p)}\} \to f_i$ $(0 \le i \le N)$ uniformly on $B(z_1, r)$ and $f = (f_0 : f_1 : \cdots : f_N)$ is a holomorphic map on $B(z_1, r)$; this means that the f_i $(0 \le i \le N)$ are without common zeros on $B(z_1, r)$.

By part (a) we know that $\{f^{(p)}\}$ has a subsequence which converges meromorphically on D to a meromorphic mapping f of D to $\mathbb{P}^N(\mathbb{C})$, which means that after passing to a subsequence we have the following.

Let z_1 be any point of D. Then there exists $r > 0$ and, for each $f^{(p)}$, an admissible representation on $B(z_1, r)$

$$f^{(p)} = (f_0^{(p)} : f_1^{(p)} : \cdots : f_N^{(p)})$$

with suitable holomorphic functions $f_i^{(p)}$ $(0 \le i \le N)$ on $B(z_1, r)$, such that $\{f_i^{(p)}\} \to f_i$ $(0 \le i \le N)$ uniformly on $B(z_1, r)$ and $f = (f_0 : f_1 : \cdots : f_N)$ is a meromorphic map on $B(z_1, r)$. Observing that the admissible representations of the holomorphic maps $f^{(p)} = (f_0^{(p)} : f_1^{(p)} : \cdots : f_N^{(p)})$ are automatically without common zeros, the only thing which remains to be proved is that under the conditions of (b) we have

$$E = \{z \in B(z_1, r) : f_0(z) = f_1(z) = \cdots = f_N(z) = 0\} = \emptyset.$$

We also recall that by the proof of part (a), we have that there exists $k_0 \in \{1, \ldots, N+1\}$ such that $Q^{(p)} = Q_{k_0}(f^{(p)})$, $p \ge 1$, converge uniformly on compact subsets of D to $Q = Q_{k_0}$, and $f(D - S) \not\subset H_{k_0}$; more precisely, we have that for any representation $f = (f_0 : \cdots : f_N)$ of the meromorphic map $f : D \to \mathbb{P}^N(\mathbb{C})$ (admissible or not) we have

$$Q(f_0, \ldots, f_N) \not\equiv 0. \tag{3.3}$$

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Now we can end the proof with an easy application of Hurwitz's theorem, as follows. By the condition of (b) we have that, for all $p \geq 1$,

$$Q^{(p)}(f_0^{(p)}, \ldots, f_N^{(p)}) \neq 0$$

on $B(z_1, r)$. And we also have that

$$Q^{(p)}(f_0^{(p)}, \ldots, f_N^{(p)}) \to Q(f_0, \ldots, f_N)$$

uniformly on compact subsets of $B(z_1, r)$. By (3.3) and Hurwitz's theorem we get that $Q(f_0, \ldots, f_N) \neq 0$ on $B(z_1, r)$. But since Q is a homogeneous polynomial, this implies that

$$E = \big\{z \in B(z_1, r) : f_0(z) = f_1(z) = \cdots = f_N(z) = 0\big\} = \emptyset. \qquad \square$$

We remark that the following corollary of Lemma 3.7(a) generalizes [2, Proposition 3.5].

COROLLARY 3.8. *Let $\{f^{(p)}\}$ be a sequence of meromorphic mappings of a domain D in $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$, and let S be a closed subset of D with $\Lambda^{2n-1}(S) = 0$. Suppose that $\{f^{(p)}\}$ meromorphically converges on $D - S$ to a meromorphic mapping f of $D - S$ into $\mathbb{P}^N(\mathbb{C})$. If there exists a moving hypersurface H in $\mathbb{P}^N(\mathbb{C})$ such that $f(D - S) \not\subset H$ and $\{\nu(f^{(p)}, H)\}$ is a convergent sequence of divisors on D, then $\{f^{(p)}\}$ is meromorphically convergent on D.*

LEMMA 3.9 ([19, Theorem 2.5]). *Let $\mathcal{F}$ be a family of holomorphic mappings of a domain D in $\mathbb{C}^n$ onto $\mathbb{P}^N(\mathbb{C})$. Then the family $\mathcal{F}$ is not normal on D if and only if there exist a compact subset $K_0 \subset D$ and sequences $\{f_i\} \subset \mathcal{F}$, $\{z_i\} \subset K_0$, $\{r_i\} \subset \mathbb{R}$ with $r_i > 0$ and $r_i \longrightarrow 0^+$, and $\{u_i\} \subset \mathbb{C}^n$ which are unit vectors such that*

$$g_i(\xi) := f_i(z_i + r_i u_i \xi),$$

where $\xi \in \mathbb{C}$ such that $z_i + r_i u_i \xi \in D$, converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic map g of $\mathbb{C}$ to $\mathbb{P}^N(\mathbb{C})$.

