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STRICT TOPOLOGY ON PARACOMPACT LOCALLY COMPACT SPACES

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In this paper, X denotes a Hausdorff paracompact locally compact space, E a Hausdorff locally convex space over K, the field of real or complex numbers (we call the elements of K *scalars*), $\{\| \|_p : p \in \mathscr{P}\}$ a filtering upwards family of semi-norms on E generating the topology of E, $C_b(X)$ the space of all continuous scalar-valued funcions on X, and $C_b(X, E)$ the space of all continuous, bounded E-valued functions. The strict topology β_0 on $C_b(X, E)$ is generated by semi-norm $\| \|_{h,p}$, $p \in \mathscr{P}$, $h \in C_0(X) = \{h \in C_b(X) : h$ vanishing at infinity$\}$, $\|f\|_{h,p} = \sup_{x\in X}\|h(x)f(x)\|_p$ ([**2**; **8**]). We denote by u the topology of uniform convergence on $C_b(X, E)$. For a Hausdorff locally convex space G, G' will denote its topological dual; G is called *Mackey* if every absolutely convex relatively compact subset of $(G', \sigma(G', G))$ is equicontinuous, and *strongly Mackey* if every relatively countably compact subset of $(G', \sigma(G', G))$ is equicontinuous. The natural bilinear mapping on $E \times E'$ or $E' \times E$ will be denoted by $\langle\ ,\ \rangle$. In this paper we prove that if E is metrizable then $(C_b(X, E), \beta_0)$ is Mackey and if, in addition, E is complete, that $(C_b(X, E), \beta_0)$ is strongly Mackey. This generalizes Conway's result ([**1**]).

We denote by $M_t(X)$ the set of all scalar, bounded, regular Borel measures on X ([**2**]), by $\mathscr{B}(X) = \mathscr{B}$, the Borel subsets of X, by $S(X, \mathscr{B}, E)$ all $\mathscr{B}$-simple, E-valued functions on X, i.e., functions of the form $\sum_{i\in I} \chi_{B_i} \otimes x_i$, $\{B_i\} \subset \mathscr{B}$, $\{x_i\} \subset E$, I finite, and by $B(X, \mathscr{B}, E)$ the closure of $S(X, \mathscr{B}, E)$, in the space of all bounded E-valued functions on X with the topology of uniform convergence on X. Any finitely additive set function $\mu : \mathscr{B} \to E'$ gives rise to a linear mapping $\mu : S(X, \mathscr{B}, E) \to K$; μ will be called a *measure* if this corresponding mapping is continuous with uniform topology on $S(X, \mathscr{B}, E)$ ([**7**, p. 375]). For a measure $\mu : \mathscr{B} \to E'$ and $x \in E$, we define $\mu_x : \mathscr{B} \to K$, $\mu_x(B) = \langle \mu(B), x\rangle = \mu(\chi_B \otimes x)$, $B \in \mathscr{B}$.

We define $M_t(X, E') = \{\mu : \mathscr{B} \to E' : \mu$ is a measure, and $\mu_x \in M_t(X)$, for all $x \in E'\}$. Take a $\mu \in M_t(X, E')$. If $p \in \mathscr{P}$ is such that $|\mu|_p$, defined by, $\forall\ B \in \mathscr{B}$, $|\mu|_p(B) = \sup \{|\sum \mu(\chi_{B_i} \otimes x_i)| : \{B_i\}$ a finite Borel partition of B, and $\{x_i\} \subset E$ with $\|x_i\|_p \leqq 1$, $\forall\ i\}$, is finite $\forall\ B \in \mathscr{B}$ (there will be at least one such $p \in \mathscr{P}$), then $|\mu|_p \in M_t(X)$ (cf. [**2**, Proposition 3.9, p. 850]). Thus for any $\mu \in M_t(X, E')$ we get a linear continuous mapping $\mu : S(X, \mathscr{B}, E) \to K$, with uniform topology on $S(X, \mathscr{B}, E)$; this can be uniquely extended to $\mu : B(X, \mathscr{B}, E) \to K$. It is easy to verify that $C_b(X) \otimes$

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$E \subset B(X, \mathscr{B}, E)$. It is known that $C_b(X) \otimes E$ is dense in $(C_b(X, E), \beta_0)$ and $(C_b(X, E), \beta_0)' = M_t(X, E')$ [**8**]. For a $p \in \mathscr{P}$, we put $M_{t,p}(X, E') = \{\mu \in M_t(X, E') : |\mu|_p \in M_t(X)\}$.

We first prove the following lemmas.

