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The Effective Cone of the Kontsevich Moduli Space

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Abstract. In this paper we prove that the cone of effective divisors on the Kontsevich moduli spaces of stable maps, $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$, stabilize when $r \geq d$. We give a complete characterization of the effective divisors on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$. They are non-negative linear combinations of boundary divisors and the divisor of maps with degenerate image.

1 Introduction

The ample and effective cones of divisors play a crucial role in the birational geometry of a variety. The study of these cones for the moduli spaces of stable curves has been especially fruitful, leading to the proof that the moduli space of stable curves $\overline{M}_g$ is of general type when $g > 23$ (see [HM, H, EH]). Recently, inspired by the work of G. Farkas, D. Khosla and M. Popa, there has been renewed interest in constructing divisors of small slope on $\overline{M}_g$ in order to understand the effective cone of $\overline{M}_g$ and to determine the Kodaira dimension of $\overline{M}_g$ in the remaining cases (see [FaP, Far1, Far3, Kh]). For instance, Farkas, using his construction of new divisors, announced a proof that $\overline{M}_{22}$ is of general type [Far2].

The aim of this paper is to describe the classes of effective divisors on a related moduli space, the Kontsevich moduli space of stable maps $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$. For $d > 1$, the scheme parameterizing smooth, degree d, rational curves in $\mathbb{P}^r$ is not proper. The Kontsevich moduli space gives a useful compactification. For integers $n, d \geq 0$, the Kontsevich moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is a smooth, proper, Deligne–Mumford stack parameterizing the data $(C, (p_1, \dots, p_n), f)$ of

(i) C, a proper, connected, at-worst-nodal curve of arithmetic genus 0,
(ii) $p_1, \dots, p_n$, an ordered sequence of distinct, smooth points of C,
(iii) $f \colon C \to \mathbb{P}^r$, a morphism with $\deg(f^*\mathcal{O}_{\mathbb{P}^r}(1)) = d$ satisfying the following stability condition: every irreducible component of C mapped to a point under f contains at least 3 special points, *i.e.,* marked points p_i and nodes of C.

In this paper we will determine the classes of all effective divisors on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ when $r \geq d$.

R. Pandharipande [Pa] proved that when $r \geq 2$, the divisor class $\mathcal{H}$ and the classes of the boundary divisors $\Delta_{k,d-k}$ for $1 \leq k \leq \lfloor d/2 \rfloor$ generate the group of $\mathbb{Q}$-Cartier divisors of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$. We recall that

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(i) $\mathcal{H}$ is the class of the divisor of maps whose images intersect a fixed codimension two linear space in $\mathbb{P}^r$ (provided $r > 1$ and $d > 0$).
(ii) $\Delta_{k,d-k}$, $1 \leq k \leq \lfloor d/2 \rfloor$ is the class of the boundary divisor consisting of maps with reducible domains, where the map has degree k on one component and degree $d - k$ on the other component.

The main problem we would like to address is the following.

Problem 1.1 Describe the cone of effective divisor classes on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ in terms of these generators of the Picard group.

Denote by P_d the $\mathbb{Q}$-vector space of dimension $\lfloor d/2 \rfloor + 1$ with basis labeled $\mathcal{H}$ and $\Delta_{k,d-k}$ for $k = 1, \dots, \lfloor d/2 \rfloor$. For each $r \geq 2$, there is a $\mathbb{Q}$-linear map

$$u_{d,r} \colon P_d \to \operatorname{Pic}(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$$

that is an isomorphism of $\mathbb{Q}$-vector spaces.

Definition 1.2 For every integer $r \geq 2$, denote by $\mathrm{Eff}_{d,r} \subset P_d$ the inverse image under $u_{d,r}$ of the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$.

A more precise version of Problem 1.1 is to describe $\mathrm{Eff}_{d,r}$. A first result is that for a fixed degree d, there is an inclusion between these cones as r increases. Furthermore, the cones stabilize for $r \geq d$.