LEMMA 3.10 ([9, Theorem 4′]). *Suppose that $q \geq 2N + 1$ hyperplanes $H_1, \ldots, H_q$ are given in general position in $\mathbb{P}^N(\mathbb{C})$ and that q positive integers (may be ∞) $m_1, \ldots, m_q$ are given such that*

$$\sum_{i=1}^{q} \Big(1 - \frac{N}{m_j}\Big) > N + 1.$$

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Then there does not exist a nonconstant holomorphic mapping $f : \mathbb{C} \longrightarrow \mathbb{P}^N(\mathbb{C})$ *such that* f *intersects* H_j *with multiplicity at least* m_j *(*$1 \le j \le q$*).*

LEMMA 3.11. *Let* $P_0, \dots, P_N$ *be* $N+1$ *homogeneous polynomials of common degree in* $\mathbb{C}[x_0, \dots, x_n]$*. Also, let* $\{Q_j\}_{j=1}^q$ *(*$q \ge N+1$*) be homogeneous polynomials in* $\mathcal{S}(\{P_i\}_{i=0}^N)$ *such that*

$$D(Q_1, \dots, Q_q) = \prod_{1 \le j_0 < \cdots < j_N \le q} \inf_{\|\omega\|=1} \left(\left|Q_{j_0}(\omega)\right|^2 + \cdots + \left|Q_{j_N}(\omega)\right|^2 \right) > 0,$$

where $Q_j(\omega) = \sum_{|I|=d_j} a_{jI} \cdot \omega^I$*.*

Then $\{Q_j\}_{j=1}^q$ *are located in general position in* $\mathcal{S}(\{P_i\}_{i=0}^N)$*, and* $\{P_i\}_{i=0}^N$ *are located in general position in* $P^N(\mathbb{C})$ *(see Sections 2.3 and 2.6).*

Proof. (a) Suppose that $\{Q_j\}_{j=1}^q$ are not located in general position in $S(\{P_i\}_{i=0}^N)$. Then there exist $N+1$ polynomials in $\{Q_j\}_{j=1}^q$ which are not linearly independent. Without loss of generality, we may assume that

$$Q_{N+1} = \sum_{j=1}^{N} b_j Q_j \quad (b_j \in \mathbb{C}).$$

Then

$$\begin{aligned} X &= \left\{ \omega \in \mathbb{C}^{N+1} \;\middle|\; Q_1(\omega) = \cdots = Q_N(\omega) = Q_{N+1}(\omega) = 0 \right\} \\ &= \left\{ \omega \in \mathbb{C}^{N+1} \;\middle|\; Q_1(\omega) = \cdots = Q_N(\omega) = 0 \right\} \end{aligned}$$

is an analytic subset in $\mathbb{C}^{N+1}$. Since $\dim X \ge 1$, there exists $\omega_0 \ne 0$ in $\mathbb{C}^{N+1}$ such that

$$Q_1(\omega_0) = \cdots = Q_N(\omega_0) = Q_{N+1}(\omega_0) = 0.$$

Moreover, since $\{Q_j\}_{j=1}^q$ are all homogeneous polynomials, we may assume that $\|\omega_0\| = 1$. Thus, we have

$$\left|Q_1(\omega_0)\right|^2 + \cdots + \left|Q_{N+1}(\omega_0)\right|^2 = 0,$$

and hence

$$D(Q_1, \dots, Q_q) = 0.$$

This is a contradiction.

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(b) Suppose that $\{P_i\}_{i=0}^N$ are not located in general position in $P^N(\mathbb{C})$. Then there exists $\omega_0 \neq 0$ in $\mathbb{C}^{N+1}$ such that

$$P_0(\omega_0) = \cdots = P_N(\omega_0) = 0.$$

Therefore, we have $Q_j(\omega_0) = 0$ for any $1 \leq j \leq q$, and thus again,

$$D(Q_1, \ldots, Q_q) = 0.$$

This is a contradiction. □

LEMMA 3.12. *Let $f = (f_0 : \cdots : f_N) : \mathbb{C} \longrightarrow \mathbb{P}^N(\mathbb{C})$ be a holomorphic mapping, and let $\{P_i\}_{i=0}^N$ be $N+1$ homogeneous polynomials in general position of common degree in $\mathbb{C}[\omega_0, \ldots, \omega_N]$. Assume that*

$$F = (F_0 : \cdots : F_n) : \mathbb{P}^N(\mathbb{C}) \longrightarrow \mathbb{P}^N(\mathbb{C})$$

is the mapping defined by

$$F_i(\omega) = P_i(\omega) \quad (0 \leq i \leq N).$$

Then, $F(f)$ is a constant map if and only if f is a constant map.

Proof. Since $\{P_i\}_{i=0}^N$ are $N+1$ homogeneous polynomials in general position of common degree in $\mathbb{C}[\omega_0, \ldots, \omega_N]$, F is a morphism. Suppose that $F(f) = a$, where $a = (a_0 : \cdots : a_n) \in \mathbb{P}^N(\mathbb{C})$. We have $f(\mathbb{C}) \subset F^{-1}(a)$. Suppose that $\dim F^{-1}(a) \geq 1$. Take H any hyperplane in $\mathbb{P}^N(\mathbb{C})$ with $a \notin H$. Then $F^{-1}(H)$ is a hypersurface in $\mathbb{P}^N(\mathbb{C})$ since the $\{P_i\}_{i=0}^N$ are in general position, so in particular they are not linearly dependent. By Bézout's theorem there exists a point $\omega_0 \in F^{-1}(a) \cap F^{-1}(H)$. Hence, $a = F(\omega_0) \in H$. This is a contradiction. Therefore, $\dim F^{-1}(a) = 0$, so $F^{-1}(a)$ is a finite set. Since f is continuous and $f(\mathbb{C}) \subset F^{-1}(a)$, it must be a constant map. □

