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A-BILINEAR FORMS AND GENERALISED A-QUADRATIC FORMS ON UNITARY LEFT A-MODULES

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In this paper we shall define a generalised A-quadratic form and prove that in some way this form and an A-bilinear form are equivalent to each other. Our result characterises that of Vukman in the sense that we use any n vectors for a fixed $n \geqslant 2$, instead of any two vectors. Consequently, a new generalisation of an inner product space among vector spaces is obtained. This also leads to a new relationship between a 2-inner product space and a 2-normed space.

1. INTRODUCTION AND DEFINITIONS

It was shown by Vukman in a very recent paper [**7**, Theorem 7] that given an A-quadratic form on a unitary left A-module, there exists an A-bilinear form with some kind of relation between them. The idea originated in a paper by the same author [**6**, Theorem 2.1]. In this paper we shall define a generalised A-quadratic form and prove that in some sense the two forms are equivalent to each other. Therefore, our result characterises that of Vukman [**6**, **7**] which in turn generalised Kurepa's extension [**4**, **5**] of Jordan-Neumann's generalisation of inner product spaces among vector spaces. It may be noted that all well-known characterisations of an inner product space in the past used any two vectors in a normed vector space, in contrast to this we shall use any n vectors for a fixed $n \geqslant 2$. In the final section we shall present a new relstionship between a 2-inner product space and a 2-normed space.

Let us recall first of all some standard definitions. A *Banach $\star$-algebra* is a $\star$-algebra (an algebra with involution) which is also a Banach algebra. Let A be a $\star$-algebra with a unity element e, and let X be a vector space which is also a left A-module. We call a mapping $B\colon X \times X \to A$ an *A-bilinear form* [**6**] if B is additive in both arguments, $B(ax,y) = aB(x,y)$ and $B(x,ay) = B(x,y)a^\star$ for all pairs x and y in X and all a in A. A left A-module X is said to be *unitary* if $ex = x$ for all x in X.

DEFINITION 1: A mapping $Q\colon X \to A$ is called a *generalised A-quadratic form* if

$$(\star) \qquad Q(ax) = aQ(x)a^\star \text{ and}$$

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$$cQ\left(\sum_{i=1}^{n} v_i\right) + \sum_{j=2}^{n} Q\left(\sum_{i=1}^{n} v_i - (n+c-1)v_j\right)$$
$$= (n+c-1)\left(Q(v_1) + c\sum_{i=2}^{n} Q(v_i) + \sum_{\substack{i<j \\ j=3}}^{n}\sum_{i=2}^{n-1} Q(v_i - v_j)\right)$$

for all a in A, all n vectors $v_1, v_2, \ldots, v_n$ in X, a constant $c \neq 0, \neq 1-n, \neq 1-\frac{1}{2}n$ and a fixed $n \geqslant 2$. Thus, when $n = 2$ and $c = 1$ it reduces to

$$(\star\star) \qquad Q(v_1 + v_2) + Q(v_1 - v_2) = 2(Q(v_1) + Q(v_2))$$

which is known as an A-*quadratic form* [6].

2. The principal result

THEOREM. *Let A be a complex Banach $\star$-algebra with a unity element e, and let X be a complex vector space which is also a unitary left A-module. If $B: X \times X \to A$ and $Q: X \to A$ are two mappings, then the following two statements are equivalent:*

(1) *B is an A-bilinear form and $Q(x) = B(x,x)$;*
(2) *The relation $B(x,y) = \frac{1}{4}(Q(x+y) - Q(x-y) + iQ(x+iy) - iQ(x-iy))$ holds and Q is a generalised A-quadratic form.*

PROOF: (1) $\Rightarrow$ (2): Starting from the righthand side, the first relation is a direct computation. That $Q(ax) = aQ(x)a^\star$ is obvious. To show the second identity in $(\star)$, since

$$cQ\left(\sum_{i=1}^{n} v_i\right) = c\sum_{i=1}^{n} Q(v_i) + c\sum_{\substack{i<j \\ j=2}}^{n}\sum_{i=1}^{n-1} (B(v_i,v_j) + B(v_j,v_i)) \text{ and}$$

$$\sum_{j=2}^{n} Q\left(\sum_{i=1}^{n} v_i - (n+c-1)v_j\right) = (n-1)\sum_{i=1}^{n} Q(v_i)$$
$$-(n+c-1)\sum_{j=2}^{n}\left(B\left(\sum_{i=1}^{n} v_i, v_j\right) + B\left(v_j, \sum_{i=1}^{n} v_i\right)\right)$$
$$+(n+c-1)^2\sum_{i=2}^{n} B(v_i,v_i)$$
$$+(n-1)\sum_{\substack{i<j \\ j=2}}^{n}\sum_{i=1}^{n-1} (B(v_i,v_j) + B(v_j,v_i)),$$

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the rest is straight-forward computation.