Proposition 1.3 *For every integer $r \geq 2$, $\mathrm{Eff}_{d,r}$ is contained in $\mathrm{Eff}_{d,r+1}$. For every integer $r \geq d$, $\mathrm{Eff}_{d,r}$ equals $\mathrm{Eff}_{d,d}$.*

In view of Proposition 1.3 it is especially interesting to understand $\mathrm{Eff}_{d,d}$. Most of our paper will concentrate on this case.

The crudest invariant one can associate with the effective cone is the slope of distinguished rays. For example, Harris and Morrison [HMo1] define the slope of $\overline{M}_g$ as the slope of the ray that bounds the effective cone in the subspace spanned by the Hodge class λ and the total boundary class δ. Determining the slope of $\overline{M}_g$ is a major open problem. In analogy with the case of $\overline{M}_g$, we define the slope $s(r, d)$ of the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ as follows.

$$s(r, d) := \sup_{\alpha} \Big\{ \alpha : \mathcal{H} - \alpha \sum_{k=1}^{\lfloor d/2 \rfloor} k(d-k)\Delta_{k,d-k} \text{ is on the effective cone} \Big\}.$$

It is possible to determine the slope for the Kontsevich moduli spaces in the stable range.

Theorem 1.4 *If $r \geq d$, then the slope $s(r, d)$ of the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ is equal to*

$$s(r, d) = \frac{1}{d+1}.$$

When $r = d$, the effective divisor that achieves the extremal slope has a simple description. Let $D_{\deg}$ denote the class of the locus parameterizing stable maps $f\colon C \to \mathbb{P}^d$ of degree d whose set theoretic image does not span $\mathbb{P}^d$. Then $D_{\deg}$ is a divisor class in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ which gives the desired slope.

The class $D_{\deg}$ plays a crucial role in describing the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$. The following theorem, which describes the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ completely, is the main theorem of our paper.

Theorem 1.5 *The class of a divisor lies in the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ if and only if it is a non-negative linear combination of the class of degenerate maps $D_{\deg}$ and the classes of the boundary divisors $\Delta_{k,d-k}$ for $1 \leq k \leq \lfloor d/2 \rfloor$.*

Theorem 1.4 follows immediately from Theorem 1.5. However, since it is easy to give an independent proof and since the curves that span the null-space of the divisor $D_{\deg}$ are interesting in their own right, we will give a simple proof of it in §2. Combining Theorem 1.5 with Proposition 1.3 and Lemma 2.1, we obtain the following corollary.

Corollary 1.6 *When $r \geq d$, the class of a divisor lies in the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ if and only if it is a non-negative linear combination of the class*

$$\mathcal{H} - \frac{1}{d+1} \sum_{k=1}^{\lfloor d/2 \rfloor} k(d-k)\Delta_{k,d-k}$$

and the classes of the boundary divisors $\Delta_{k,d-k}$ for $1 \leq k \leq \lfloor d/2 \rfloor$.

The space of curves of a given degree and genus has many distinguished subvarieties defined by imposing geometric conditions on the curves. Examples of such subvarieties are given by curves that have an unexpected secant linear space or curves with an unexpected osculating linear space or curves with a point of unexpected ramification. An informal way of restating Theorem 1.5 is to say that "geometric conditions" do not give new divisors on the space of rational curves of degree d in $\mathbb{P}^d$. Rational normal curves are too predictable.

We now briefly outline the proof of Theorem 1.5. Since $D_{\deg}$ and the classes of the boundary divisors are effective, their non-negative linear combinations also lie in the effective cone. The main content of the theorem is to show that there are no other effective divisor classes.

Definition 1.7 A reduced, irreducible curve C on a scheme X is a *moving curve* if the deformations of C cover a Zariski open subset of X. More precisely, a curve C is a moving curve if there exists a flat family of curves $\pi\colon \mathcal{C} \to T$ on X such that $\pi^{-1}(t_0) = C$ for $t_0 \in T$ and for a Zariski open subset $U \subset X$ every point $x \in U$ is contained in $\pi^{-1}(t)$ for some $t \in T$. We call the class of a moving curve a *moving curve class*.