LEMMA 3.13. *Let $P_0, \ldots, P_N$ be $N+1$ homogeneous polynomials of common degree in $\mathbb{C}[\omega_0, \ldots, \omega_N]$, and let $\{Q_j\}_{j=1}^q$ ($q \geq 2N+1$) be homogeneous polynomials in $\mathcal{S}(\{P_i\}_{i=0}^N)$ such that*

$$D(Q_1, \ldots, Q_q) > 0.$$

Assume that $m_1, \ldots, m_q$ are positive integers (may be ∞) such that

$$\sum_{j=1}^{q} \Big(1 - \frac{N}{m_j}\Big) > N + 1.$$

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Then there does not exist a nonconstant holomorphic mapping

$$f : \mathbb{C} \longrightarrow \mathbb{P}^N(\mathbb{C})$$

such that f intersects Q_j with multiplicity at least m_j $(1 \le j \le q)$.

Proof. Suppose that $f : \mathbb{C} \longrightarrow \mathbb{P}^N(\mathbb{C})$ is a holomorphic mapping such that f intersects Q_i with multiplicity at least m_i $(1 \le i \le q)$. For each $1 \le i \le q$, we define

$$Q_j = \sum_{i=0}^{N} b_{ji} P_i$$

and

$$H_j = \Big\{ \omega \in \mathbb{C}^{N+1} \;\Big|\; \sum_{i=0}^{N} b_{ji}\omega_i = 0 \Big\}.$$

Let $\widetilde{f} = (f_0, \dots, f_N)$ be an admissible representation of f on $\mathbb{C}$ (i.e., the $f_0, \dots, f_N$ have no common zeros), and denote $F = (P_0(\widetilde{f}) : \cdots : P_N(\widetilde{f}))$. By Lemma 3.11, $\{P_i\}_{i=0}^N$ are in general position in $\mathbb{P}^N(\mathbb{C})$, and $\{Q_j\}_{j=1}^q$ are located in general position in $\mathcal{S}(\{P_i\}_{i=0}^N)$. This means that the hyperplanes $\{H_j\}_{j=1}^q$ are located in general position in $\mathbb{P}^N(\mathbb{C})$. Since f intersects Q_j with multiplicity at least m_j and

$$Q_j(\tilde{f}) = \Big(\sum_{i=0}^{N} b_{ji} P_i\Big)(\tilde{f}) = \sum_{i=0}^{N} b_{ji}\big(P_i(\tilde{f})\big),$$

F also intersects H_j with multiplicity at least m_j $(1 \le j \le q)$. By Lemma 3.10, F is a constant map, and by Lemma 3.12, f is a constant map, too. ∎

LEMMA 3.14. *Let natural numbers N and $q \ge N+1$ be fixed. Let $T_i^{(p)}$ $(0 \le i \le N,\ p \ge 1)$ be moving hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ of common degree $d^{(p)}$, and let $H_j^{(p)} \in \widetilde{\mathcal{S}}(\{T_i^{(p)}\}_{i=0}^N)$ $(1 \le j \le q,\ p \ge 1)$ such that the following conditions are satisfied.*

(i) *For each $0 \le i \le N$, the coefficients of the homogeneous polynomials $P_i^{(p)}$ which define the $T_i^{(p)}$ are bounded above uniformly for all $p \ge 1$ on compact subsets of D, and for all $1 \le j \le q$, the coefficients $b_{ij}^{(p)}(z)$ of the linear combinations of the $P_i^{(p)}$, $i = 0, \dots, N$, which define the homogeneous polynomials $Q_j^{(p)} = \sum_{i=0}^N b_{ij}^{(p)} P_i^{(p)}$ which define the $H_j^{(p)}$ are bounded above uniformly for all $p \ge 1$ on compact subsets of D.*

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(ii) *There exists* $z_0 \in D$ *such that*

$$\inf_{p\in\mathbb{N}}\{D(Q_1^{(p)},\dots,Q_q^{(p)})(z_0)\} > 0.$$

Then, we have the following.

(a) *There exists a subsequence* $\{j_p\} \subset \mathbb{N}$ *such that for* $0 \le i \le N$, $P_i^{(j_p)}$ *converge uniformly on compact subsets of* D *to not identically zero homogeneous polynomials* P_i *(meaning that the* $P_i^{(j_p)}$ *and* P_i *are homogeneous polynomials in* $\widetilde{\mathcal{H}}_D[\omega_0,\dots,\omega_N]$ *of the same degree* d*, and all their coefficients converge uniformly on compact subsets of* D*), and the* $b_{ij}^{(j_p)}$ *converge uniformly on compact subsets of* D *to* $b_{ij} \in \mathcal{H}_D$ *for all* $0 \le i \le N$, $1 \le j \le q$.

