1

Fundamental Functional Equations

Throughout this book, we will make contact with the venerable subject of functional equations. A functional equation is an equation in an unknown function satisfied at all values of its arguments; or more generally, it is an equation relating several functions to each other in this way.

To set the scene, we give some brief indicative examples. Viewing sequences as functions on the set of positive integers, the Fibonacci sequence $(F_n)_{n\geq 1}$ satisfies the functional equation

$$F_{n+2} = F_n + F_{n+1}$$

($n \geq 1$). Together with the boundary conditions $F_1 = F_2 = 1$, this functional equation uniquely characterizes the sequence. But more typically, one is concerned with functions of *continuous* variables. For instance, one might notice that the function

$$\begin{array}{rccc} f\colon & \mathbb{R} \cup \{\infty\} & \to & \mathbb{R} \cup \{\infty\} \\ & x & \mapsto & \dfrac{1}{1-x} \end{array}$$

satisfies the functional equation

$$f(f(f(x))) = x \tag{1.1}$$

($x \in \mathbb{R} \cup \{\infty\}$). The natural question, then, is whether f is the *only* function satisfying equation (1.1) for all x. In this case, it is not. (This can be shown by constructing an explicit counterexample or via the theory of Möbius transformations.) So, it is then natural to seek the whole set of solutions f, perhaps restricting the search to just those functions that are continuous, differentiable, etc.

https://doi.org/10.1017/9781108963558.003 Published online by Cambridge University Press

A more sophisticated example is the functional equation

$$\zeta(1-s) = \frac{2^{1-s}}{\pi^s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s)\, \zeta(s)$$

($s \in \mathbb{C}$) satisfied by the Riemann zeta function ζ (Theorem 12.7 of Apostol [16], for instance). Here Γ is Euler's gamma function. This functional equation, proved by Riemann himself, is a fundamental property of the zeta function.

In this chapter, we solve three classical, fundamental, functional equations. The first is Cauchy's equation on a function $f : \mathbb{R} \to \mathbb{R}$:

$$f(x + y) = f(x) + f(y)$$

($x, y \in \mathbb{R}$) (Section 1.1). Once we have solved this, we will easily be able to deduce the solutions of related equations such as

$$f(xy) = f(x) + f(y) \tag{1.2}$$

($x, y \in (0, \infty)$).

The second is the functional equation

$$f(mn) = f(m) + f(n)$$

($m, n \geq 1$) on a *sequence* $(f(n))_{n\geq 1}$. Despite the resemblance to equation (1.2), the shift from continuous to discrete makes it necessary to develop quite different techniques (Section 1.2).

Third and finally, we solve the functional equation

$$f(xy) = f(x) + g(x)f(y)$$

in two unknown functions $f, g : (0, \infty) \to \mathbb{R}$. The nontrivial, measurable solutions f turn out to be the constant multiples of the so-called q-logarithms (Section 1.3), a one-parameter family of functions of which the ordinary logarithm is just the best-known member.

1.1 Cauchy's Equation

A function $f : \mathbb{R} \to \mathbb{R}$ is **additive** if

$$f(x + y) = f(x) + f(y) \tag{1.3}$$

for all $x, y \in \mathbb{R}$. This is **Cauchy's functional equation**, some of whose long history is recounted in Section 2.1 of Aczél [2]. Let us say that f is **linear** if there exists $c \in \mathbb{R}$ such that

$$f(x) = cx$$

https://doi.org/10.1017/9781108963558.003 Published online by Cambridge University Press

for all $x \in \mathbb{R}$. Putting $x = 1$ shows that if such a constant c exists then it must be equal to $f(1)$.

Evidently any linear function is additive. The question is to what extent the converse holds. If we are willing to assume that f is differentiable then the converse is very easy.

Proposition 1.1.1 *Every differentiable additive function $\mathbb{R} \to \mathbb{R}$ is linear.*

Proof Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable additive function. Differentiating equation (1.3) with respect to y gives

$$f'(x + y) = f'(y)$$

for all $x, y \in \mathbb{R}$. Taking $y = 0$ then shows that f' is constant. Hence there are constants $c, d \in \mathbb{R}$ such that $f(x) = cx + d$ for all $x \in \mathbb{R}$. Substituting this expression back into equation (1.3) gives $d = 0$. □

However, differentiability is a stronger condition than we will want to assume for our later purposes. It is, in fact, unnecessarily strong. In the rest of this section, we prove that additivity implies linearity under a succession of ever weaker regularity conditions, starting with continuity and finishing with mere measurability.

We begin with a lemma that needs no regularity conditions at all.

Lemma 1.1.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function. Then $f(qx) = qf(x)$ for all $q \in \mathbb{Q}$ and $x \in \mathbb{R}$.*

Proof First, $f(0 + 0) = f(0) + f(0)$, so $f(0) = 0$. Then, for all $x \in \mathbb{R}$,

$$0 = f(0) = f(-x + x) = f(-x) + f(x),$$

so $f(-x) = -f(x)$.

Let $x \in \mathbb{R}$. By induction,

$$f(nx) = nf(x) \tag{1.4}$$

for all integers $n > 0$, and we have just shown that equation (1.4) also holds when $n = 0$. Moreover, when $n < 0$,

$$f(nx) = f\big(-(-n)x\big) = -f\big((-n)x\big) = -(-n)f(x) = nf(x),$$

using equation (1.4) for positive integers. Hence (1.4) holds for all integers n.