LEMMA 1. *For a* $\mu \in M_{t,p}(X, E')$ *and* $f \in C_b(X), f \geqq 0$,

$$|\mu|_p(f) = \sup \{|\mu(g)| : g \in C_b(X) \otimes E, ||g||_p \leqq f\}.$$

(*Here* $||f||_p : X \to R$, $||f||_p(x) = ||f(x)||_p$.)

Proof of this involves routine arguments and is omitted.

LEMMA 2. *Let* 2^N *denote all subsets of* N, *with product topology. If* $\lambda_n : 2^N \to K$ *is a sequence of countably additive measures (this implies they are continuous) and* $\lim \lambda_n(M) = \lambda(M)$ *exists for all* $M \subset N$, *then* $\lambda_n \to \lambda$ *uniformly on* 2^N. *In particular,* $\lambda_n(\{n\}) \to 0$.

Proof. This lemma is a particular case of [**4**, Lemma 1].

LEMMA 3. *A subset* $A \subset M_t(X, E')$ *is equicontinuous on* $(C_b(X, E), \beta_0)$ *if and only if given* $\epsilon > 0$, *there exists a* $p \in \mathscr{P}$ *and a compact* $K \subset X$ *such that* $A \subset M_{t,p}(X, E')$, $|\mu|_p(X \backslash K) \leqq \epsilon$ *for all* $\mu \in A$, *and*

$$\sup \{|\mu|_p(X) : \mu \in A\} = \alpha_0 < \infty.$$

Proof. Suppose first that A is β_0-equicontinuous. This means there exists a $\varphi \in C_0(X)$ and a $p \in \mathscr{P}$ such that

$$\{f \in C_b(X, E) : ||\varphi f||_p \leqq 1\} \subset \{g \in C_b(X, E) : |(\mu)g| \leqq 1, \text{ for all } \mu \in A\}.$$

Suppose $\sup_{x \in X} |\varphi(x)| \leqq \alpha_1 > 0$. This means

$$\sup \{|\mu(f)| : \mu \in A, f \in C_b(X, E), ||f||_p \leqq 1\} \leqq 1/\alpha_1.$$

By Lemma 1, $\sup \{|\mu|_p(X) : \mu \in A\} = \alpha_0 \leqq 1/\alpha_1 < \infty$. Take a compact set $K \subset X$ with the property that $K \supset \{x \in X : |\varphi(x)| \geqq \epsilon\}$. If $|\mu|_p(X \backslash K) > \epsilon$ for some $\mu \in A$, then there exists an $f \in C_b(X)$, $0 \leqq f \leqq \chi_{X \backslash K}$, with $|\mu|_p(f) > \epsilon$. By Lemma 1, there exists a $g \in C_b(X) \otimes E$ with $||g||_p \leqq f$ and $|\mu(g)| > \epsilon$. This is a contradiction since $||\varphi g||_p < \epsilon$ implies $|\mu(g)| \leqq \epsilon$.

Conversely, suppose A satisfies the given conditions. Let $(E_p, ||\cdot||_p)$ be the normed space associated with the semi-normed space $(E, ||\cdot||_p)$. Then $C_b(X) \otimes E_p \subset \pi_p \circ C_b(X, E) \subset C_b(X, E_p)$, $\pi_p : E \to E_p$ being the canonical mapping, and $C_b(X) \otimes E_p$ is dense in $(C_b(X, E_p), \beta_0)$ ([**2**, p. 851]). Thus the elements of A can be considered as elements of $M_t(X, E_p') = (C_b(X, E_p), \beta_0)'$. We shall prove A is equicontinuous on $(C_b(X, E_p), \beta_0)$; this will prove the result. Since β_0 is the finest locally convex topology coinciding with compact-open topology on the norm-bounded subsets of $C_b(X, E_p)$ [**2**], it is enough to prove that for any $k > 0$ there exists a compact subset K of X and some $\eta > 0$ such that $Z = \{f \in C_b(X, E_p) : ||f||_p \leqq k, ||f||_K \leqq \eta\} \subset \{g \in C_b(X, E_p) : |\mu(g)| \leqq 1, \text{ for all } \mu \in A\}$. By hypothesis there exists a compact $K \subset X$ with $|\mu|_p(X\backslash K) < 1/(2k+1)$, for all $\mu \in A$. Take $\eta = 1/(2(1+\alpha_0))$. For an $f \in Z$ and $\mu \in A$,

$$\begin{aligned}|\mu(f)| &\leqq \int ||f||_p \, d|\mu|_p \quad ([\mathbf{2}, \text{p. } 851]) \\ &= \int_K ||f||_p \, d|\mu|_p + \int_{X\backslash K} ||f||_p \, d|\mu|_p \\ &\leqq \frac{1}{2(1+\alpha_0)}\alpha_0 + \frac{k}{2k+1} \leqq 1.\end{aligned}$$

This proves the result.

