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# LINEAR SYMMETRY CLASSES

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**Summary.** A formula is derived for the dimension of a symmetry class of tensors (over a finite dimensional complex vector space) associated with an arbitrary finite permutation group $G$ and a linear character of $\chi$ of $G$. This generalizes a result of the first author [**3**] which solved the problem in case $G$ is a cyclic group.

First, symmetry classes are introduced and some of the known results about them are reviewed. In case $\chi$ is the principal character of $G$, the dimension of a symmetry class turns out to be the number of $G$-inequivalent functions from the object set of $G$ into a finite set. Such numbers can be found immediately using Pólya's counting theorem, which is stated and then applied to several examples.

In the next section Pólya's theorem is applied to count the number of $G$-inequivalent functions, the stabilizer subgroups of which have a prescribed image under a linear character $\chi$ of $G$. This is a special case of the enumeration of orbits of $G$ having given stabilizer subgroups, a problem solved by Klass [**5**], Stockmeyer [**10**], and White [**12**]. The solutions to this general problem are computationally difficult, since they involve the full table of marks of the subgroups of $G$ and also the number of functions left fixed by each subgroup. In our case the linearity of $\chi$ allows the numbers to be found from a slight generalization of Burnside's lemma which is easy to evaluate in practice. This method is applied in the last section to the dimension problem for linear symmetry classes, which corresponds to counting just those $G$-inequivalent functions which have $\chi$-trivial stabilizer subgroups.

**1. Introduction.** Throughout $V$ denotes a complex vector space of finite dimension $n$. We consider pairs $(G, \chi)$ where $G$ is a subgroup of the symmetric group $S_m$ and $\chi$ is a character of $G$. Let $m > 1$ be a positive integer. A multilinear mapping $f : \times^m V \to W$ of the Cartesian product of $m$ copies of $V$ to another complex vector space $W$ is said to be *symmetric with respect to $G$ and $\chi$* if

$$f(x_{g(1)}, \ldots, x_{g(m)}) = \chi(g)f(x_1, \ldots, x_m)$$

for every $g \in G$ and $m$-tuple $(x_1, \ldots, x_m)$ of vectors from $V$.

If $\tau : \times^m V \to P$ is multilinear and symmetric with respect to $G$ and $\chi$ then the pair $(P, \tau)$ is a *symmetry class of tensors associated with $G$ and $\chi$*, or, briefly, a $(G, \chi)$ space provided

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(i) the range of $\tau$ generates $P$,

(ii) (universal factoring property) if $f : \times^m V \to W$ is any multilinear mapping symmetric with respect to $G$ and $\chi$ then there exists a linear mapping $g : P \to W$ such that $f = g \circ \tau$; i.e., the following diagram is commutative.

$$\begin{tikzcd} & W & \\ \times^m V \arrow[ur, "f"] \arrow[rr, "\tau"'] & & P \arrow[ul, dashed, "g"'] \end{tikzcd}$$

For each subgroup $G < S_m$ and any character $\chi$ of $G$ a $(G, \chi)$ space exists and is unique to within a canonical isomorphism. Any $(G, \chi)$ space will be denoted by $V_\chi^m(G)$. If $\chi$ is the principal character assigning 1 to each element of $G$ then we will sometimes write $V_1^m(G)$. In any symmetry class the elements of $\tau(\times^m V)$ are called *decomposable*. If $x_1, \ldots, x_m$ are vectors of $V$ then $\tau(x_1, \ldots, x_m)$ will be denoted by $x_1 * \ldots * x_m$.

If $G = \{e\}$ consists of the identity permutation alone and $\chi(e) = 1$ then $V_\chi^m(G) = \otimes^m V$, the $m$th tensor product of $V$, and has dimension $n^m$. If $G = S_m$ and $\chi$ is the alternating character of $S_m$ then $V_\chi^m(G) = \Lambda^m V$, the $m$th component of the exterior algebra over $V$, and its dimension is the binomial coefficient $\binom{n}{m}$. If $G = S_m$ and $\chi$ is the principal character then $V_1^m(G) = V^m V$, the $m$th component of the symmetric algebra over $V$, and its dimension is $\binom{n+m-1}{m}$.