Now let $x \in \mathbb{R}$ and $q \in \mathbb{Q}$. Write $q = m/n$, where $m, n \in \mathbb{Z}$ with $n \neq 0$. Then by two applications of equation (1.4),

$$f(qx) = \tfrac{1}{n} f(nqx) = \tfrac{1}{n} f(mx) = \tfrac{m}{n} f(x) = qf(x),$$

as required. □

Remark 1.1.3 The same argument proves that any additive function between vector spaces over $\mathbb{Q}$ is linear over $\mathbb{Q}$. In the case of functions $\mathbb{R} \to \mathbb{R}$, our question is whether (or under what conditions) $\mathbb{Q}$-linearity implies $\mathbb{R}$-linearity, which here we are just calling 'linearity'.

Lemma 1.1.2 enables us to improve Proposition 1.1.1, relaxing differentiability to continuity. The following result was known to Cauchy himself (cited in Hardy, Littlewood and Pólya [137], proof of Theorem 84).

Proposition 1.1.4 *Every continuous additive function* $\mathbb{R} \to \mathbb{R}$ *is linear.*

Proof Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous additive function, and write $c = f(1)$. By Lemma 1.1.2, $f(q) = cq$ for all $q \in \mathbb{Q}$. Thus, the two functions f and $x \mapsto cx$ are equal when restricted to $\mathbb{Q}$. But both are continuous, so they are equal on all of $\mathbb{R}$. □

It is now straightforward to relax continuity of f to a much weaker condition.

Proposition 1.1.5 *Every additive function* $\mathbb{R} \to \mathbb{R}$ *that is continuous at one or more point is linear.*

In other words, every additive function is linear unless, perhaps, it is discontinuous everywhere.

Proof Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function continuous at a point $x \in \mathbb{R}$. By Proposition 1.1.4, it is enough to show that f is continuous. Let $y, t \in \mathbb{R}$: then by additivity,

$$f(y + t) - f(y) = f(t) = f(x + t) - f(x) \to 0$$

as $t \to 0$, as required. □

Next we show that mere measurability suffices: every measurable additive function is linear.

Remark 1.1.6 Readers unfamiliar with measure theory may wish to read the rest of this remark then resume at Corollary 1.1.11. Measurability is an extremely weak condition. In the usual logical framework for mathematics, there do exist nonmeasurable functions and nonlinear additive functions (Remark 1.1.9). However, every function for which anyone has ever written down an explicit formula, or ever will, is measurable (by Remark 1.1.10). So it is not too dangerous to assume that every function is measurable and, therefore, that every additive function is linear.

https://doi.org/10.1017/9781108963558.003 Published online by Cambridge University Press

There are several proofs that every measurable additive function is linear. The first was published by Maurice Fréchet in his 1913 paper 'Pri la funkcia ekvacio $f(x + y) = f(x) + f(y)$' [110]. (Fréchet wrote many papers in Esperanto, and served three years as the president of the Internacia Scienca Asocio Esperantista.) Here we give the proof by Banach [27]. It is based on a standard measure-theoretic result of Lusin [235], which makes precise Littlewood's maxim that every measurable function is 'nearly continuous' [233].

Write λ for Lebesgue measure on $\mathbb{R}$.

Theorem 1.1.7 (Lusin) *Let $a \leq b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a measurable function. Then for all $\varepsilon > 0$, there exists a closed subset $V \subseteq [a, b]$ such that $f|_V$ is continuous and $\lambda([a, b] \setminus V) < \varepsilon$.*

Proof See Theorem 7.5.2 of Dudley [85], for instance. □

Following Banach, we deduce:

Theorem 1.1.8 *Every measurable additive function $\mathbb{R} \to \mathbb{R}$ is linear.*

Proof Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable additive function. By Lusin's theorem, we can choose a closed set $V \subseteq [0, 1]$ such that $f|_V$ is continuous and $\lambda(V) > 2/3$. Since V is compact, $f|_V$ is uniformly continuous.

By Proposition 1.1.5, it is enough to prove that f is continuous at 0. Let $\varepsilon > 0$. We have to show that $|f(x)| < \varepsilon$ for all x in some neighbourhood of 0.

By uniform continuity, we can choose $\delta > 0$ such that for $v, v' \in V$,

$$|v - v'| < \delta \implies |f(v) - f(v')| < \varepsilon.$$

I claim that $|f(x)| < \varepsilon$ for all $x \in \mathbb{R}$ such that $|x| < \min\{\delta, 1/3\}$. Indeed, take such an x. Then, writing $V - x = \{v - x : v \in V\}$, the inclusion-exclusion property of Lebesgue measure λ gives

$$\lambda(V \cap (V - x)) = \lambda(V) + \lambda(V - x) - \lambda(V \cup (V - x)).$$