Remark. This lemma evidently holds for any Hausdorff completely regular space X.

LEMMA 4. *Let $F = C_b(X, E)$, $F' = M_t(X, E')$, and let A, a relatively countably compact subset of $(F', \sigma(F', F))$, be equicontinuous on $(C_b(X, E), u)$. Then A is equicontinuous on (F, β_0).*

Proof. There exists a $p \in \mathscr{P}$ and $\alpha_0 > 0$ such that

$$\alpha_0 = \sup \{|\mu|_p(X) : \mu \in A\} < \infty.$$

Thus $A \subset M_{t,p}(X, E')$. First we prove that for any partition of unity $\{f_\alpha\}_{\alpha \in I}$, in X, and $\epsilon > 0$, there exists a finite subset $I_0 \subset I$ such that $\sum_{\alpha \in I_0} |\mu|_p(f_\alpha) > |\mu|_p(1) - \epsilon$, for all $\mu \in A$. If this is not true, there exists a sequence $\{\mu_n\} \subset A$ and a strictly increasing sequence $\{q(n)\} \subset N$ with $q(0) = 1$, and a distinct countable set $\{\alpha_n\} \subset I$ such that

$$|\mu_n|_p\left(\sum_{i=0}^{q(n+1)-q(n)-1} f_{\alpha_{q(n)+i}}\right) > \epsilon/2, \quad \text{for all } n.$$

This means there exists a sequence $\{g_n\} \subset C_b(X) \otimes E$,

$$||g_n||_p \leqq \sum_{i=0}^{q(n+1)-q(n)-1} f_{\alpha_{q(n)+i}}, \quad \text{and} \quad |\mu_n(g_n)| > \epsilon/2, \quad \text{for all } n.$$

For any subset $M \subset N$, $g_M = \sum_{n \in M} g_n \in C_b(X, E)$ and $||g_M||_p \leqq 1$. The countable set $\{g_M : M \text{ finite}\}$ is dense in $P = \{g_M : M \subset N\} \subset (C_b(X, E), \beta_0)$ and so if μ, in $(F', \sigma(F', F))$, is an adherent point of $\{\mu_n\}$, there exists a subsequence of $\{\mu_n\}$, which for notational convenience we denote by $\{\mu_n\}$, such that $\mu_n \to \mu$, pointwise on P [**5**]. Define $\lambda_n : 2^N \to K$, $\lambda_n(M) = \mu_n(g_M)$. Using the dominated convergence theorem and the relation $|\mu_n(g)| \leqq |\mu_n|_p(||g||_p)$, for all $g \in C_b(X, E)$ [**2**, p. 851], we prove that the λ_n's are countably additive; also $\{\lambda_n(M)\}$ is convergent for every $M \subset N$. By Lemma 2, $\mu_n(g_n) =$

$\lambda_n(g_n) \to 0$, which is a contradiction. Let $\{V_\alpha\}_{\alpha \in I}$ be a family of relatively compact, open subsets of X filtering upwards such that $\bigcup_{\alpha \in I} V_\alpha = X$, and let $\{f_\alpha\}$ be a partition of unit subordinated to this converging. Using the above notations we have $|\mu|_p(\sum_{\alpha \in I_0} f_\alpha) > |\mu|_p(1) - \epsilon$. This means there exists $\alpha_1 \in I$ such that $|\mu_p|(V_{\alpha_1}) > |\mu_p|(1) - \epsilon$. Taking $K = \bar{V}_{\alpha_1}$ and using Lemma 3, we get the result.

THEOREM 5. *If E is metrizable, then $(C_b(X, E), \beta_0)$ is Mackey; if, in addition, E is complete $(C_b(X, E), \beta_0)$ is strongly Mackey.*

Proof. Let $F = C_b(X, E)$, $F' = M_t(X, E')$. Take A to be an absolutely convex and compact subset of $(F', \sigma(F, F'))$. Since E is metrizable $(C_b(X, E), u)$ is metrizable. Using $\beta_0 \leqq u$ we prove that A is equicontinuous on $(C_b(X, E), u)$. By Lemma 4, A is equicontinuous on $(C_b(X, E), \beta_0)$. If E is a Fréchet space then $(C_b(X, E), u)$ is barreled and so if A is relatively countably compact subset of $(F', \sigma(F', F))$, then A is equicontinuous on $(C_b(X, E), u)$ [**6**, p. 127, 1.6 Corollary]. The result follows from Lemma 4.

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