An obvious observation is that the intersection pairing between the class of an effective divisor and a moving curve class is always non-negative. Intersecting divisors with a moving curve class gives an inequality for the coefficients of an effective divisor class. The strategy for the proof of Theorem 1.5 is to produce enough moving curves to force the effective divisor classes to be a non-negative linear combination of $D_{\deg}$ and the boundary classes.

Moving curves in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ are easy to recognize by the following lemma.

Lemma 1.8 *If $C \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ is a reduced, irreducible curve that intersects the complement in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ of the union of the boundary divisors and $D_{\deg}$, then C is a moving curve.*

Proof The automorphism group of $\mathbb{P}^d$ acts transitively on rational normal curves. An irreducible curve of degree d that spans $\mathbb{P}^d$ is a rational normal curve. Hence, a curve $C \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ that intersects the complement in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ of the boundary divisors and the divisor of maps whose image is degenerate, contains a point that represents a map that is an embedding of $\mathbb{P}^1$ as a rational normal curve. The translations of C by $\mathbb{P}GL(d+1)$ cover a Zariski open set of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$. ■

In §3, using certain linear systems on blow-ups of $\mathbb{P}^1 \times \mathbb{P}^1$, we will construct one-parameter families of rational curves whose general member is a rational normal curve. By Lemma 1.8, these will be moving curves in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$. These moving curves will give us enough inequalities on the effective cone to deduce Theorem 1.5.

Remark 1.9. After we posted our article, S. Keel provided a different proof of Theorem 1.5. Keel's argument, although beautiful, does not construct moving curves dual to effective divisors. Most applications of Theorem 1.5 we have in mind rely on the existence of the moving curves we construct. For instance, using the moving curves one can characterize the effective cones of the space of stable maps to other homogeneous varieties (see [CS] for a discussion of the case of Grassmannians).

2 Preliminaries

In this section we prove Proposition 1.3 and collect basic facts about the divisor class $D_{\deg}$.

2.1 The Stability of the Effective Cone

In this subsection we prove that $\mathrm{Eff}_{d,r}$ is contained in $\mathrm{Eff}_{d,r+1}$ and that $\mathrm{Eff}_{d,r} = \mathrm{Eff}_{d,d}$ for $r \geq d$. Recall that $\mathrm{Eff}_{d,r}$ is the image of the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$ when $\mathrm{Pic}(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$ is identified with the vector space that has a basis labeled by $\mathcal{H}$ and $\Delta_{k,d-k}$ for $1 \leq k \leq \lfloor d/2 \rfloor$.

Proof of Proposition 1.3 Let $p \in \mathbb{P}^{r+1}$ be a point, denote $U = \mathbb{P}^{r+1} - \{p\}$, and let $\pi \colon U \to \mathbb{P}^r$ be a linear projection from p. This induces a smooth 1-morphism

$$\overline{\mathcal{M}}_{0,0}(\pi, d) \colon \overline{\mathcal{M}}_{0,0}(U, d) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d).$$

Let $i \colon U \to \mathbb{P}^{r+1}$ be the open immersion. This induces a 1-morphism

$$\overline{\mathcal{M}}_{0,0}(i, d) \colon \overline{\mathcal{M}}_{0,0}(U, d) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^{r+1}, d)$$

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relatively representable by open immersions. The complement of the image of $\overline{\mathcal{M}}_{0,0}(i,d)$ has codimension r, which is greater than 2. Therefore, the pull-back morphism

$$\overline{\mathcal{M}}_{0,0}(i,d)^*\colon\ \mathrm{Pic}(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^{r+1},d)) \to \mathrm{Pic}(\overline{\mathcal{M}}_{0,0}(U,d))$$

is an isomorphism. So there is a unique homomorphism

$$h\colon\ \mathrm{Pic}(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^{r},d)) \to \mathrm{Pic}(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^{r+1},d))$$

such that $\overline{\mathcal{M}}_{0,0}(\pi,d)^* = \overline{\mathcal{M}}_{0,0}(i,d)^* \circ h$.