(2) $\Rightarrow$ (1): since $Q(0) = 0$ by $(\star)$ (the condition that $c \neq 1 - \frac{1}{2}n$ is required to assure this result), $Q(x) = B(x,x)$ is evident from the relation. To prove that $B(x+y,z) = B(x,z) + B(y,z)$, we first assert that

$$\text{(a)}\quad \begin{aligned} &cB\left(u + \frac{1}{n-1}\sum_{i=1}^{n-1} w_i, z\right) + \sum_{j=1}^{n-1} B\left(u + \frac{1}{n-1}\sum_{i=1}^{n-1} w_i - \frac{n+c-1}{n-1} w_j, z\right) \\ &\qquad = (n-1)B\left(\frac{n+c-1}{n-1}u, z\right) = cB\left(\frac{n+c-1}{c}u, z\right) \end{aligned}$$

for all u, z and w_i in X, with $i = 1, 2, \dots, n-1$. To this end, let $v_1 = u + z$ and $v_i = w_{i-1}/(n-1)$ for $i = 2, 3, \dots, n$ in the second identity of $(\star)$, so

$$\text{(b)}\quad \begin{aligned} &cQ\left(u + z + \frac{1}{n-1}\sum_{i=1}^{n-1} w_i\right) + \sum_{j=1}^{n-1} Q\left(u + z + \frac{1}{n-1}\sum_{i=1}^{n-1} w_i - \frac{n+c-1}{n-1} w_j\right) \\ &\quad = (n+c-1)\left(Q(u+z) + \frac{c}{(n-1)^2}\sum_{i=1}^{n-1} Q(w_i) + \frac{1}{(n-1)^2}\sum_{\substack{i<j \\ j=2}}^{n-1}\sum_{i=1}^{n-2} Q(w_i - w_j)\right). \end{aligned}$$

If z is replaced by $-z$, by iz and by $-iz$ in (b), we shall get three equations. From these three equations together with (b) we obtain easily an identity expressed in terms of the mapping B, namely

$$\text{(c)}\quad \begin{aligned} &cB\left(u + \frac{1}{n-1}\sum_{i=1}^{n-1} w_i, z\right) + \sum_{j=1}^{n-1} B\left(u + \frac{1}{n-1}\sum_{i=1}^{n-1} w_i - \frac{n+c-1}{n-1} w_j, z\right) \\ &\qquad = (n+c-1)B(u,z). \end{aligned}$$

Let $w_i = \frac{n-1}{c}u$ in (c) for $i = 1, 2, \dots, n-1$. Then

$$\text{(d)}\quad cB\left(\frac{n+c-1}{c}u, z\right) = (n+c-1)B(u,z).$$

Also let $w_i = -u$ in (c) for $i = 1, 2, \dots, n-1$. Then

$$\text{(e)}\quad (n-1)B\left(\frac{n+c-1}{n-1}u, z\right) = (n+c-1)B(u,z).$$

Thus, the identities (c), (d) and (e) constitute our assertion.

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Next, let $u = \dfrac{n-1}{n+c-1}y$ in (d) and (e). Then $cB\left(\dfrac{n-1}{c}y, z\right) = (n-1)B(y,z)$. Also let $u = \dfrac{n-1}{n+c-1}(x+y)$ and $w_i = \dfrac{1-n}{n+c-1}x + \dfrac{(n-1)^2}{c(n+c-1)}y$ in (a) for $i = 1, 2, \ldots, n-1$. Then

$$cB\left(\frac{n-1}{c}y, z\right) + (n-1)B(x,z) = (n-1)B(x+y,z),$$

that is, $B(y,z) + B(x,z) = B(x+y,z)$. Analogously one can show the additivity in the second argument and we shall omit the details.