Symmetry classes were introduced by H. Weyl in the context of quantum mechanics [**11**]. There a $(G, \chi)$ space was defined as the range in $\otimes^m$ of the symmetry operator

$$(1) \qquad \frac{1}{|G|} \sum_{g \in G} \chi(g) P(g)$$

when $P(g)$ is the linear transformation satisfying

$$P(g^{-1}) x_1 \otimes \ldots \otimes x_m = x_{g(1)} \otimes \ldots \otimes x_{g(m)}.$$

If $\chi$ is an irreducible character then the orthogonality relations may be used [**8**] to prove that (1), if multiplied by the degree of the character, is both idempotent and Hermitian with respect to the inner product on $\otimes^m V$:

$$(x_1 \otimes \ldots \otimes x_m, y_1 \otimes \ldots \otimes y_m) = \prod_{i=1}^m (x_i, y_i) \quad x_i, y_i \in V.$$

Here ( , ) is the usual inner product. Thus, the trace of (1) will be the dimension of its range when $\chi$ is irreducible.

A *linear character* of a group $G$ is the character $\chi$ of a 1-dimensional representation; i.e., $\chi : G \to C^*$ is a group homomorphism of $G$ to the group of nonzero complex numbers. When $G$ is finite the values of a linear character of $G$ form a subgroup of the group of roots of unity.

Botta and Williamson [**1**] proved that if $\chi$ is a linear character then the trace of (1) is

$$(2) \qquad \dim V_\chi^m(G) = \frac{1}{|G|}\sum_{g\in G} \chi(g)n^{c(g)}$$

where $c(g)$ is the number of cycles (counting fixed points) in the cyclic decomposition of the permutation $g$. The formula (2) is difficult to use because it requires finding sums of roots of unity. Using (2) the first author has shown [**3**] that if $G = \langle(1 \dots m)\rangle$ and $\chi$ is the principal character then for any $n$-dimensional vector space $V$

$$\dim V_1^m(G) = \frac{1}{m}\sum_{d|m} \phi(m/d)n^d,$$

where $\phi$ is the Euler totient function. He further showed that if $G = \langle(1 \dots m)\rangle$ and $\chi(1 \dots m)$ generates the group of $m$th roots of unity then for any $n$-dimensional vector space $V$

$$\dim V_\chi^m(G) = \frac{1}{m}\sum_{d|m} \mu(m/d)n^d,$$

where $\mu$ is the Mobius function.

Let $D = \{1, \dots, m\}$ and $R = \{1, \dots, n\}$. If $e_1, \dots, e_n$ is a basis of the vector space $V$ then the decomposable elements $e_\alpha = e_{\alpha(1)}* \dots *e_{\alpha(m)}$, $\alpha \in R^D$, always span $V_\chi(G)$. When $\chi$ is linear, Marcus and Minc [**6**] proved that

$$(3) \qquad \{e_\alpha : \alpha \in \bar{\Delta}\}$$

is a basis of $V_\chi(G)$ where $\Delta$ is any transversal of the orbits in $R^D$ under the action of $G$ on $D$ and

$$\bar{\Delta} = \{\alpha \in \Delta \mid G_\alpha < \ker \chi\}$$

where $G_\alpha = \{g \in G \mid \alpha \circ g = \alpha\}$.

When $\chi$ is a nonlinear character Merris and Pierce [**7**] have shown that (3) may be completed to a basis of $V_\chi(G)$. In this paper we are concerned with the linear case.

**2. Pólya's theorem and linear symmetry classes.** Every subgroup $G < S_m$ acts on the set of $n^m$ functions in $R^D$ if we define for each $g \in G$

$$(4) \qquad g\alpha(i) = \alpha(g^{-1}(i)) \quad \alpha \in R \text{ and } i \in D.$$

For each permutation $g \in G$ let $j_k(g)$ be the number of cycles of length $k$ in the cyclic decomposition of $g$. The *cycle type* of $g$ is the monomial

$$Z(g) = Z(g; a_1, \ldots, a_m) = \prod_{k=1}^{m} a_k^{j_k(g)}$$

in the variables $a_1, \ldots, a_m$. The *cycle index* of $G$ is the polynomial

$$Z(G) = Z(G; a_1, \ldots, a_m) = \frac{1}{|G|} \sum_{g \in G} Z(g).$$

If there is a mapping $w : R \rightarrow A$ where $A$ is any commutative ring then the elements $w(i)$, $i \in R$, are called *weights*. The *weight of a function* $\alpha \in R^D$ is defined as $W(\alpha) = \prod_{i=1}^{m} W(\alpha(i))$. Clearly two functions in the same orbit (or *pattern*) of $R^D$ under the action of $G$ have the same weights. If $W(F)$ is the weight of any representative of an orbit $F$ then $\sum_F W(F)$ is called the *pattern inventory*. With these definitions we may state:

THEOREM 1. (Pólya).