(b) *The* $Q_j^{(j_p)} = \sum_{i=0}^N b_{ij}^{(j_p)} P_i^{(j_p)}$ *converge, for all* $0 \le i \le N$, $1 \le j \le q$, *uniformly on compact subsets of* D *to* $Q_j := \sum_{i=0}^N b_{ij} P_i \in \widetilde{\mathcal{S}}(\{P_i\}_{i=0}^N)$*, and we have*

$$D(Q_1,\dots,Q_q)(z_0) > 0.$$

In particular, the moving hypersurfaces $Q_1(z_0),\dots,Q_q(z_0)$ *are located in general position, and the moving hypersurfaces* $Q_1(z),\dots,Q_q(z)$ *are located in (weakly) general position.*

Proof. Since, by our conditions on the coefficients of the $P_i^{(p)}$ and on the $b_{ij}^{(p)}$, for all $1 \le j \le q$ the coefficients of the homogeneous polynomials $Q_j^{(p)}$ of degree $d^{(p)}$ are locally bounded uniformly for all $p \ge 1$ on compact subsets of D, all conditions of Lemma 3.6 are satisfied, and we get, after passing to a subsequence (which we denote for simplicity again by $\{p\} \subset \mathbb{N}$), that for $1 \le j \le q$, $Q_j^{(p)}$ converge uniformly on compact subsets of D to not identically vanishing homogeneous polynomials Q_j (meaning that the $Q_j^{(p)}$ and Q_j are homogeneous polynomials in $\widetilde{\mathcal{H}}_D[\omega_0,\dots,\omega_N]$ of the same degree d_j, and all their coefficients converge uniformly on compact subsets of D). Moreover (still by Lemma 3.6), we have that

$$D(Q_1,\dots,Q_q)(z_0) > 0,$$

so the hypersurfaces $Q_1(z_0),\dots,Q_q(z_0)$ are located in general position, and the moving hypersurfaces $Q_1(z),\dots,Q_q(z)$ are located in (weakly) general position. Observe, moreover, that since all the $Q_j^{(p)}$, $1 \le j \le q$, were of the same degree $d^{(p)}$, we have $d = d_j$ independent of j for our subsequence.

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Hence, we have, for all $0 \le i \le N$, $p \ge 1$,

$$P_i^{(p)}(z)(\omega) = \sum_{|I|=d} a_{ipI}(z) \cdot \omega^I.$$

Now, since the $a_{jpI}(z)$ and the $b_{ij}^{(p)}(z)$ are locally bounded uniformly for all $p \ge 1$ on D, by using Montel's theorem and a standard diagonal argument with respect to an exhaustion of D with compact subsets, after passing another time to a subsequence (which we denote for simplicity again by $\{p\} \subset \mathbb{N}$), we also can assume that $\{a_{ipI}(z)\}_{p=1}^{\infty}$ converges uniformly on compact subsets of D to a_{iI} for each i, I, and that $\{b_{ij}^{(p)}(z)\}_{p=1}^{\infty}$ converges uniformly on compact subsets of D to $b_{ij}(z)$ for each i, j. Denote

$$P_i(z)(\omega) := \sum_{|I|=d} a_{iI}(z) \cdot \omega^I.$$

Since the limit is unique, then we have $Q_j = \sum_{i=0}^{N} b_{ij} P_i$ for $1 \le j \le q$, and in particular, we have that none of the $P_0(z), \ldots, P_N(z)$ is identically vanishing. (Otherwise, they could not be in (weakly) general position, which contradicted the general position of the $Q_1(z_0), \ldots, Q_q(z_0)$; in fact, if the $P_i(z_0)(\omega)$ had a nonzero solution ω_0 in common, so would the $Q_j(z_0)(\omega)$.) Hence, $Q_j \in \widetilde{\mathcal{S}}(\{P_i\}_{i=0}^{N})$, which completes the proof. □

§4. Proofs of the Theorems

Proofs of Theorem 1.1 and Theorem 1.2. Let $\{f^{(p)}\}$ be a sequence of meromorphic mappings in $\mathcal{F}$. We have to prove that after passing to a subsequence (which we denote again by $\{f^{(p)}\}$), the sequence $\{f^{(p)}\}$ converges meromorphically on D to a meromorphic mapping f. Moreover, under the stronger conditions of Theorem 1.2, we have to show that $\{f^{(p)}\}$ converges uniformly on compact subsets of D to a holomorphic mapping f.

In order to simplify notation, we denote, for $1 \le k \le q$,

$$Q_k^{(p)} := Q_k(f^{(p)}) \qquad \text{and} \qquad H_k^{(p)} := H_k(f^{(p)}).$$

By Lemma 3.6, after passing to a subsequence, for all $1 \le k \le q$, $Q_k^{(p)}$ converge uniformly on compact subsets of D to Q_k, meaning that

$$Q_k^{(p)} = Q_k^{(p)}(z)(\omega) = \sum_{|I|=d_k} a_{kpI}(z) \cdot \omega^I$$

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and

$$Q_k = Q_k(z)(\omega) = \sum_{|I|=d_k} a_{kI}(z)\cdot\omega^I$$

are homogeneous polynomials in $\widetilde{\mathcal{H}}_D[\omega_0,\dots,\omega_N]$ of the same degree d_k, and that all their coefficients a_{kpI} converge uniformly on compact subsets of D to a_{kI}. Moreover, $Q_1,\dots,Q_q$ are located in (weakly) general position.