Consider the right-hand side. For the first two terms, we have $\lambda(V) > 2/3$ and so $\lambda(V - x) > 2/3$. For the last, if $x \geq 0$ then $V \cup (V - x) \subseteq [-1/3, 1]$, if $x \leq 0$ then $V \cup (V - x) \subseteq [0, 4/3]$, and in either case, $\lambda(V \cup (V - x)) \leq 4/3$. Hence

$$\lambda(V \cap (V - x)) > 2/3 + 2/3 - 4/3 = 0.$$

In particular, $V \cap (V - x)$ is nonempty, so we can choose an element y. Then $y, x + y \in V$ with $|y - (x + y)| = |x| < \delta$, so $|f(y) - f(x + y)| < \varepsilon$ by definition of δ. But since f is additive, this means that $|f(x)| < \varepsilon$, as required. □

The regularity condition can be weakened still further; see Reem [292] for a recent survey. However, measurability is as weak a condition as we will need.

https://doi.org/10.1017/9781108963558.003 Published online by Cambridge University Press

Remark 1.1.9 Assuming the axiom of choice, there do exist additive functions $\mathbb{R} \to \mathbb{R}$ that are not linear. To see this, first note that the real line $\mathbb{R}$ is a vector space over $\mathbb{Q}$ in the evident way. Choose a basis B for $\mathbb{R}$ over $\mathbb{Q}$. Choose an element b of B, and let $\phi\colon B \to \mathbb{R}$ be the function taking value 1 at b and 0 elsewhere. By the universal property of bases, ϕ extends uniquely to a $\mathbb{Q}$-linear map $f\colon \mathbb{R} \to \mathbb{R}$.

Certainly f is additive. On the other hand, we can show that f is not $\mathbb{R}$-linear (that is, not 'linear' in the terminology of this section). Indeed, any $\mathbb{R}$-linear function $\mathbb{R} \to \mathbb{R}$ either is identically zero or vanishes nowhere except at 0. Now f is not identically zero, since $f(b) = \phi(b) = 1$. But also, for any $b' \neq b$ in B, we have $f(b') = \phi(b') = 0$ with $b' \neq 0$, so f vanishes at some point other than 0. Hence f is a nonlinear, additive function $\mathbb{R} \to \mathbb{R}$.

Remark 1.1.10 It is consistent with the Zermelo–Fraenkel axioms of set theory (that is, ZFC without the axiom of choice) that all functions $\mathbb{R} \to \mathbb{R}$ are measurable. This is a 1970 theorem of Solovay [318]. If all functions $\mathbb{R} \to \mathbb{R}$ are measurable then by Theorem 1.1.8, all additive functions are linear.

On the other hand, the axiom of choice is also consistent with ZF. If the axiom of choice holds then by Remark 1.1.9, not all additive functions are linear.

Hence, starting from ZF, one may consistently assume *either* that every additive function is linear *or* that not every additive function is linear.

Theorem 1.1.8 classifies the measurable functions that convert addition into addition. One can easily adapt it to classify the functions that convert addition into multiplication, multiplication into multiplication, and so on:

Corollary 1.1.11 *i. Let $f\colon \mathbb{R} \to (0, \infty)$ be a measurable function. The following are equivalent:*

a. $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$;
b. there exists $c \in \mathbb{R}$ such that $f(x) = e^{cx}$ for all $x \in \mathbb{R}$.

ii. Let $f\colon (0, \infty) \to \mathbb{R}$ be a measurable function. The following are equivalent:

a. $f(xy) = f(x) + f(y)$ for all $x, y \in (0, \infty)$;
b. there exists $c \in \mathbb{R}$ such that $f(x) = c \log x$ for all $x \in (0, \infty)$.

iii. Let $f\colon (0, \infty) \to (0, \infty)$ be a measurable function. The following are equivalent:

a. $f(xy) = f(x)f(y)$ for all $x, y \in (0, \infty)$;
b. there exists $c \in \mathbb{R}$ such that $f(x) = x^c$ for all $x \in (0, \infty)$.

Proof For (i), evidently (b) implies (a). Assuming (a), define $g: \mathbb{R} \to \mathbb{R}$ by $g(x) = \log f(x)$. Then g is measurable and additive, so by Theorem 1.1.8, there is some constant $c \in \mathbb{R}$ such that $g(x) = cx$ for all $x \in \mathbb{R}$. It follows that $f(x) = e^{cx}$ for all $x \in \mathbb{R}$, as required.

Parts (ii) and (iii) are proved similarly, putting $g(x) = f(e^x)$ and $g(x) = \log f(e^x)$. □

Remark 1.1.12 In this book, the notation log means the natural logarithm $\ln = \log_e$. However, the choice of base for logarithms is usually unimportant, as it is in Corollary 1.1.11(ii): changing the base amounts to multiplying the logarithm by a positive constant, which is in any case absorbed by the free choice of the constant c.

Theorem 1.1.8 also allows us to classify the additive functions that are defined on only half of the real line.