Recalling from the introduction that $u(r,d)$ is the map that identifies the Picard group of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r,d)$ with the vector space spanned by $\mathcal{H}$ and the boundary divisors $\Delta_{k,d-k}$, we see that $h \circ u_{d,r}$ equals $u_{d,r+1}$. So to prove $\mathrm{Eff}_{d,r}$ is contained in $\mathrm{Eff}_{d,r+1}$, it suffices to prove that $\overline{\mathcal{M}}_{0,0}(\pi,d)$ pulls back effective divisors to effective divisor classes, which follows since $\overline{\mathcal{M}}_{0,0}(\pi,d)$ is smooth.

Next assume $r \geq d$. Let D be any effective divisor in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r,d)$. A general point in the complement of D parameterizes a stable map $f\colon C \to \mathbb{P}^r$ such that $f(C)$ spans a d-plane. Denote by $j\colon \mathbb{P}^d \to \mathbb{P}^r$ a linear embedding whose image is this d-plane. There is an induced 1-morphism

$$\overline{\mathcal{M}}_{0,0}(j,d)\colon \overline{\mathcal{M}}_{0,0}(\mathbb{P}^d,d) \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^r,d).$$

The map $\overline{\mathcal{M}}_{0,0}(j,d)^* \circ u_{d,r}$ equals $u_{d,d}$. By construction, $\overline{\mathcal{M}}_{0,0}(j,d)^*([D])$ is the class of the effective divisor $\overline{\mathcal{M}}_{0,0}(j,d)^{-1}(D)$, *i.e.,* $[D]$ is in $\mathrm{Eff}_{d,d}$. Thus $\mathrm{Eff}_{d,d}$ contains $\mathrm{Eff}_{d,r}$, which in turn contains $\mathrm{Eff}_{d,d}$ by the last paragraph. Therefore $\mathrm{Eff}_{d,r}$ equals $\mathrm{Eff}_{d,d}$. ■

2.2 The Divisor Class $D_{\mathbf{deg}}$

In this subsection we determine the class of the divisor of degenerate maps in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d,d)$. We then give a basis of moving curves that span the null-space of D_{deg} in the cone of curves. This completes the proof of Theorem 1.4.

Lemma 2.1 *The class* D_{deg} *equals*

$$D_{\mathrm{deg}} = \frac{1}{2d}\Big[(d+1)\mathcal{H} - \sum_{k=1}^{\lfloor d/2 \rfloor} k(d-k)\Delta_{k,d-k}\Big]. \tag{2.1}$$

Proof We will prove the equality (2.1) by intersecting D_{deg} with test curves. This well-developed method was first applied to Kontsevich moduli spaces by Pandharipande [Pa]. Fix a general rational normal surface scroll S of degree i and a general rational normal curve R of degree $d-i-1$ intersecting S in one point p. A general rational normal surface scroll is the image of $\mathbb{P}^1 \times \mathbb{P}^1$ (resp., the blow-up of $\mathbb{P}^2$ at one point) if the degree i is even (resp., odd). Let f denote the class of a fiber of the scroll. Let e denote the class of the other fiber (resp., of the exceptional curve) when S is $\mathbb{P}^1 \times \mathbb{P}^1$ (resp., the blow-up of $\mathbb{P}^2$ at one point). Choose a general pencil of degree

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$i+1$ rational curves in the class $\mathcal{O}_S(e + \lfloor \frac{i+3}{2} \rfloor f)$ on S having p as a base point. Consider the one-parameter family C_i of degree d rational curves consisting of the union of the fixed curve R with the elements of the pencil.

The intersection number $C_i \cdot \mathcal{H}$ is the degree of the surface component swept out by the rational curves parameterized by C_i. By construction, these curves sweep out the degree i scroll S. Hence, $C_i \cdot \mathcal{H} = i$.

On a rational surface scroll of degree $i \neq 1$, a pencil of degree $i+1$ rational curves is determined by $i+2$ general points (see [C, Lemma 2.5]). When $i = 1$, a pencil of conics is determined by 4 base points. One of these points (p) lies on R. The other $i+1$ base points together with the fixed curve R span $\mathbb{P}^d$. Hence, all the curves parameterized by C_i are non-degenerate. We conclude that $C_i \cdot D_{\deg} = 0$.