$$\sum_F W(F) = Z\left(G; \sum_{i=1}^{n} W(i), \sum_{i=1}^{n} W(i)^2, \ldots, \sum_{i=1}^{n} (Wi)^m\right).$$

COROLLARY 2. *If $R = \{1, \ldots, n\}$ and $G$ is a group of permutations on $D = \{1, \ldots, m\}$, then the number of $G$-orbits in $R^D$ is $Z(G; n, n, \ldots, n)$.*

THEOREM 3. *Let $V_1^m(G)$ be any linear symmetry class associated with the principal character. If $V$ is any $n$-dimensional vector space then*

$$\dim V_1^m(G) = \frac{1}{|G|} \sum_{g \in G} Z(g; n, \ldots, n). \tag{5}$$

*Proof.* If $\chi$ is the principal character then $\Delta = \bar{\Delta}$ and (3) is a basis of $V_\chi^m(G)$. Therefore (5) is an immediate consequence of Corollary 2.

In particular, if $G = \langle (12 \ldots m) \rangle$ then

$$\dim V_1^m(g) = \frac{1}{|G|} \sum_{d|m} \phi(m/d) n^d$$

which is a result previously obtained in [**3**]. When $m = 6$ one verifies that for any $n$-dimensional vector space $V$.

$$\dim V_1^6(G) = \tfrac{1}{6}[2n + 2n^2 + n^3 + n^6].$$

However, the cyclic group $H = \langle (12)(345) \rangle$ which is abstractly isomorphic to $G$ yields

$$\dim V_1^6(H) = \tfrac{1}{6}[2n^3 + 2n^4 + n^5 + n^6].$$

Of course every permutation of $H$ fixes 6. If $H$ is viewed as acting on $\{1, \ldots, 5\}$ instead we obtain

$$\dim V_1^5(H) = \tfrac{1}{6}[2n^3 + 2n^3 + n^4 + n^5].$$

These simple examples suggest that many properties of symmetry classes are inherently combinatorial rather than algebraic in character. In Section 4 we extend Theorem 3 to obtain a useable formula for $V_\chi^m(G)$ when $\chi$ is a linear character other than the principal one.

**3. Counting restricted orbits.** In this section we consider a finite permutation group $G$ acting on any finite set $X$ and supplied with a linear character $\chi$. Then in the next section the results will be applied to $X = R^D$ under the induced action (4).

For $\alpha$ in $X$, let $G_\alpha = \{g \in G \mid g(\alpha) = \alpha\}$ be the stabilizer of $\alpha$. For any $g \in G$ let $F(g)$ be the number of objects left fixed by $g$. It is well known by Burnside's lemma [**2**, p. 191] that the number of orbits of $G$ is just

$$(6) \qquad \frac{1}{|G|} \sum_{g \in G} F(g).$$

For any positive integer $k$, let $C_k$ denote the group of complex $k$th roots of unity; since $\chi$ is a linear character of a finite group, we have $\chi(G) = C_c$ for some $c$. For $\alpha$ in $X$, let $\sigma(\alpha)$ be determined by

$$\chi(G_\alpha) = C_{\sigma(\alpha)}.$$

If $\alpha$ and $\beta$ are in the same $G$-orbit then $G_\alpha$ and $G_\beta$ are conjugate and $\chi(G_\alpha) = \chi(G_\beta)$, so $\sigma(\alpha) = \sigma(\beta)$. It is convenient to denote the order of $\chi(g)$ by $\chi(g)^o$. A character $\chi_k$ is obtained from $\chi$ by defining

$$\chi_k(g) = \chi(g)^k$$

for each $g \in G$.

THEOREM 4. *Let $G$ be a permutation group acting on a finite set $X$ with a linear character $\chi$. If $\chi(G) = C_c$ and $k|c$, then the number of $G$-orbits consisting of $\alpha \in X$ with $\sigma(\alpha) = k$ is given by*

$$a_k = \frac{1}{|G|} \sum_{g \in G} F(g) \sum_{v|k} \frac{\mu(\chi_v(g)^o)}{\phi(\chi_v(g)^o)} \mu(k/v).$$

*Proof.* We first find $a_1$, since that is of interest in the determination of $\dim V_\chi^m(G)$.