Corollary 1.1.13 *Let $f: [0, \infty) \to \mathbb{R}$ be a measurable function satisfying $f(x+y) = f(x)+f(y)$ for all $x, y \in [0, \infty)$. Then there exists $c \in \mathbb{R}$ such that $f(x) = cx$ for all $x \in [0, \infty)$.*

Proof First we extend $f: [0, \infty) \to \mathbb{R}$ to a measurable additive function $g: \mathbb{R} \to \mathbb{R}$. By the hypothesis on f, for all $a^+, a^-, b^+, b^- \in [0, \infty)$,

$$a^+ - a^- = b^+ - b^- \implies f(a^+) - f(a^-) = f(b^+) - f(b^-).$$

We can, therefore, consistently define a function $g: \mathbb{R} \to \mathbb{R}$ by

$$g(a^+ - a^-) = f(a^+) - f(a^-)$$

($a^+, a^- \in [0, \infty)$). To prove that g is additive, let $x, y \in \mathbb{R}$, and choose $a^\pm, b^\pm \in [0, \infty)$ such that

$$x = a^+ - a^-, \qquad y = b^+ - b^-.$$

Then

$$x + y = (a^+ + b^+) - (a^- + b^-)$$

with $a^+ + b^+, a^- + b^- \in [0, \infty)$. Hence

$$\begin{aligned} g(x + y) &= f(a^+ + b^+) - f(a^- + b^-) \\ &= f(a^+) + f(b^+) - f(a^-) - f(b^-) \\ &= f(a^+) - f(a^-) + f(b^+) - f(b^-) \\ &= g(x) + g(y), \end{aligned}$$

as required. To prove that g is measurable, note that

$$g(x)=\begin{cases} f(x) & \text{if } x \geq 0, \\ -f(-x) & \text{if } x \leq 0 \end{cases}$$

($x \in \mathbb{R}$), as if $x \geq 0$ then we can take $a^+ = x$ and $a^- = 0$ in the definition of g, and similarly for $x \leq 0$. Since f is measurable, so is g.

By Theorem 1.1.8, there exists a constant c such that $g(x) = cx$ for all $x \in \mathbb{R}$. It follows that $f(x) = cx$ for all $x \in [0, \infty)$. □

The techniques and results of this section can be assembled in several ways to derive variant theorems. Rather than attempting to catalogue all the possibilities, we illustrate the point with two particular variants needed later.

Corollary 1.1.14 *Let $f: (0, 1] \to \mathbb{R}$ be a measurable function. The following are equivalent:*

i. $f(xy) = f(x) + f(y)$ *for all* $x, y \in (0, 1]$*;*
ii. *there exists a constant* $c \in \mathbb{R}$ *such that* $f(x) = c \log x$ *for all* $x \in (0, 1]$.

Proof Trivially, (ii) implies (i). Now assuming (i), define $g: [0, \infty) \to \mathbb{R}$ by $g(u) = f(e^{-u})$. Then g is measurable and $g(u + v) = g(u) + g(v)$ for all $u, v \in [0, \infty)$, so by Corollary 1.1.13, $g(u) = bu$ for some real constant b. It follows that $f(x) = -b \log x$ for all $x \in (0, 1]$, as required. □

The moral of Corollary 1.1.14 is that for the Cauchy-like functional equation $f(xy) = f(x) + f(y)$, there is no substantial difference between solving it on the domain $(0, \infty)$ and solving it on the domain $(0, 1]$ (or $[1, \infty)$, similarly). But matters become very different when we seek solutions on the discrete domain $\{1, 2, 3, \ldots\}$, as we will discover in the next section.

Remark 1.1.15 In this text, we always use the terms 'increasing' and 'decreasing' in their non-strict senses. Thus, a function $f: S \to \mathbb{R}$ on a subset $S \subseteq \mathbb{R}$ is **increasing** if

$$x \leq y \implies f(x) \leq f(y)$$

($x, y \in S$), and **decreasing** if $-f$ is increasing. It is **strictly** increasing or decreasing if $x < y$ implies $f(x) < f(y)$ or $f(x) > f(y)$, respectively. The same terminology applies to sequences.

Corollary 1.1.16 *Let $f: (0, 1) \to (0, \infty)$ be an increasing function. The following are equivalent:*

i. $f(xy) = f(x)f(y)$ *for all* $x, y \in (0, 1)$*;*
ii. *there exists a constant* $c \in [0, \infty)$ *such that* $f(x) = x^c$ *for all* $x \in (0, 1)$.

https://doi.org/10.1017/9781108963558.003 Published online by Cambridge University Press

Proof Trivially, (ii) implies (i). Assuming (i), define $g: (0, \infty) \to \mathbb{R}$ by $g(u) = -\log f(e^{-u})$. Then $g(u + v) = g(u) + g(v)$ for all $u, v \in (0, \infty)$, and g is also increasing.

By the same argument as in the proof of Lemma 1.1.2, $g(qu) = qg(u)$ for all $q, u \in (0, \infty)$ with q rational. Define $\widetilde{g}: (0, \infty) \to \mathbb{R}$ by $\widetilde{g}(u) = g(1)u$. Then $g(q) = \widetilde{g}(q)$ for all $q \in (0, \infty) \cap \mathbb{Q}$. Since g is increasing and $\widetilde{g}$ is either increasing or decreasing (depending on the sign of $g(1)$), it follows that $\widetilde{g}$ is increasing. But now $g, \widetilde{g}: (0, \infty) \to \mathbb{R}$ are increasing functions that are equal on the positive rationals, so $g = \widetilde{g}$. Hence $f(x) = x^{g(1)}$ for all $x \in (0, 1)$. □

1.2 Logarithmic Sequences

A sequence $f(1), f(2), \ldots$ of real numbers is **logarithmic** if

$$f(mn) = f(m) + f(n) \tag{1.5}$$

for all $m, n \geq 1$. Certainly the sequence $(c \log n)_{n \geq 1}$ is logarithmic, for any real constant c. But in contrast to the situation for functions $f: (0, \infty) \to \mathbb{R}$ satisfying $f(xy) = f(x) + f(y)$ (Corollary 1.1.11(ii)), it is easy to write down logarithmic sequences that are not of this simple form. Indeed, we can choose $f(p)$ arbitrarily for each prime p, and these choices uniquely determine a logarithmic sequence, generally not of the form $(c \log n)$.

