ON TWO EARLY UNIFIED FIELD THEORIES

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1. Introduction

In this article we shall compare two unified field theories namely, Einstein's first presentation of a four dimensional theory (refs. 1, 2) and Levi-Civita's reformulation of it (ref. 3). It is well known that the problem of a unified field theory (of macroscopic world) is to find a suitable entity within a proposed geometry to represent the electromagnetic field. Gravitation is usually built into the theory since most of theories utilising a four dimensional space time as a model of the physical world consist in a direct extension of General Relativity.

Although Einstein himself abandoned later his original choice of an electromagnetic vector potential, our remarks are prompted by two considerations. First, no unified field theory has ever been 'disproved'. In view of the numerous attempts to formulate one it is likely that if one should eventually be accepted, it will in some sense contain the ideas previously discussed. Secondly, we wish to consider the assertion that Einstein's and Levi Civita's theories are equivalent. This has been made by the latter Author and we may find a similar statement in a recent work of Mme Tonnelat (ref. 4, pp. 280—81).

Two theories may be regarded as equivalent either if they lead to the same field equations or if the basic geometry of the space time is the same in both. It is clear of course, that the latter sense does not apply in our case. Levi Civita's theory employs an ordinary Riemann geometry while Einstein's is, from the outset, non riemannian. We shall show below that the first equivalence does not hold either. At least the electromagnetic field is basically distinct in the two theories. Thus, any similarity between them derives only from the first approximation in which a wave equation for the field quantities is obtained in both.

2. The n-uplet rotation

We use Greek indices to refer to space-time coordinates and Latin ones to a lattice of lines which we assume can be constructed at any point of the manifold. The theory is formulated independently of whether the manifold

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is four-dimensional but if the geometrical space time is n-dimensional there are n congruences at every point also. Thus both Greek and Latin indices go from 1 to n (or, tacitly, from 1 to 4). The Greek, coordinate indices are raised or lowered with the help of a symmetric metric tensor but there is no corresponding operation for the n-uplet and Latin indices are shifted purely for notational convenience. Both Einstein and Levi-Civita set up an n-uplet (tetrad) in their manifold but Einstein uses it to define a nonsymmetric affine connection $\Gamma^\lambda_{\mu\nu}$ while Levi-Civita constructs with its help the Ricci coefficients of rotation γ_{ijk}.

We shall assume that both theories start with the same Riemannian manifold V_n. The n-uplets cannot then be identical for, with Einstein's connection, the γ_{ijk}'s would vanish identically.

Let us denote Einstein's n-uplet parameters by h^i_μ and Levi-Civita's by $\lambda_{i/\mu}$. The stroke merely separates Greek and Latin indices and we use the semicolon notation for covariant derivatives. The summation convention applies to both kinds of indices. Then the metric tensor $g_{\mu\nu}$ is defined either by

(1)   or by

$$g_{\mu\nu} = h^a_\mu h^a_\nu, \qquad g^{\mu\nu} = h^\mu_a h^\nu_a,$$

$$g_{\mu\nu} = \lambda_{a/\mu}\lambda_{a/\nu}, \; g^{\mu\nu} = \lambda^\mu_a \lambda^\nu_a.$$

There exists a transformation

$$(2) \qquad\qquad dx'_a = h^\mu_a dx_\mu, \; dx_\mu = h^\mu_a dx_a,$$

for which the metric is Cartesian in the 'n-uplet' frame. However, (2) must be non-integrable if the space is to remain curved.

We define, with Einstein, an affine connection, as

$$(3) \qquad\qquad \Gamma^\sigma_{\mu\nu} = -h^a_\mu h^\sigma_{a,\nu} = h^\sigma_a h^a_{\mu,\nu},$$

whence

$$(4) \qquad\qquad g_{\mu\nu;\lambda} = g_{\mu\nu,\lambda} - \Gamma^\rho_{\mu\lambda} g_{\rho\nu} - \Gamma^\rho_{\nu\lambda} g_{\mu\rho} = 0.$$

Similarly

$$(5) \qquad\qquad \gamma_{ij\mu} = \lambda_{i/\mu;\nu} \lambda^\mu_j \lambda^\nu_k,$$

so that (4) ensures its skew-symmetry in i and j:

$$(6) \qquad\qquad \gamma_{ijk} + \gamma_{jik} = 0.$$

If we identify the two metric tensors of (1) with each other, it follows that there exists a (local, that is, applying to the Latin, n-uplet indices) transformation

$$(7) \qquad\qquad \lambda^\mu_i = A_{ia} h^\mu_a, \; h^a_\mu = \lambda_{ij\mu} A_{ia}, \; h^a_\mu = \lambda^\mu_i A_{ia}.$$

Clearly (under the assumption of orthogonality of the n-uplets), we have

(8)
$$A_{ia} = \lambda_{ij\mu} h_a^{\mu}.$$

The transformation coefficients A_{ia} form a representation of the rotation group which operates on the elements of the abstract space of local (that is, n-uplet or 'Latin' index) tensors. Indeed

(9)
$$A_{ia} A_{ja} = \delta_{ij} = A_{ai} A_{aj}.$$

A new form for the Ricci coefficients (5) is obtained if we use Einstein's connection (3).