For each $q|c$, $G_q = \chi^{-1}(C_q)$ is a normal subgroup of $G$. Let

$$(7) \qquad A_q = \frac{1}{|G_q|} \sum_{g \in G_q} F(g).$$

By (6) $A_q$ is the number of $G_q$-orbits in $X$. We claim that if $k = \sigma(\alpha)$, then the $G$-orbit of $\alpha$ contains exactly $(c/k, c/q)$ $G_q$-orbits. To see this, note first that the subgroup of $G$ mapping $\alpha$ to its own $G_q$-orbit is $G_qG_\alpha$. Thus the number of $G_q$-orbits of $\alpha$ is

$$[G : G_qG_\alpha] = [G : G_q]/[G_\alpha : G_q \cap G_\alpha].$$

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Clearly $[G : G_q] = c/q$ since $G = \chi^{-1}(C_c)$ and $G_q = \chi^{-1}(C_q)$. Let $\chi_\alpha$ denote the restriction of $\chi$ to $G_\alpha$. Then, as before,

$$G_\alpha = \chi_\alpha^{-1}(C_k)$$

and $G_\alpha \cap G_q = \chi_\alpha^{-1}(C_k \cap C_q) = \chi_\alpha^{-1}(C_{(k,q)})$. Consequently $[G_\alpha : G_q \cap G_\alpha] = k/(k, q)$ and hence

$$[G : G_q G_\alpha] = c(k, q)/kq = (c/k, c/q).$$

Summing over all $G$-orbits we obtain

$$(8) \qquad A_q = \sum_{k|c} (c/k, c/q) a_k.$$

Using the Mobius inversion formula of elementary number theory, we claim that (8) inverts to yield

$$(9) \qquad a_1 = \frac{1}{\phi(c)} \sum_{q|c} \mu(q) A_q$$

where $\phi$ is Euler's totient function.

To verify this, use (8) to express the summation of (9) in the form

$$(10) \qquad \sum_{k|c} a_k \sum_{q|c} (c/k, c/q)\mu(q).$$

The coefficient of $a_1$ in (10) is $\sum_{q|c} c/q\mu(q)$, which is seen to equal $\phi(c)$ upon inversion of the more obvious relation $c = \sum_{k|c}\phi(k)$. The coefficient of $a_k$ in (10) is 0 whenever $k > 0$. To see this, consider for any fixed $r$ the subsum obtained by restricting the summation to $q$ such that $(c/k, c/q) = r$. If there is some prime $p$ dividing both $k$ and $c/kr$, then $p^2|q$ for any such $q$, so $\mu(q)$ is always 0 in the subsum. If there is no such prime $p$ then $(k, c/kr) = 1$. In this case the restriction on $q$ is just that $q = (c/kr)q'$ where $q'|k$. Factoring out $r$, the subsum becomes

$$r \sum_{q'|k} \mu\left(\frac{c}{kr} q'\right) = r\mu\left(\frac{c}{kr}\right) \sum_{q'|k} \mu(q') = 0.$$

Substituting (7) into (9) we obtain

$$(11) \quad a_1 = \frac{1}{\phi(c)} \sum_{q|c} \mu(q) \frac{1}{|G_q|} \sum_{g \in G_q} F(g).$$

Since $g \in G_q$ if and only if $\chi(g)^o|q$, and $|G|/|G_q| = c/q$, we can rearrange (11) in the form

$$a_1 = \frac{1}{|G|} \sum_{g \in G} F(g) \frac{c}{\phi(c)} \sum \frac{\mu(q)}{q},$$

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where the last summation is over $q$ such that $(g)^o|q|c$. It is straightforward that

$$\frac{c}{\phi(c)}\sum_{j|q|c}\frac{\mu(q)}{q} = \frac{\mu(j)}{\phi(j)}$$

holds for each divisor $j$ of $c$, and so (11) takes the form

$$(12)\quad a_1 = \frac{1}{|G|}\sum_{g\in G}\frac{\mu(\chi(g)^o)}{\phi(\chi(g)^o)}F(g).$$

To find $a_k$ for arbitary $k$, note that

$$(13)\quad a_1^{(k)} = \sum_{v|k} a_v,$$

where $a_1^{(k)}$ is determined using the character $\chi_k$ defined by $\chi_k(g) = \chi(g)^k$. Inverting (13) we have

$$a_k = \sum_{v|k} a_1^{(v)}\mu(k/v),$$

and applying (12) with $\chi_v$ in place of $\chi$ we have

$$(14)\quad a_k = \frac{1}{|G|}\sum_{g\in G}F(g)\sum_{v|k}\frac{\mu(\chi_v(g)^o)}{\phi(\chi_v(g)^o)}\mu(k/v).$$

This completes the proof.

In connection with evaluating (14), it should perhaps be noted that if $\chi(g)^o = j$ then $\chi_v(g)^o = j/(j, v)$.