However, there are reasonable conditions on a logarithmic sequence $(f(n))$ guaranteeing that it is of the form $(c \log n)$. One such condition is that f is increasing:

$$f(1) \leq f(2) \leq \cdots .$$

An alternative condition is that

$$\lim_{n \to \infty} (f(n + 1) - f(n)) = 0.$$

We will prove a single theorem implying both of these results. But a direct proof of the result on increasing sequences is short enough to be worth giving separately, even though it is not logically necessary.

Theorem 1.2.1 (Erdős) *Let $(f(n))_{n \geq 1}$ be an increasing sequence of real numbers. The following are equivalent:*

i. *f is logarithmic;*
ii. *there exists a constant $c \geq 0$ such that $f(n) = c \log n$ for all $n \geq 1$.*

https://doi.org/10.1017/9781108963558.003 Published online by Cambridge University Press

This was first proved by Erdős [92]. In fact, he showed more: as is customary in number theory, he only required equation (1.5) to hold when m and n are relatively prime. But since we will not need the extra precision of that result, we will not prove it.

The argument presented here follows Khinchin ([188], p. 11).

Proof Certainly (ii) implies (i). Now assume (i). By the logarithmic property,

$$f(1) = f(1 \cdot 1) = f(1) + f(1),$$

so $f(1) = 0$. Since f is increasing, $f(n) \geq 0$ for all n. If $f(n) = 0$ for all n then (ii) holds with $c = 0$. Assuming otherwise, we can choose some $N > 1$ such that $f(N) > 0$.

Let $n \geq 1$. For each integer $r \geq 1$, there is an integer $\ell_r \geq 1$ such that

$$N^{\ell_r} \leq n^r \leq N^{\ell_r+1}$$

(since $N > 1$). As f is increasing and logarithmic,

$$\ell_r f(N) \leq r f(n) \leq (\ell_r + 1) f(N),$$

which since $f(N) > 0$ implies that

$$\frac{\ell_r}{r} \leq \frac{f(n)}{f(N)} \leq \frac{\ell_r + 1}{r}. \tag{1.6}$$

As log is also increasing and logarithmic, the same argument gives

$$\frac{\ell_r}{r} \leq \frac{\log n}{\log N} \leq \frac{\ell_r + 1}{r}. \tag{1.7}$$

Inequalities (1.6) and (1.7) together imply that

$$\left| \frac{f(n)}{f(N)} - \frac{\log n}{\log N} \right| \leq \frac{1}{r}.$$

But this conclusion holds for all $r \geq 1$, so

$$\frac{f(n)}{f(N)} = \frac{\log n}{\log N}.$$

Hence $f(n) = c \log n$, where $c = f(N)/\log N$. And since this is true for all $n \geq 1$, we have proved (ii). □

We now prove the unified theorem promised above. Before stating it, let us recall the concept of **limit inferior**. Given a real sequence $(g(n))_{n \geq 1}$, define

$$h(n) = \inf\{g(n), g(n+1), \ldots\} \in [-\infty, \infty)$$

https://doi.org/10.1017/9781108963558.003 Published online by Cambridge University Press

$(n \geq 1)$. The sequence $(h(n))_{n\geq 1}$ is increasing and therefore has a limit (perhaps $\pm\infty$), written as

$$\liminf_{n\to\infty} g(n) = \lim_{n\to\infty} h(n) \in [-\infty, \infty].$$

If the ordinary limit $\lim_{n\to\infty} g(n)$ exists then $\liminf_{n\to\infty} g(n) = \lim_{n\to\infty} g(n)$. However, the limit inferior exists whether or not the limit does. For instance, the sequence $1, -1, 1, -1, \ldots$ has a limit inferior of -1, but no limit.

If $(f(n))$ is a sequence that either is increasing or satisfies $f(n+1)-f(n) \to 0$ as $n \to \infty$, then

$$\liminf_{n\to\infty}(f(n+1) - f(n)) \geq 0.$$

The following theorem therefore implies both of the results mentioned above.

Theorem 1.2.2 (Erdős, Kátai, Máté) *Let $(f(n))_{n\geq 1}$ be a sequence of real numbers such that*

$$\liminf_{n\to\infty}(f(n+1) - f(n)) \geq 0.$$

The following are equivalent:

i. f is logarithmic;
ii. there exists a constant c such that $f(n) = c \log n$ for all $n \geq 1$.

This result was stated without proof by Erdős in 1957 [93], then proved independently by Kátai [183] and by Máté [245], both in 1967. Again, the logarithmic condition can be relaxed by only requiring that (1.5) holds when m and n are relatively prime, but again, we have no need for this extra precision.