It remains to verify that $B(ax,y) = aB(x,y)$ and $B(x,ay) = B(x,y)a^*$ for all pairs x and y in X and all a in A. We shall omit the proof since the result was mentioned in [**7**, Theorem 7], and was proved in detail in [**6**, Theorem 2.1]. This completes the proof of the theorem. ∎

3. Corollaries

When X is a real (complex) normed vector space and A is the field of real (complex) numbers, we have

Corollary 1. *X is a real (complex) inner product space if and only if the norm in X satisfies the condition:*

(i)
$$\begin{aligned} &c\Big\|\sum_{i=1}^{n} v_i\Big\|^2 + \sum_{j=2}^{n}\Big\|\sum_{i=1}^{n} v_i - (n+c-1)v_j\Big\|^2 \\ &= (n+c-1)\left(\|v_1\|^2 + c\sum_{i=2}^{n}\|v_i\|^2 + \sum_{\substack{i<j \\ j=3}}^{n}\sum_{i=2}^{n-1}\|v_i - v_j\|^2\right)\end{aligned}$$

for any n vectors $v_1, v_2, \ldots, v_n$ in X, a constant $c \neq 0$, $\neq 1-n$, and a fixed $n \geqslant 2$, and the inner product is defined by

(ii)
$$(x,y) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$$
$$\left((x,y) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)\right).$$

Proof: Let us consider the real case first. The proof of the relation $(x+y,z) = (x,z) + (y,z)$ is merely changes of notations in our Theorem: $B(x,y) = \frac{1}{4}(Q(x+y) - Q(x-y))$, where $B(x,y) = (x,y)$ the usual real inner product of x and y, and $Q(x) = \|x\|^2$. This same relation implies the identity $(ax,y) = a(x,y)$ for all real a which can be found in [**1**, p.175]. As for the complex version it can easily be derived from the real case [**1**, p.176] and we shall omit the details. ∎

When $n = 2$ and $c = 1$ the equation (i) is reduced to

$$\text{(iii)} \qquad \|x+y\|^2 + \|x-y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big)$$

which is the classical condition of the Jordan-Neumann generalisation of an inner product space. It is also known as the parallelogram law in an inner product space. This leads, with the aid of Corollary 1, to:

COROLLARY 2. *Statements in Corollary 1 still hold if (i) is replaced by*

$$\text{(i}'\text{)} \qquad \|x+y\|^2 + \|x-y\|^2 + \|x+z\|^2 + \|x-z\|^2 = 4\|x\|^2 + \|y+z\|^2 + \|y-z\|^2$$

for all triplets x, y *and* z *in* X.

PROOF: In virtue of Corollary 1, we shall prove the real case only.
($\Rightarrow$): obviously (iii) implies (i$'$).
($\Leftarrow$): after interchanging x and y in (i$'$) we have

$$\|x+y\|^2 + \|x-y\|^2 + \|y+z\|^2 + \|y-z\|^2 = 4\|y\|^2 + \|x+z\|^2 + \|x-z\|^2.$$

Adding this to equation (i$'$) we get equation (iii). ∎

Corresponding to Corollary 2 we have

COROLLARY 3. *Statements in the Theorem still hold if* Q *in (2), instead of being a generalised* A*-quadratic form, satisfies the conditions:*

$$(\star') \qquad \begin{gathered} Q(ax) = aQ(x)a^\star \text{ and} \\ Q(x+y) + Q(x-y) + Q(x+z) + Q(x-z) = 4Q(x) + Q(y+z) + Q(y-z) \end{gathered}$$

for all triplets x, y *and* z *in* X.

4. REMARKS

1) When $n = 2$ and $c = 1$ the implication (1) $\Rightarrow$ (2) in the Theorem is immediate, and that (2) $\Rightarrow$ (1) is precisely [**7**, Theorem 7].

2) If $c = 1$ in particular, we can show that

$$B\left(\sum_{i=1}^{n} x_i, z\right) = \sum_{i=1}^{n} B(x_i, z)$$

in the proof of (2) $\Rightarrow$ (1) in the Theorem. This goes as follows: from our assertion in (a) we have, when $c = 1$,

$$B\left(u + \frac{1}{n-1}\sum_{i=1}^{n-1} w_i, z\right) + \sum_{j=1}^{n-1} B\left(u + \frac{1}{n-1}\sum_{i=1}^{n-1} w_i - \frac{n}{n-1}w_j, z\right) = B(nu, z).$$

The desired result follows by setting $u = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $w_i = \frac{(n-1)(x_1 - x_{i+1})}{n}$ for $i = 1, 2, \ldots, n-1$ in the above.

3) A closer look ad Kurepa's papers ([4] and [5]) shows that many results appearing there can be generalised by just replacing the quadratic form by our generalised quadratic form (i).