A general pencil of degree $i+1$ curves on S becomes reducible $i+2$ times, breaking into the union of a curve of degree i and a fiber passing through one of the $i+2$ base points. Suppose that $2 \leq i \leq \lfloor d/2 \rfloor - 1$. When the fiber passes through p, then C_i intersects $\Delta_{i,d-i}$. When the fiber passes through one of the other base points, then C_i intersects $\Delta_{1,d-1}$. Since the total space of the family is smooth at the corresponding nodes, it is standard that both intersections are transverse. Therefore,

$$C_i \cdot \Delta_{i,d-i} = 1, \quad C_i \cdot \Delta_{1,d-1} = i+1.$$

The curve C_i is contained in the boundary divisor $\Delta_{i+1,d-i-1}$. The intersection is given by the sum of the self-intersections of the sections given by the attaching points. On R the section is trivial. On the blow-up of the scroll at p, the section obtained by p has self-intersection -1. Hence, $C_i \cdot \Delta_{i+1,d-i-1} = -1$. Finally, the intersection number of C_i with all the other boundary divisors is zero since C_i misses them. When $i = 1$, we have to modify the intersection number of C_1 with $\Delta_{1,d-1}$ to read $C_1 \cdot \Delta_{1,d-1} = 3$, since a general pencil of plane conics contains three reducible members.

Next consider the one-parameter family B_1 of rational curves of degree $d > 2$ that contain $d+2$ general points and intersect a general line l. Since the points always span $\mathbb{P}^d$, the curves never become degenerate. Hence, $B_1 \cdot D_{\deg} = 0$. The intersection number of B_1 with all the boundary divisors but $\Delta_{1,d-1}$ is zero. Suppose there were reducible curves of degree $i > 1$ and $d - i > 1$. A rational curve of degree i spans at most $\mathbb{P}^i$. Therefore, at most $i+1$ (respectively, $d-i+1$) of the points could be contained in the curve of degree i (respectively, $d-i$). Hence, exactly $i+1$ (respectively, $d-i+1$) of the points are contained in them and the components of the curves lie in the linear spaces spanned by these points. However, if the line l is general, it cannot meet either of these linear spaces. This is a contradiction. Similar reasoning yields that the reducible curves that contain $d+2$ points and intersect l consist of a line $\tilde{l}$ containing two of the points and a degree $d-1$ curve that contains the other d points and intersects l and $\tilde{l}$. Using the fact that $d+2$ linearly general points in $\mathbb{P}^{d-1}$ determine a unique rational normal curve, we conclude that

$$B_1 \cdot \Delta_{1,d-1} = \frac{(d+2)(d+1)}{2}.$$

Finally, to determine $B_1 \cdot \mathcal{H}$, we need to count the number of rational curves of degree d that contain $d+2$ general points, intersect a general line, and intersect a general $\mathbb{P}^{d-2}$. This number may be easily determined to be

$$B_1 \cdot \mathcal{H} = \frac{d^2 + d - 2}{2}$$

by the algorithm proved in [V, Theorem 6.1].

Briefly, if we specialize the linear space $\mathbb{P}^{d-2}$ to the span Λ of d of the points, then the degree d curves become reducible by Bezout's Theorem. Vakil [V, Theorem 6.1] describes the possible limits and their multiplicities. There is either a component of degree $d-1$ or of degree 1 contained in Λ. (We already proved that the curves cannot break any other way.) If there is a line in Λ, it can join any of the $d(d-1)/2$ pairs of points in Λ. The complementary curve of degree $d-1$ is uniquely determined as before. If there is a curve of degree $d-1$ in Λ, then the complementary line $\tilde{l}$ is the line joining the two points that are not contained in Λ. The curve of degree $d-1$ is uniquely determined because it must contain d points in Λ and intersect l and $\tilde{l}$. It counts with multiplicity $d-1$ for the choice of intersection with the linear space $\mathbb{P}^{d-2}$ that we specialized to Λ. We obtain the number claimed.