The proof below follows Aczél and Daróczy's adaptation of Kátai's argument (Theorem 0.4.3 of [3]). The strategy is to put $c = \liminf_{n\to\infty} f(n)/\log n$ and show that $f(N)/\log N = c$ for all N.

Proof It is trivial that (ii) implies (i). Now assume (i). I claim that for all $N \geq 2$,

$$\liminf_{n\to\infty} \frac{f(n)}{\log n} = \frac{f(N)}{\log N}. \tag{1.8}$$

Let $N \geq 2$. First we show that the left-hand side of (1.8) is less than or equal to the right. For each $r \geq 1$, the logarithmic property of f implies that

$$\frac{f(N^r)}{\log(N^r)} = \frac{rf(N)}{r\log N} = \frac{f(N)}{\log N}.$$

Since $N^r \to \infty$ as $r \to \infty$, it follows from the definition of limit inferior that

$$\liminf_{n\to\infty} \frac{f(n)}{\log n} \leq \frac{f(N)}{\log N}.$$

https://doi.org/10.1017/9781108963558.003 Published online by Cambridge University Press

Now we prove the opposite inequality,

$$\liminf_{n\to\infty} \frac{f(n)}{\log n} \geq \frac{f(N)}{\log N}. \tag{1.9}$$

Let $\varepsilon > 0$. By hypothesis, we can choose $k \geq 1$ such that for all $n \geq N^k$,

$$f(n+1) - f(n) \geq -\varepsilon. \tag{1.10}$$

Any integer $n \geq N^k$ has a base N expansion

$$n = c_\ell N^\ell + \cdots + c_1 N + c_0$$

with $c_0, \ldots, c_\ell \in \{0, \ldots, N-1\}$, $c_\ell \neq 0$, and $\ell \geq k$. Then

$$\begin{aligned} f(n) &\geq f(c_\ell N^\ell + \cdots + c_1 N) - c_0 \varepsilon && (1.11)\\ &\geq f(c_\ell N^\ell + \cdots + c_1 N) - N\varepsilon && (1.12)\\ &= f(c_\ell N^{\ell-1} + \cdots + c_1) + f(N) - N\varepsilon, && (1.13) \end{aligned}$$

where inequality (1.11) follows from (1.10) using induction and the fact that $\ell \geq k$, inequality (1.12) holds because $c_0 \leq N$, and equation (1.13) follows from the logarithmic property of f. As long as $\ell - 1 \geq k$, we can apply the same argument again with $c_\ell N^{\ell-1} + \cdots + c_1$ in place of $n = c_\ell N^\ell + \cdots + c_0$, giving

$$f(c_\ell N^{\ell-1} + \cdots + c_1) \geq f(c_\ell N^{\ell-2} + \cdots + c_2) + f(N) - N\varepsilon$$

and so

$$f(n) \geq f(c_\ell N^{\ell-2} + \cdots + c_2) + 2(f(N) - N\varepsilon).$$

Repeated application of this argument gives

$$f(n) \geq f(c_\ell N^{k-1} + \cdots + c_{\ell-k+1}) + (\ell - k + 1)(f(N) - N\varepsilon).$$

Hence, writing $A = \min\{f(1), f(2), \ldots, f(N^k)\}$,

$$f(n) \geq A + (\ell - k + 1)(f(N) - N\varepsilon). \tag{1.14}$$

In (1.14), the only term on the right-hand side that depends on n is ℓ, which is equal to $\lfloor \log_N n \rfloor$, and $\lfloor \log_N n \rfloor / \log_N n \to 1$ as $n \to \infty$. Hence

$$\begin{aligned} \liminf_{n\to\infty} \frac{f(n)}{\log_N n} &\geq \liminf_{n\to\infty} \left\{ \frac{A}{\log_N n} + \left(\frac{\lfloor \log_N n \rfloor}{\log_N n} + \frac{-k+1}{\log_N n} \right) (f(N) - N\varepsilon) \right\} \\ &= f(N) - N\varepsilon. \end{aligned}$$

This holds for all $\varepsilon > 0$, so

$$\liminf_{n\to\infty} \frac{f(n)}{\log_N n} \geq f(N).$$

https://doi.org/10.1017/9781108963558.003 Published online by Cambridge University Press

Since $\log_N n = (\log n)/(\log N)$, this proves the claimed inequality (1.9) and, therefore, equation (1.8).

Putting $c = \liminf_{n\to\infty} f(n)/\log n \in \mathbb{R}$, we have $f(N) = c \log N$ for all $N \geq 2$. Finally, the logarithmic property of f implies that $f(1) = 0$, so $f(1) = c \log 1$ too. □

Corollary 1.2.3 *Let $(f(n))_{n\geq 1}$ be a sequence such that*

$$\lim_{n\to\infty} (f(n+1) - f(n)) = 0. \tag{1.15}$$

The following are equivalent:

i. f is logarithmic;
ii. there exists a constant c such that $f(n) = c \log n$ for all $n \geq 1$. □

To apply this corollary, we will need to be able to verify the limit condition (1.15). The following improvement lemma will be useful.