5. 2-NORMED AND 2-INNER PRODUCT SPACES

The following standard defintions are from [3] and [2]. If X is a real linear space of dimension greater than one, and if $\|.,.\|$ and $(.,.|.)$ are real functions on $X \times X$ and $X \times X \times X$ respectively, then X is called a *real 2-normed space* with a *2-norm* $\|.,.\|$ if the following conditions are satisfied:

(1) $\|x, y\| = 0$ if and only if x and y are linearly dependent;
(2) $\|x, y\| = \|y, x\|$;
(3) $\|ax, y\| = |a|\,\|x, y\|$ for every real a;
(4) $\|x + y, z\| \leqslant \|x, z\| + \|y, z\|$.

X is called a *real 2-inner product space* with a 2-inner producet $(.,.|.)$ if the following conditions are satisfied:

(1′) $(x, x|y) \geqslant 0$; $(x, x|y) = 0$ if and only if x and y are linearly dependent;
(2′) $(x, x|y) = (y, y|x)$;
(3′) $(x, y|z) = (y, x|z)$;
(4′) $(ax, y|z) = a(x, y|z)$ for every real a;
(5′) $(x + y, z|s) = (x, z|s) + (y, z|s)$.

It may be noted that in other papers, including [3] and [2], these are simply called a 2-normed space and a 2-inner product space. In this last section we shall present a new generalisation of a real 2-inner product space among real 2-normed spaces. We also define in an obvious fashion a complex 2-normed space and a complex 2-inner product space, and give a similar generalisation.

COROLLARY 4. *The following two statements are equivalent:*

(I) *X is a real 2-inner product space and*

(f) $$\|x, y\| = (x, x|y)^{\frac{1}{2}};$$

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(II) *X is a real 2-normed space and*

$$(x, y|z) = \frac{1}{4}\left(\|x+y, z\|^2 - \|x-y, z\|^2\right), \tag{g}$$

and the 2-norm in X satisfies the relation

$$\begin{aligned} &c\|\sum_{i=1}^{n} v_i, s\|^2 + \sum_{j=2}^{n}\|\sum_{i=1}^{n} v_i - (n+c-1)v_j, s\|^2 \\ &= (n+c-1)\left(\|v_1, s\|^2 + c\sum_{i=2}^{n}\|v_i, s\|^2 + \sum_{\substack{i<j \\ j=3}}^{n}\sum_{i=2}^{n-1}\|v_i - v_j, s\|^2\right) \end{aligned} \tag{h}$$

for any vectors $v_1, v_2, \ldots, v_n$ and s in X, a constant $c \neq 0$, $\neq 1-n$ and $n \geqslant 2$.

PROOF: (I) $\Rightarrow$ (II). It is known [**2**, Theorem 4] that X is a real 2-normed space with the 2-norm in (f), and that (g) holds. The equality (h) is similar to (i) in Corollary 1.

(II) $\Rightarrow$ (I). That (g) implies (f) is obvious. It may be noted that a 2-norm is nonnegative [**3**], and the condition (5′) implies (4′) [**2**, Theorem 5]. Thus it suffices to show that (h) implies (5′). We remark that the claim (a) in the Theorem is now

$$\begin{aligned} c\left(u + \frac{1}{n-1}\sum_{i=1}^{n-1} w_i, z|s\right) + \sum_{j=1}^{n-1}\left(u + \frac{1}{n-1}\sum_{i=1}^{n-1} w_i - \frac{n+c-1}{n-1} w_j, z|s\right) \\ = (n-1)\left(\frac{n+c-1}{n-1}u, z|s\right) = c\left(\frac{n+c-1}{c}u, z|s\right) \end{aligned}$$

for u, z, s and w_i in X, and $i = 1, 2, \ldots, n-1$. The rest of the proof is clear. ∎

It seems natural to consider the complex variant of the space X, and that of Corollary 4. We shal begin with two definitions.

DEFINITION 2: Let X be a complex linear space of dimension greater than one, and let $\|.,.\|$ be a real function on $X \times X$, and $[.,.|.]$ a complex function on $X \times X \times X$, then X is called a *complex 2-normed space* with a 2-norm $\|.,.\|$ if all four conditions in a real 2-normed space are satisfied, where a is a complex number in (3). X is called a *complex 2-inner product space* with a 2-inner product $[.,.|.]$ if the following conditions are satisfied:

(1″) $[x, x|y] \geqslant 0$; $[x, x|y] = 0$ if and only if x and y are linearly dependent;
(2″) $[x, x|y] = [y, y|x]$;

(3″) $[x,y|z]=[y,x|z]^\star$;

(4″) $[ax,y|z]=a[x,y|z]$ for every complex a;

(5″) $[x+y,z|s]=[x,z|s]+[y,z|s]$; where * denotes the conjugate of a complex number.