This determines the class of D_{deg} up to a constant multiple. In order to determine the multiple, consider the curve C that consists of a fixed degree $d-1$ curve and a pencil of lines in a general plane intersecting the curve in one point. The curve C has intersection number zero with all the boundary divisors but $\Delta_{1,d-1}$. Arguing as above, it is easy to see that C has the following intersection numbers: $C \cdot \mathcal{H} = 1$, $C \cdot D_{\text{deg}} = 1$, and $C \cdot \Delta_{1,d-1} = -1$. The lemma follows from these intersection numbers. ■

Consider the one-parameter family B_k of rational curves of degree d in $\mathbb{P}^d$ that contain $d+2$ general fixed points and intersect a general linear space $\mathbb{P}^k$ and a general linear space $\mathbb{P}^{d-k}$ for $1 \leq k \leq \lfloor d/2 \rfloor$. When $k = 1$, we omit the linear space $\mathbb{P}^{d-1}$. A general member of B_k is a rational normal curve. This follows, for example, from [FP, Lemma 14]. By Lemma 1.8, it follows that B_k is a moving curve for every k. The only reducible elements of B_1 are unions of curves of degree 1 and $d-1$. For $k > 1$, the only reducible members of B_k have degrees $(1, d-1)$ or $(k, d-k)$. Since the $d+2$ points always span $\mathbb{P}^d$, $B_k \cdot D_{\text{deg}} = 0$ for every k. Since the curves B_k are independent, they must span the null-space of D_{deg} in the cone of curves. Observe that these curves give a proof of Theorem 1.4.

Proof of Theorem 1.4 By Lemma 2.1, the divisor class D_{deg} lies in the plane spanned by the divisor classes $\mathcal{H}$ and $\sum_{k=1}^{\lfloor d/2 \rfloor} k(d-k)\Delta_{k,d-k}$. Hence, it determines a ray in the intersection of this plane with the effective cone. By Lemma 2.1, the slope of this ray is $\frac{1}{d+1}$. We conclude that the slope of $\text{Eff}_{d,d}$ is at least $\frac{1}{d+1}$. On the other hand, there are moving curves that have intersection number zero with D_{deg}. Hence, the ray determined by D_{deg} is extremal in the intersection of the effective cone with the plane spanned by $\mathcal{H}$ and $\sum_{k=1}^{\lfloor d/2 \rfloor} k(d-k)\Delta_{k,d-k}$. Therefore, the slope of $\text{Eff}_{d,d}$ is at most $\frac{1}{d+1}$. ■

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2.3 A Digression on the Slope of $\overline{M}_g$

Recall that the slope $s(g)$ of $\overline{M}_g$ is defined by

$$s(g) := \inf_{\alpha}\{\alpha\lambda - \delta \text{ is on the effective cone}\},$$

where λ is the Hodge class and δ is the total boundary class. Harris and Morrison [HMo1] proved that the slope of the moduli space of curves $\overline{M}_g$ is non-negative. However, the lower bounds they obtain on the slope tend to zero as the genus tends to infinity. On the other hand, none of the currently known effective divisors on $\overline{M}_g$ produce a ray in the effective cone of slope less than 6. There are families of effective divisors, such as the Brill–Noether divisors, whose slopes tend to 6 from above as the genus tends to infinity. Determining the slope, even giving a positive lower bound for it, is an important problem with applications to the Schottky problem and the Kodaira dimension of $\overline{M}_g$. One method for proving lower bounds on the slope is to produce moving curves on $\overline{M}_g$. As discussed after Definition 1.7, each moving curve gives a lower bound on the slope. To the best of our knowledge, currently known moving curves in $\overline{M}_g$ give lower bounds on the slope that tend to zero with the genus.

The proof of Theorem 1.4 suggests a family of moving curves that might improve known bounds. Recall that the component of the Hilbert scheme parameterizing canonically embedded curves of genus g in $\mathbb{P}^{g-1}$ has dimension $g^2 + 3g - 4$. We can impose g^2+3g-5 conditions on canonical curves by requiring them to intersect "the appropriate number" of general linear spaces in $\mathbb{P}^{g-1}$. We thus obtain one-parameter families of canonical curves depending on the numerical data of the linear spaces. It is not hard to see that by varying the linear spaces, we can arrange the one-parameter families to contain a general canonical curve. Hence, each of these one-parameter families induce moving curves in $\overline{M}_g$. These moving curves are especially interesting when as many of the the linear spaces as possible are points. (A dimension count shows that when $g \geq 8$, this amounts to considering the one-parameter family of canonical curves that contain $g + 5$ general points and intersect a general $\mathbb{P}^{g-7}$.)