Following Levi-Civita we introduce first invariant differentiation along the lattice lines. Thus

(10)
$$\frac{d}{ds_k} = \lambda_k^{\sigma} \frac{\partial}{\partial x_{\sigma}}, \quad \frac{\partial}{\partial x_{\mu}} = \lambda_{i/\mu} \frac{d}{ds_i}.$$

Then

$$\gamma_{ijk} = \lambda_{i/\mu;\,\nu} \lambda_j^{\mu} \lambda_k^{\nu} = (A_{ia})_{;\,\nu} A_{jb} A_{kc} h_b^{\mu} h_c^{\nu} h_{\mu}^{a}$$

(11)
$$= A_{ec} A_{ja} A_{kb} \frac{dA_{ia}}{ds_e} h_{\nu}^{c} h_b^{\nu}$$

$$= A_{ja} \frac{dA_{ia}}{ds_k},$$

since local quantities are space-time invariants. The fact that the Ricci coefficients can be expressed exclusively in terms of the A-transformation coefficients and their lattice-derivatives, plays a large part in our proof of non-equivalence of Einstein's and Levi-Civita's theories.

3. The electromagnetic field tensor

Levi Civita assumes the localised electromagnetic field tensor f_{ij} to be proportional to the space invariant

(12)
$$\zeta_{ij} = \frac{d\gamma_{ijk}}{ds_k}$$

The constant of proportionality is a numerical multiple of an elementary electric charge. On the other hand, Einstein identifies

(13)
$$\Gamma_{\mu} = 2\Gamma_{\mu\sigma}^{\sigma} = \Gamma_{\mu\sigma}^{\mu} - \Gamma_{\sigma\mu}^{\mu} = h_a^{\sigma}(h_{\mu,\sigma}^{a} - h_{\sigma,\mu}^{a}),$$

with the electromagnetic potential vector. His field tensor, F_{ij} say (in localised form) is therefore

(14)
$$F_{ij} = \lambda_i^{\mu} \lambda_j^{\nu} (\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu}).$$

It is not clear yet whether (12) and (14) can not be reconciled. We prove first that Γ_μ can be defined equivalently from either theory.

Indeed, from the definition (5) of a Ricci coefficient, we have

$$2\Gamma^\sigma_{\mu\nu} = \lambda^\sigma_i(\lambda_{i/\mu,\nu} - \lambda_{i/\nu,\mu} - (\gamma_{ijk} - \gamma_{ikj})\lambda_{j/\mu}\lambda_{k/\nu}),$$

whence

(15) $$\Gamma_\mu = \lambda^\sigma_i(\lambda_{i/\mu,\sigma} - \lambda_{i/\sigma,\mu}) - \gamma_{iji}\lambda_{j/\mu}.$$

But, from (3) using the rotation (7), and equation (11)

$$2\Gamma^\sigma_{\mu\nu} = \lambda^\sigma_i(\lambda_{i/\mu,\nu} - \lambda_{i/\nu,\mu}) + \lambda^\sigma_i\gamma_{jik}(\lambda_{j/\mu}\lambda_{k/\nu} - \lambda_{j/\nu}\lambda_{k/\mu}).$$

Since

$$\gamma_{iij} = 0,$$

a contraction of (16) leads back to (15) and this proves our assertion.

It follows that we can use either definition of Γ_μ to calculate Einstein's, localised electromagnetic tensor. In fact a simple reduction gives

$$F_{ij} = -(\lambda^\mu_i\lambda^\nu_j - \lambda^\nu_i\lambda^\mu_j)(\lambda^\sigma_{k,\nu}\lambda_{k/\sigma,\mu} + \lambda^\sigma_{k,\nu}\lambda_{k/\mu,\sigma}$$
$$+ \lambda_{k/\mu}\lambda^\sigma_{k,\nu\sigma} + \gamma_{klk}\lambda_{k/\mu,\nu}) - \frac{d}{ds_j}\lambda_{kik} + \frac{d}{ds_i}\gamma_{kjk}.$$