By condition (ii) of Theorems 1.1 and 1.2 and Lemmas 3.1 and 3.2, and by condition (iii) of Theorems 1.1 and 1.2 and Lemmas 3.3 and 3.4, after passing to a subsequence, we may assume that the sequence $\{f^{(p)}\}$ satisfies

$$\lim_{p\to\infty}(f^{(p)})^{-1}(H_k^{(p)}) = S_k \quad (1\le k\le N+1)$$

as a sequence of closed subsets of D, where S_k are either empty or pure $(n-1)$-dimensional analytic sets in D, and

$$\lim_{p\to\infty}(f^{(p)})^{-1}(H_k^{(p)}) - S = S_k \quad (N+2\le k\le q)$$

as a sequence of closed subsets of $D-S$, where S_k are either empty or pure $(n-1)$-dimensional analytic sets in $D-S$.

Let $T=(\dots,t_{kI},\dots)$ $(1\le k\le q,\ |I|\le M:=\max\{d_1,\dots,d_q\})$ be a family of variables. Set $\widetilde{Q}_k=\sum_{|I|\le M}t_{kI}\omega^I\in\mathbb{Z}[T,\omega]$ $(1\le k\le q)$. For each subset $L\subset\{1,\dots,q\}$ with $|L|=n+1$, take $\widetilde{R}_L$ is the resultant of the $\widetilde{Q}_k$ $(k\in L)$. Since $\{Q_k\}_{k\in L}$ are in the (weakly) general position, $\widetilde{R}_L(\dots,a_{kI},\dots)\not\equiv 0$ (where we put $a_{kI}=0$ for $|I|\ne d_k$). We set

$$\widetilde{S} := \big\{z\in D \,\big|\, \widetilde{R}_L(\dots,a_{kI},\dots)=0 \text{ for some } L\subset\{1,\dots,q\} \text{ with } |L|=n+1\big\}.$$

Let $E=(\bigcup_{k=1}^q S_k\cup\widetilde{S})-S$. Then E is either empty or a pure $(n-1)$-dimensional analytic set in $D-S$.

Fix any point z_1 in $(D-S)-E$. Choose a relatively compact neighborhood U_{z_1} of z_1 in $(D-S)-E$. Then $\{f^{(p)}|_{U_{z_1}}\}\subset \mathrm{Hol}(U_{z_1},\mathbb{P}^N(\mathbb{C}))$. We now prove that the family $\{f^{(p)}|_{U_{z_1}}\}$ is a holomorphically normal family. Indeed, suppose that the family $\{f^{(p)}|_{U_{z_1}}\}$ is not holomorphically normal. By Lemma 3.9, there exist a subsequence (again denoted by $\{f^{(p)}|_{U_{z_1}}\}_{p=1}^\infty$) and $P_0\in U_{z_1}$, $\{P_p\}_{p=1}^\infty\subset U_{z_1}$ with $P_p\to P_0$, $\{r_p\}\subset(0,+\infty)$ with $r_p\to 0^+$, and $\{u_p\}\subset\mathbb{C}^n$, which are unit vectors, such that $g_p(z):=f^{(p)}(P_p+r_pu_pz)$

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converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic map g of $\mathbb{C}$ into $\mathbb{P}^N(\mathbb{C})$. Then, there exist admissible representations $g^{(p)} = (g_0^{(p)} : \cdots : g_N^{(p)})$ of $g^{(p)}$ and an admissible representation $g = (g_0 : \cdots : g_N)$ of g such that the $\{g_i^{(p)}\}$ converge uniformly on compact subsets of $\mathbb{C}$ to g_i $(0 \le i \le N)$. (Observe that an admissible representation of a holomorphic map is automatically without common zeros.) This implies that $Q_k^{(p)}(P_p + r_p u_p z)(g_0^{(p)}(z), \ldots, g_n^{(p)}(z))$ converges uniformly on compact subsets of $\mathbb{C}$ to $Q_k(P_0)(g_0(z), \ldots, g_N(z))$, $(1 \le k \le q)$. Thus, by Hurwitz's theorem, one of the following two assertions holds:

(i) $Q_k(P_0)(g_0(z), \ldots, g_N(z)) = 0$ on $\mathbb{C}$, that is, $g(\mathbb{C}) \subset H_k(P_0)$;

(ii) $Q_k(P_0)(g_0(z), \ldots, g_N(z)) \ne 0$ on $\mathbb{C}$, that is, $g(\mathbb{C}) \cap H_k(P_0) = \emptyset$.