Lemma 1.2.4 *Let $(a_n)_{n\geq 1}$ be a real sequence such that $a_{n+1} - \frac{n}{n+1} a_n \to 0$ as $n \to \infty$. Then $a_{n+1} - a_n \to 0$ as $n \to \infty$.*

Our proof of Lemma 1.2.4 follows that of Feinstein [99] (pp. 6–7), and uses the following standard result.

Proposition 1.2.5 (Cesàro) *Let $(x_n)_{n\geq 1}$ be a real sequence, and for $n \geq 1$, write*

$$\overline{x}_n = \tfrac{1}{n}(x_1 + \cdots + x_n).$$

Suppose that $\lim_{n\to\infty} x_n$ exists. Then $\lim_{n\to\infty} \overline{x}_n$ exists and is equal to $\lim_{n\to\infty} x_n$.

Proof This can be found in introductory analysis texts such as Apostol [15] (Theorem 12-48). □

Proof of Lemma 1.2.4 It is enough to prove that $a_n/(n+1) \to 0$ as $n \to \infty$. Write $b_1 = a_1$ and $b_n = a_n - \frac{n-1}{n} a_{n-1}$ for $n \geq 2$; then by hypothesis, $b_n \to 0$ as $n \to \infty$. We have $na_n = nb_n + (n-1)a_{n-1}$ for all $n \geq 2$, so

$$na_n = nb_n + (n-1)b_{n-1} + \cdots + 1b_1$$

for all $n \geq 1$. Dividing through by $n(n+1)$ gives

$$\begin{aligned} \frac{a_n}{n+1} &= \frac{1}{2} \cdot \frac{1}{\frac{1}{2}n(n+1)} (b_1 + b_2 + b_2 + b_3 + b_3 + b_3 + \cdots + \underbrace{b_n + \cdots + b_n}_{n}) \\ &= \frac{1}{2} \cdot M_1(b_1, b_2, b_2, b_3, b_3, b_3, \ldots, \underbrace{b_n, \ldots, b_n}_{n}), \end{aligned} \tag{1.16}$$

https://doi.org/10.1017/9781108963558.003 Published online by Cambridge University Press

where M_1 denotes the arithmetic mean. Since $b_n \to 0$ as $n \to \infty$, the sequence

$$b_1, b_2, b_2, b_3, b_3, b_3, \ldots, \underbrace{b_n, \ldots, b_n}_{n}, \ldots$$

also converges to 0. Proposition 1.2.5 applied to this sequence then implies that

$$M_1(b_1, b_2, b_2, b_3, b_3, b_3, \ldots, \underbrace{b_n, \ldots, b_n}_{n}) \to 0 \text{ as } n \to \infty.$$

But by equation (1.16), this means that $a_n/(n+1) \to 0$ as $n \to \infty$, completing the proof. □

Remark 1.2.6 Lemma 1.2.4 can also be deduced from the Stolz–Cesàro theorem (Section 3.1.7 of Mureşan [258], for instance). This is a discrete analogue of l'Hôpital's rule, and states that given a real sequence (x_n) and a strictly increasing sequence (y_n) diverging to ∞, if

$$\frac{x_{n+1} - x_n}{y_{n+1} - y_n} \to \ell$$

as $n \to \infty$ then $x_n/y_n \to \ell$ as $n \to \infty$. Lemma 1.2.4 follows by taking $x_n = na_n$ and $y_n = \frac{1}{2}n(n+1)$. (I thank Xīlíng Zhāng for this observation.)

1.3 The q-Logarithm

The q-logarithms ($q \in \mathbb{R}$) form a continuous one-parameter family of functions that include the ordinary natural logarithm as the case $q = 1$. They can be regarded as deformations of the natural logarithm. We will show that as a family, they are characterized by a single functional equation.

For $q \in \mathbb{R}$, the **q-logarithm** is the function

$$\ln_q : (0, \infty) \to \mathbb{R}$$

defined by

$$\ln_q(x) = \int_1^x t^{-q}\, dt$$

($x \in (0, \infty)$). Thus,

$$\ln_1(x) = \log(x)$$

and for $q \neq 1$,

$$\ln_q(x) = \frac{x^{1-q} - 1}{1 - q}. \tag{1.17}$$

Then $\ln_q(x) \to \ln_1(x)$ as $q \to 1$, by l'Hôpital's rule.

https://doi.org/10.1017/9781108963558.003 Published online by Cambridge University Press

Let $q \in \mathbb{R}$. The q-logarithm shares with the natural logarithm the property that

$$\ln_q(1) = 0.$$

However, in general

$$\ln_q(xy) \neq \ln_q(x) + \ln_q(y).$$

One can see this without calculation: for by Corollary 1.1.11(ii), the only measurable functions that transform multiplication into addition are the multiples of the natural logarithm. There is nevertheless a simple formula for $\ln_q(xy)$ in terms of $\ln_q(x)$ and $\ln_q(y)$:

$$\ln_q(xy) = \ln_q(x) + \ln_q(y) + (1-q)\ln_q(x)\ln_q(y).$$

Later, we will use a second formula for $\ln_q(xy)$:

$$\ln_q(xy) = \ln_q(x) + x^{1-q}\ln_q(y). \tag{1.18}$$

Similarly, in general

$$\ln_q(1/x) \neq -\ln_q(x),$$

but instead we have the following three formulas for $\ln_q(1/x)$:

$$\begin{aligned} \ln_q(1/x) &= \frac{-\ln_q(x)}{1+(1-q)\ln_q(x)} \\ &= -x^{q-1}\ln_q(x) \\ &= -\ln_{2-q}(x). \end{aligned} \tag{1.19}$$