**4. Evaluating** dim $V_\chi^m(G)$**.** If $G$ acts on $D = \{1, \ldots, m\}$ originally, and induces the action (4) on $R^D$ where $R = \{1, \ldots, n\}$, then the basic result of Marcus and Minc [**5**] is that dim $V_\chi^m(G) = a_1$. In this case $F(g) = n^{c(g)}$, where $c(g)$ is the number of cycles in the disjoint cycle decomposition of $g$ over $D$. Thus we obtain:

THEOREM 5. *If $\chi$ is a complex linear character of $G < S_m$ then for any $n$ dimensional vector space $V$*

$$(15)\quad \dim V_\chi^m(G) = \frac{1}{|G|}\sum_{g\in G}\frac{\mu(\chi(g)^o)}{\phi(\chi(g)^o)}n^{c(g)}.$$

This has the advantage over (2) that in general there is considerable cancellation of complex numbers in (2), which is avoided in (15).

We could derive (15) from (2) directly as follows: Partition $G$ by the equivalence relation: $g \sim h$ if $\langle g\rangle = \langle h\rangle$, [**9**, p. 148]. If $g \sim h$ then $Z(g) = Z(h)$ and *a fortiori* $c(g) = c(h)$. Thus the contributions to (2) due to the cyclic equivalence class of $g$ can be collected, and $n^{c(g)}$ factored out to leave $\sum_{h\sim g}\chi(h)$. Now, if $g$ has order $a$ in $G$ and $\epsilon$ is some primitive $a$th root of unity then $\chi(g) = \epsilon^b$

for some $0 \leqq b < a$,

$$\sum_{h \sim g} \chi(h) = \sum_{1 \leqq i \leqq a:(i,a)=1} \chi(g^i) = \sum_{1 \leqq i \leqq a:(i,a)=1} \epsilon^{ib},$$

being the sum of the $b$th powers of the primitive $a$th roots of unity. From this it is clear that the sum is the same whatever primitive $a$th root $\epsilon$ of unity is taken as the starting point, and in the sum each primitive $a/(b, a)$th root of unity appears the same number of times. Now if $a/(a, b) = l$ then $l = \chi(g)^o$. Moreover, the primitive $l$th roots of unity sum to $\mu(l)$, as was first proved by Gauss [**4**, p. 52]. As there are $\phi(l)$ of these primitive $l$th roots of unity, the same sum is obtained if we replace $\chi(g)$ by $\mu(\chi(g)^o)/\phi(\chi(g)^o)$ as in (15).

We can obtain an analogue to (5) by defining the generalized cycle index of $G$ with respect to $\chi$ as

$$Z_\chi(G) = \frac{1}{|G|} \sum_{g \in G} \frac{\mu(\chi(g)^o)}{\phi(\chi(g)^o)} Z(g),$$

for then (14) can be put in the form

$$\dim V_\chi^m(G) = Z_\chi(G; n, n, \ldots, n).$$

It is seen from our derivation of (15) from (2) that in fact $Z_\chi(G)$ is identical to the generalized cycle index $P(G, D; \chi)$ defined by Rudvalis and Snapper [**9**, p. 146] by the sum (in our notation)

$$P(G, D; \chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g) Z(g).$$

The computational value of (15) over (1) is evident in the following example: For $m = 24$ set

$$\begin{aligned} a &= (1 \ldots 12)(13 \ldots 24) \\ b &= (c_1 \ldots c_{12})(d_1 \ldots d_{12}) \end{aligned}$$

where

$$c_i = \begin{cases} i & i \text{ odd} \\ i + 11 & i \text{ even} \end{cases} \quad \text{and} \quad d_i = \begin{cases} i + 1 & i \text{ odd;} \\ i + 12 & i \text{ even} \end{cases} \quad i = 1, \ldots, 12.$$

The group $G = \langle a, b \rangle$ is a commutative group of order 24. Since $a^2 = b^2$ its elements may be listed as

$$(16) \quad 1, a, a^2, \ldots, a^{11}, ab, a^2b, \ldots, a^{11}b, b.$$

A linear character on $G$ is obtained by defining $x(a) = \xi$, $x(b) = \xi^7$ where $\xi$ is a primitive 12th root of unity. If the group elements are ordered according to (16) then the function $\mu(\chi(g)^o)/\phi(\chi(g)^o)$ vanishes on every other group element. Computing the values for $c(g)$ we quickly conclude

$$\dim V_\chi^{24}(G) = \tfrac{1}{24}[2n^2 - n^8 - n^{12} + n^{24}].$$

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