Denote by J the set of all indices $k \in \{1, \ldots, q\}$ with $g(\mathbb{C}) \subset H_k(P_0)$. Set $X = \bigcap_{k \in J} H_k(P_0)$ if $J \ne \emptyset$ and $X = \mathbb{P}^N(\mathbb{C})$ if $J = \emptyset$. Since $\mathbb{C}$ is irreducible, there exists an irreducible component Z of X such that $g(\mathbb{C}) \subset Z - (\bigcup_{k \notin J} H_k(P_0))$. Since $P_0 \in U_{z_1}$, it implies that $\{H_k(P_0)\}_{k=1}^q$ are in general position in $\mathbb{P}^N(\mathbb{C})$, meaning that

$$\text{for all } I \subset \{1, \ldots, q\} \text{ with } \#I = N+1, \bigcap_{k \in I} H_k(P_0) = \emptyset. \tag{4.1}$$

Put $J^c := \{1, \ldots, q\} \setminus J$, and put $m := \dim_{\mathbb{C}} Z$. We claim that for the hypersurfaces $\{H_k(P_0)\}_{k \in J^c}$ in $\mathbb{P}^N(\mathbb{C})$ we have

$$\#J^c \ge 2m+1; \text{ for all } I \subset J^c \text{ with } \#I = m+1, Z \cap \Big(\bigcap_{k \in I} H_k(P_0)\Big) = \emptyset. \tag{4.2}$$

In fact, if $J = \emptyset$, so $X = \mathbb{P}^N(\mathbb{C})$, this holds since $q \ge 2N+1$ and by (4.1). If $J \ne \emptyset$, the key observation is that by (4.1) and by Bézout's theorem we have

$$\begin{aligned} &\text{for all } 1 \le l \le q, \text{for all } I \subset \{1, \ldots, q\} \text{ with } \#I = l, \\ &\bigcap_{k \in I} H_k(P_0) \text{ is of pure dimension } \dim_{\mathbb{C}} \bigcap_{k \in I} H_k(P_0) = \max\{N - l, -1\} \end{aligned} \tag{4.3}$$

(in particular, all irreducible components of $\bigcap_{k \in I} H_k(P_0)$ are of the same dimension), where $\dim_{\mathbb{C}}(\emptyset) = -1$. From that we first get

$$m = \dim_{\mathbb{C}} Z = \dim_{\mathbb{C}} \bigcap_{k \in J} H_k(P_0) = \max\{N - \#J, -1\}.$$

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Since $g(\mathbb{C}) \subset Z$, so $m \geq 0$, we get

$$\#J = N - m. \tag{4.4}$$

Hence,

$$\#J^c = q - \#J \geq (2N+1) - (N-m) = N + m + 1 \geq 2m + 1.$$

Moreover, if $I \subset J^c$ with $\#I = m+1$, then by (4.4)

$$\#(I \cup J) = (m+1) + (N-m) = N+1$$

and

$$Z \cap \bigcap_{k\in I} H_k(P_0) \subset \bigcap_{k\in J} H_k(P_0) \cap \bigcap_{k\in I} H_k(P_0) = \bigcap_{k\in I\cup J} H_k(P_0) = \emptyset,$$

where the last equality follows from (4.1). This proves (4.2) in the case $J \neq \emptyset$. By (4.2) and by [3, Corollary 1] (or by the more general [10, Theorem 7.3.4]), we get that $Z - (\bigcup_{k\notin J} H_k(P_0))$ is complete hyperbolic and hyperbolically embedded, and hence g is constant. This is a contradiction.

Thus, $\{f^{(p)}\}$ is a holomorphically normal family on U_{z_1}. By the usual diagonal argument, we can find a subsequence (again denoted by $\{f^{(p)}\}$) which converges uniformly on compact subsets of $(D-S)-E$ to a holomorphic mapping f of $(D-S)-E$ into $\mathbb{P}^N(\mathbb{C})$.

By Lemma 3.7(a), $\{f^{(p)}\}$ has a meromorphically convergent subsequence (again denoted by $\{f^{(p)}\}$) on $D-S$, and again by Lemma 3.7(a), $\{f^{(p)}\}$ has a meromorphically convergent subsequence on D. Then $\mathcal{F}$ is a meromorphically normal family on D. The proof of Theorem 1.1 is completed.

Under the additional conditions of Theorem 1.2 by Lemma 3.7(b), $\{f^{(p)}\}$ has a subsequence which converges uniformly on compact subsets of D to a holomorphic mapping of D to $\mathbb{P}^N(\mathbb{C})$. The proof of Theorem 1.2 is completed. □

Proof of Theorem 1.3. Suppose that $\mathcal{F}$ is not normal on D. Then, by Lemma 3.9, there exists a subsequence denoted by $\{f^{(p)}\} \subset \mathcal{F}$ and $z_0 \in D$, $\{z_p\}_{p=1}^{\infty} \subset D$ with $z_p \to z_0$, $\{r_p\} \subset (0,+\infty)$ with $r_p \to 0^+$, and $\{u_p\} \subset \mathbb{C}^n$, which are unit vectors, such that $g^{(p)}(\xi) := f^{(p)}(z_p + r_p u_p \xi)$ converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic map g of $\mathbb{C}$ into $\mathbb{P}^N(\mathbb{C})$.