It follows easily that $[x,ay|z] = a^\star[x,y|z]$ for every complex a, $|[x,y|z]| \leqslant [x,x|z]^{\frac{1}{2}}[y,y|z]^{\frac{1}{2}}$ (the proof is a slight change in that of [**2**, Lemma 1]), and $[x,y|y]=0$.

Examples of such spaces can easily be found. In fact, it is not difficult to show that every inner product space, that is, a complex pre-Hilbert space, of dimension greater than one with the usual inner product $(.|.)$ is a complex 2-inner product space if the 2-inner space is defined by

$$[x,y|z]=(x|y)\|z\|^2-(x|z)(z|y).$$

On the other hand, if X is a complex 2-inner product space, then it is a complex 2-normed space if the 2-norm is defined by

$$\|x,y\|=[x,x|y]^{\frac{1}{2}}.$$

COROLLARY 5. *The following two statements are equivalent:*

(I) *X is a complex 2-inner product space and*

(f′) $$\|x,y\|=[x,x|y]^{\frac{1}{2}}.$$

(II) *X is a complex 2-normed space and*

(g′) $$[x,y|z]=\frac{1}{4}\left(\|x+y,z\|^2-\|x-y,z\|^2+i\|x+iy,z\|^2-i\|x-iy,z\|^2\right),$$

and the 2-norm in X satisfies the relation in (h).

PROOF: (I) $\Rightarrow$ (II). With the 2-norm in (f′) it follows easily that X is a complex 2-normed space (here, we use the aforementioned inequality $|[x,y|z]| \leqslant [x,x|z]^{\frac{1}{2}}[y,y|z]^{\frac{1}{2}}$ to show the inequality (4)). That (f′) implies (g′) is straightforward. The relation (h) holds exactly as in Corollary 4 except replacing $(.,.|.)$ by $[.,.|.]$.

(II) $\Rightarrow$ (I). That (g′) implies (f′) is immediate, and hence conditions (1″) and (2″) hold. We shall use Corollary 4 and adapt the device developed in [**1**, p.176] to verify the other three conditions, and proceed as follows: we may regard X as a real 2-inner product space; in view of Corolary 4, (g) defines a real-bilinear form with $\|x,y\|=(x,x|y)^{\frac{1}{2}}$ and $(y,x|z)=(x,y|z)$. Rewriting (g′) in the form

$$[x,y|z]=(x,y|z)+i(x,iy|z),$$

then $[x, y|z]$ is also real-bilinear, and hence (5″) holds.

Using (g) in Corollary 4, that is, $(x, y|z) = (ix, iy|z)$, (3″) follows from

$$\begin{aligned}[y, x|z]^* &= (y, x|z) - i(y, ix|z) = (x, y|z) + i(ix, -y|z) \\ &= (x, y|z) + i(x, iy|z) = [x, y|z].\end{aligned}$$

To show (4″) at last, in view of real-bilinearity it will be sufficient if we prove that $[ix, y|z] = i[x, y|z]$; indeed

$$\begin{aligned}[ix, y|z] &= (ix, y|z) + i(ix, iy|z) = -(x, iy|z) + i(x, y|z) \\ &= i((x, y|z) + i(x, iy|z)) = i[x, y|z].\end{aligned}$$

∎

Corresponding to Corollary 2, we have

COROLLARY 6. *The following two statements are equivalent:*

(I) *X is a real (complex) 2-inner product space and (f) holds (respectively, (f′) holds);*

(II) *X is a real (complex) 2-normed space and (g) holds (respectively, (g′) holds), and the 2-norm in X satisfies the relation*

$$\begin{aligned}4\|x, z\|^2 = \|y + z, x\|^2 + \|y - z, x\|^2 + \|x + y, z\|^2 + \|x - y, z\|^2 \\ -\|x + z, y\|^2 - \|x - z, y\|^2\end{aligned}$$

for any vectors x, y and z in X.

In conclusion, it should be noted that when $c = 1$ and $n = 2$ in particular, (h) becomes

$$\|v_1 + v_2, s\|^2 + \|v_1 - v_2, s\|^2 = 2\left(\|v_1, s\|^2 + \|v_2, s\|^2\right).$$

Thus, Theorem 4 and 5 in [2] are special cases of our Corollary 4. When $c = 1$ we can show that

$$\left(\sum_{i=1}^{n} x_i, z|s\right) = \sum_{i=1}^{n} (x_i, z|s).$$

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