Since

$$\lambda_{k/\sigma,\mu} = \gamma_{kmn}\lambda_{m/\sigma}\lambda_{n/\mu} + \Gamma^\tau_{\sigma\mu}\lambda_{k/\tau},$$

and

$$\lambda^\sigma_{k,\nu} = \gamma_{kmn}\lambda^\sigma_m\lambda_{n/\nu} - \Gamma^\sigma_{\tau\nu}\lambda^\tau_k,$$

we have

$$\lambda_{k/\mu,\nu\sigma} = \gamma_{kmn}(\gamma_{mpq}\lambda_{p/\mu}\lambda_{q/\sigma}\lambda_{n/\nu} + \gamma_{npq}\lambda_{p/\nu}\lambda_{q/\sigma}\lambda_{m/\mu}$$
$$+ \Gamma^\rho_{\mu\sigma}\lambda_{n/\nu}\lambda_{m/\rho} + \Gamma^\rho_{\nu\sigma}\lambda_{n/\rho}\lambda_{m/\mu})$$
$$+ \Gamma^\rho_{\mu\nu}(\gamma_{kpq}\lambda_{p/\rho}\lambda_{q/\sigma} + \Gamma^\tau_\sigma\lambda_{k/\tau}) + \Gamma_{\mu\nu,\sigma}\lambda_{k/}$$
$$+ \lambda_{r/\sigma}\lambda_{m/\mu}\lambda_{n/\nu}\frac{d}{ds_r}(\gamma_{kmn}).$$

Using these expressions we find that

$$F_{ij} = \mathscr{E}_{ij} + \frac{d}{ds_k}(\gamma_{kij} - \gamma_{kji}) - \frac{d}{ds_j}\gamma_{kik} + \frac{d}{ds_i}\gamma_{kjk}$$
$$+ 2\lambda^\mu_i\lambda^\nu_j\Gamma^\sigma_{\mu\nu;\sigma} + 2\gamma_{kmn}\Gamma_{\mu\sigma}\lambda^\sigma_k(\lambda_{m/}\ \lambda^\mu_i\delta_{jn}$$
$$- \lambda_{m/}\ \lambda^\mu_j\delta_{in} - \lambda_{m/}\ \lambda^\mu_i\delta_{jm} + \lambda_{n/}\ \lambda^\mu_j\delta_{im}),$$

where

$$\mathscr{E}_{ik} = (\delta_{ip}\delta_{jn} - \delta_{in}\delta_{jp})(\gamma_{kmn}\gamma_{mpk} + \gamma_{kpm}\gamma_{pmn}$$
$$+ \gamma_{kmn}\gamma_{kpm} + \gamma_{mkn}\gamma_{kmp}).$$

Furthermore, if we introduce Levi-Civita's four index symbols

$$\gamma_{ij,hk} = \frac{d_{\gamma ijn}}{ds_k} - \frac{d_{\gamma ijk}}{ds_h} + \gamma_{ijl}(\gamma_{lhk}-\gamma_{lkh})+\gamma_{lih}\gamma_{ljh}-\gamma_{lih}\gamma_{ljk},$$

we find that

(16) $\mathscr{E}_{ij}+ \dfrac{d}{ds_k}(\gamma_{kij}-\gamma_{kji})- \dfrac{d}{ds_j}\gamma_{kik}+ \dfrac{d}{ds_i}\gamma_{kjk} = \gamma_{ki,jk}-\gamma_{kj,ik} = 0,$

whence finally

(17) $\frac{1}{2}F_{ij} = \gamma_i^\alpha \lambda_j^\beta \Gamma_{\alpha\beta;\sigma}^\sigma + \Gamma_{\mu\sigma}^\rho \lambda_{m/\rho} \lambda_k^\sigma (\delta_{jn}\lambda_i^\mu - \delta_{in}\lambda_j^\mu)(\gamma_{kmn}-\gamma_{knm}).$

4. Conclusions

We can deduce now easily that the theories of Einstein and of Levi-Civita are incompatible. Let us for example, choose

(18) $\lambda_i^\mu = h_i^\mu$

as a special case. Then, as mentioned above, the Ricci coefficients vanish identically. However, Einstein's field tensor as given by (17) becomes

$$F_{ij} = \lambda_i^\alpha \lambda_j \Gamma_{\alpha\beta;\sigma}^\sigma$$

and is non zero. In Levi-Civita's theory however, this case corresponds to an absence of electromagnetic fields.

Summary

It is shown that Einstein's 1929 Unified Field Theory and that of Levi-Civita are incompatible.

References

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