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By condition (i) of Theorem 1.3 and by Lemma 3.14, there exists a subsequence (which we denote again by $\{p\} \subset \mathbb{N}$) such that for $0 \le i \le N$, $P_i^{(p)} := P_i(f^{(p)})$ converge uniformly on compact subsets of D to P_i; that the $b_{ij}^{(p)} := b_{ij}(f^{(p)})$ converge uniformly on compact subsets of D to b_{ij} for all $0 \le i \le N$, $1 \le j \le q$; that the $Q_j^{(p)} := Q_j(f^{(p)}) = \sum_{i=0}^N b_{ij}^{(p)} P_i^{(p)}$ converge, for all $0 \le i \le N$, $1 \le j \le q$, uniformly on compact subsets of D to $Q_j := \sum_{i=0}^N b_{ij} P_i \in \widetilde{\mathcal{S}}(\{P_i\}_{i=0}^N)$; and that we have, for any fixed $z = z_0 \in D$,

$$D(Q_1, \dots, Q_q)(z) > \delta(z) > 0$$

(in particular, the moving hypersurfaces $Q_1(z), \dots, Q_q(z)$ are located in (pointwise) general position). We finally recall that with writing both variables $z \in D$ and $\omega \in \mathbb{P}^N(\mathbb{C})$, we thus have that

$$P_i^{(p)}(z)(\omega) \to P_i(z)(\omega); \qquad Q_j^{(p)}(z)(\omega) \to Q_j(z)(\omega); \qquad b_{ij}^{(p)}(z) \to b_{ij}(z)$$

uniformly on compact subsets in the variable $z \in D$.

For any fixed $\xi_0 \in \mathbb{C}$, there exists a ball $B(\xi_0, r_0)$ in $\mathbb{C}$ and an index i such that $g(B(\xi_0, r_0)) \subset \{\omega \in \mathbb{P}^N(\mathbb{C}) : \omega_i \ne 0\}$. Without loss of generality, we may assume that $i = 0$. Therefore, there exist admissible representations

$$\tilde{g}^{(p)}(\xi) = \big(1, g_1^{(p)}(\xi), \dots, g_N^{(p)}(\xi)\big),$$
$$\tilde{g}(\xi) = \big(1, g_1(\xi), \dots, g_N(\xi)\big)$$

of $g^{(p)}$ and g on $B(\xi_0, r_0)$.

Because of the convergence of $\{g^{(p)}\}$ on $B(\xi_0, r_0)$, $\{g_i^{(p)}\}$ converges uniformly on compact subsets of $B(\xi_0, r_0)$ to g_i for each $1 \le i \le N$. This implies that $Q_j^{(p)}(z_p + r_p u_p \xi)(\widetilde{g}^{(p)}(\xi))$ converges uniformly on compact subsets of $\mathbb{C}$ to $Q_j(z_0)(\widetilde{g}(\xi))$ and that $P_i^{(p)}(z_p + r_p u_p \xi)(\widetilde{g}^{(p)}(\xi))$ converges uniformly on compact subsets of $\mathbb{C}$ to $P_i(z_0)(\widetilde{g}(\xi))$.

By Hurwitz's theorem, there exists a positive integer N_0 such that $Q_j^{(p)}(z_p + r_p u_p \xi)(\widetilde{g}^{(p)}(\xi))$ and $Q_j(z_0)(\widetilde{g}(\xi))$ have the same number of zeros with counting multiplicities on $B(\xi_0, r_0)$ for each $p \ge N_0$. Since the map $g^{(p)}$ of $B(\xi_0, r_0)$ into $\mathbb{P}^N(\mathbb{C})$ intersects $Q_j^{(p)}$ with multiplicity at least m_j, it implies that any zero ξ of $Q_j(z_0)(\widetilde{g}(\xi))$ has multiplicity at least m_j. Hence, g intersects $Q_j(z_0)$ with multiplicity at least m_j for each $1 \le j \le q$.

Since we have that $Q_1, \dots, Q_q$ are in $\widetilde{\mathcal{S}}(\{P_i\}_{i=0}^N)$ and that

$$D(Q_1, \dots, Q_q)(z) > 0 \quad \text{for any } z \in D,$$

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we have in particular that $Q_1(z_0),\dots,Q_q(z_0)$ are in $\mathcal{S}(\{P_i(z_0)\}_{i=0}^N)$ and that

$$D(Q_1,\dots,Q_q)(z_0)>0.$$

Thus, by Lemma 3.13, g is a constant mapping of $\mathbb{C}$ into $\mathbb{P}^N(\mathbb{C})$. This is a contradiction. □

Proofs of Theorem 1.4 and Theorem 1.5. Let $\{f^{(p)}\}$ be a sequence of meromorphic mappings in $\mathcal{F}$. We have to prove that after passing to a subsequence (which we denote again by $\{f^{(p)}\}$), the sequence $\{f^{(p)}\}$ converges meromorphically on D to a meromorphic mapping f. Moreover, under the stronger conditions of Theorem 1.5, we have to show that $\{f^{(p)}\}$ converges uniformly on compact subsets of D to a holomorphic mapping f.