By (1.19), replacing $\ln_q$ by the function $x \mapsto -\ln_q(1/x)$ defines an involution $\ln_q \leftrightarrow \ln_{2-q}$ of the family of q-logarithms, with a fixed point at the classical logarithm $\ln_1$. Finally, there is a quotient formula

$$\ln_q(x/y) = y^{q-1}(\ln_q(x) - \ln_q(y)), \tag{1.20}$$

obtained from equation (1.18) by substituting y for x and x/y for y.

Remark 1.3.1 The history of the q-logarithms as an *explicit* object of study goes back at least as far as a 1964 paper of Box and Cox in statistics (Section 3 of [49]). The notation $\ln_q$ appeared in a 1994 article of Tsallis [332], and the name 'q-logarithm' has been used since at least the late 1990s (as in Borges [45]).

But there is more than one system of q-analogues of the classical notions of calculus. For instance, there is the system developed by the early twentieth-century clergyman F. H. Jackson [155] (a modern account of which can be found in Kac and Cheung [175]). In particular, this has given rise to a different

https://doi.org/10.1017/9781108963558.003 Published online by Cambridge University Press

notion of q-logarithm, as developed in Chung, Chung, Nam and Kang [70]. Ernst [94] gives a full historical treatment of the various branches of q-calculus. In any case, none of the developments just mentioned use the q-logarithms considered here.

We now prove that the q-logarithms are characterized by a simple functional equation. The proof is essentially the argument behind Theorem 84 in the classic text of Hardy, Littlewood and Pólya [137].

Theorem 1.3.2 *Let $f : (0, \infty) \to \mathbb{R}$ be a measurable function. The following are equivalent:*

i. *there exists a function $g : (0, \infty) \to \mathbb{R}$ such that for all $x, y \in (0, \infty)$,*

$$f(xy) = f(x) + g(x)f(y); \tag{1.21}$$

ii. *$f = c \ln_q$ for some $c, q \in \mathbb{R}$, or f is constant.*

Proof First suppose that (ii) holds. If $f = c \ln_q$ for some $c, q \in \mathbb{R}$ then equation (1.21) holds with $g(x) = x^{1-q}$, by equation (1.18). Otherwise, f is constant, so (1.21) holds with $g \equiv 0$.

Now assume (i). Since $f(xy) = f(yx)$, equation (1.21) implies that

$$f(x) + g(x)f(y) = f(y) + g(y)f(x),$$

or equivalently

$$f(x)(1 - g(y)) = f(y)(1 - g(x)), \tag{1.22}$$

for all $x, y \in (0, \infty)$. If $f \equiv 0$ then f is constant and (ii) holds. Assuming otherwise, we can choose $y_0 \in (0, \infty)$ such that $f(y_0) \neq 0$. Taking $y = y_0$ in (1.22) and putting $a = (1 - g(y_0))/f(y_0)$ gives

$$g(x) = 1 - af(x) \tag{1.23}$$

($x \in \mathbb{R}$). Since f is measurable, so is g. There are now two cases: $a = 0$ and $a \neq 0$.

If $a = 0$ then $g \equiv 1$, so the original functional equation (1.21) states that $f(xy) = f(x) + f(y)$. Since f is measurable, Corollary 1.1.11(ii) implies that $f = c \log = c \ln_1$ for some $c \in \mathbb{R}$.

If $a \neq 0$ then equation (1.23) can be rewritten as

$$f(x) = \tfrac{1}{a}(1 - g(x)) \tag{1.24}$$

($x \in (0, \infty)$). Substituting this into the original functional equation (1.21) gives

$$g(xy) = g(x)g(y) \tag{1.25}$$

https://doi.org/10.1017/9781108963558.003 Published online by Cambridge University Press

($x, y \in (0, \infty)$). In particular, $g(x) = g(\sqrt{x})^2 \geq 0$ for all $x \in (0, \infty)$. There are now two subcases: g either sometimes vanishes or never vanishes.

If $g(x_0) = 0$ for some $x_0 \in (0, \infty)$ then

$$g(x) = g(x_0)g(x/x_0) = 0$$

for all $x \in (0, \infty)$, so $g \equiv 0$. Hence by equation (1.24), f is constant.

Otherwise, $g(x) > 0$ for all $x \in (0, \infty)$. Since g is measurable and satisfies the multiplicativity condition (1.25), Corollary 1.1.11(iii) implies that there is some constant $t \in \mathbb{R}$ such that $g(x) = x^t$ for all $x \in (0, \infty)$. We have assumed that $f \not\equiv 0$, so $g \not\equiv 1$ (by equation (1.24)), so $t \neq 0$. Hence

$$f(x) = \tfrac{1}{a}(1 - x^t) = \tfrac{-t}{a} \ln_{1-t}(x)$$

for all $x \in (0, \infty)$, completing the proof. $\square$

https://doi.org/10.1017/9781108963558.003 Published online by Cambridge University Press