By condition (i) of the theorems and by Lemma 3.14, there exists a subsequence (which we denote again by $\{f^{(p)}\}$) such that for $0\le i\le N$, $P_i^{(p)}:=P_i(f^{(p)})$ are homogeneous polynomials of the same degree d and converge uniformly on compact subsets of D to P_i; that the $b_{ij}^{(p)}:=b_{ij}(f^{(p)})$ converge uniformly on compact subsets of D to b_{ij} for all $0\le i\le N$, $1\le j\le q$; that the $Q_j^{(p)}:=Q_j(f^{(p)})=\sum_{i=0}^N b_{ij}^{(p)}P_i^{(p)}$ converge, for all $0\le i\le N$, $1\le j\le q$, uniformly on compact subsets of D to $Q_j:=\sum_{i=0}^N b_{ij}P_i\in\widetilde{\mathcal{S}}(\{P_i\}_{i=0}^N)$; and that

$$D(Q_1,\dots,Q_q)(z_0)>0.$$

In particular, the moving hypersurfaces $Q_1(z_0),\dots,Q_q(z_0)$ are located in general position, and the moving hypersurfaces $Q_1(z),\dots,Q_q(z)$ are located in (weakly) general position.

By condition (ii) of Theorem 1.4 and Lemmas 3.1 and 3.2, and by condition (iii) of the theorems and Lemmas 3.3 and 3.4, after passing to a subsequence, we may assume that the sequence $\{f^{(p)}\}$ satisfies

$$\lim_{p\to\infty}(f^{(p)})^{-1}(H_k^{(p)})=S_k\quad(1\le k\le N+1)$$

as a sequence of closed subsets of D, where S_k are either empty or pure $(n-1)$-dimensional analytic sets in D, and satisfies

$$\lim_{p\to\infty}\overline{\left\{z\in\operatorname{Supp}\nu(f^{(p)},H_k^{(p)})\;\middle|\;\nu(f^{(p)},H_k^{(p)})(z)<m_k\right\}}-S=S_k$$
$$(N+2\le k\le q)$$

as a sequence of closed subsets of $D-S$, where S_k are either empty or pure $(n-1)$-dimensional analytic sets in $D-S$.

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Let $T = (\dots, t_{kI}, \dots)$ $(1 \le k \le q, |I| = d)$ be a family of variables. Set $\widetilde{Q}_k = \sum_{|I|=d} t_{kI}\omega^I \in \mathbb{Z}[T, \omega]$ $(1 \le k \le q)$. For each subset $L \subset \{1, \dots, q\}$ with $|L| = n+1$, take $\widetilde{R}_L$ is the resultant of the $\widetilde{Q}_k$ $(k \in L)$. Since $\{Q_k\}_{k\in L}$ are in (weakly) general position, $\widetilde{R}_L(\dots, a_{kI}, \dots) \not\equiv 0$ (where we put $a_{kI} = 0$ for $|I| \ne d$). We set

$$\widetilde{S} := \big\{ z \in D \,\big|\, \widetilde{R}_L(\dots, a_{kI}, \dots) = 0 \text{ for some } L \subset \{1, \dots, q\} \text{ with } |L| = n+1 \big\}.$$

Let $E = (\bigcup_{k=1}^q S_k \cup \widetilde{S}) - S$. Then E is either empty or a pure $(n-1)$-dimensional analytic set in $D - S$.

Fix any point z_1 in $(D - S) - E$. Choose a relatively compact neighborhood U_{z_1} of z_1 in $(D - S) - E$. Then $\{f^{(p)}|_{U_{z_1}}\} \subset \mathrm{Hol}(U_{z_1}, \mathbb{P}^N(\mathbb{C}))$. We now prove that the family $\{f^{(p)}|_{U_{z_1}}\}$ is a holomorphically normal family. For this it is sufficient to observe that the family $\{f^{(p)}|_{U_{z_1}}\}$ now satisfies all conditions of Theorem 1.3. In fact, there exists N_0 such that, for $p \ge N_0$, $\{f^{(p)}|_{U_{z_1}}\}$ does not intersect $H_k^{(p)}$ for $1 \le k \le N+1$ and $\{f^{(p)}|_{U_{z_1}}\}$ intersects $H_k^{(p)}$ of order at least m_k for $N+2 \le k \le q$; and for all $z \in U_{z_1}$, we have $D(Q_1, \dots, Q_q)(z) > 0$. So if we still put $m_k = \infty$ for $1 \le k \le N+1$, the conditions of Theorem 1.3 are satisfied, and so the family $\{f^{(p)}|_{U_{z_1}}\}$ is a holomorphically normal family. By the usual diagonal argument, we can find a subsequence (again denoted by $\{f^{(p)}\}$) which converges uniformly on compact subsets of $(D - S) - E$ to a holomorphic mapping f of $(D - S) - E$ into $\mathbb{P}^N(\mathbb{C})$.

By Lemma 3.7(a), $\{f^{(p)}\}$ has a meromorphically convergent subsequence (again denoted by $\{f^{(p)}\}$) on $D - S$, and again by Lemma 3.7(a), $\{f^{(p)}\}$ has a meromorphically convergent subsequence on D. Then $\mathcal{F}$ is a meromorphically normal family on D. The proof of Theorem 1.4 is completed.

Under the additional conditions of Theorem 1.5, by Lemma 3.7(b) $\{f^{(p)}\}$ has a subsequence which converges uniformly on compact subsets of D to a holomorphic mapping of D to $\mathbb{P}^N(\mathbb{C})$. The proof of Theorem 1.5 is completed. □

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