In low genus, these moving curves provide the (previously known) sharp lower bounds on the slope. We will describe this for $3 \leq g \leq 6$. As in the proof of Theorem 1.4, to check that the bound is sharp, it suffices to produce an effective divisor proportional to $a\lambda-\delta$ that has intersection number zero with a moving curve. In all our examples, the moving curves will have intersection number zero with all the boundary divisors except for δ_0, the divisor of irreducible nodal curves. In the expressions of the effective divisors, all the coefficients of the boundary divisors will be negative. Furthermore, the coefficient of δ_0 will have the smallest absolute value. In all cases, we can add positive multiples of the other boundary divisors to obtain an effective divisor proportional to $a\lambda - \delta$ without changing the intersection numbers with our moving curves. We will refer the reader to the literature for the details about the divisors we invoke and leave most of the (easy) verifications to the reader. The classical facts we use about canonical curves can be found in [Ha, Ch. IV.5] and [ACGH, Ch. III, V].

Let C_3 be the one-parameter family of genus 3 canonical curves that contain 13 general points in $\mathbb{P}^2$. The canonical image of a non-hyperelliptic genus 3 curve is a quartic plane curve. Hence, C_3 is a general pencil of quartic plane curves. Every member of such a pencil is a non-hyperelliptic stable curve. In $\overline{M}_3$ the closure of the locus of hyperelliptic curves forms a divisor whose class is computed in [HMo2, Ch. 3H]. Hence, C_3 has intersection number zero with the divisor of hyperelliptic curves. By the argument outlined in the previous paragraph, C_3 gives the sharp lower bound 9 on the slope of $\overline{M}_3$.

Let C_4 be the one-parameter family of genus 4 canonical curves that contain 9 general points and intersect 5 general lines in $\mathbb{P}^3$. A genus 4 canonical curve is a $(2,3)$ complete intersection in $\mathbb{P}^3$. Since nine general points determine a unique smooth quadric, all of these curves lie on a unique smooth quadric surface. Consequently, C_4 has intersection number zero with the Petri divisor of curves whose canonical image lies on a singular quadric (see [EH] for details about the Petri divisor). Hence, C_4 gives the sharp lower bound 17/2 on the slope of $\overline{M}_4$.

Let C_5 be the one-parameter family of genus 5 canonical curves in $\mathbb{P}^4$ that contain 11 general points and intersect a general line. Canonical images of non-hyperelliptic and non-trigonal curves are complete intersections of type $(2,2,2)$ in $\mathbb{P}^4$. Canonical images of trigonal genus 5 curves lie on a cubic scroll. Since the dimension of cubic scrolls in $\mathbb{P}^4$ is 18, there cannot be any cubic scrolls in $\mathbb{P}^4$ that contain 11 general points. Hence, C_5 has intersection number zero with the Brill–Noether divisor of trigonal genus 5 curves (see [HMo2, Ch. 6F] for the class of the Brill–Noether divisor). Thus, C_5 gives the sharp lower bound 8 on the slope of $\overline{M}_5$.

Finally, let C_6 be the one-parameter family of genus 6 canonical curves in $\mathbb{P}^5$ that contain 11 general points and intersect a general line and a general plane. The canonical image of a non-hyperelliptic genus 6 curve lies on a Del Pezzo surface of degree 5 in $\mathbb{P}^5$. An easy dimension count shows that there are only finitely many such Del Pezzo surfaces that contain 11 general points and intersect a general line and a general plane. Furthermore, these Del Pezzo surfaces will be smooth. Hence, C_6 has intersection number zero with the Petri divisor of curves that lie on a singular quintic Del Pezzo surface (see [EH] for the class), leading to the sharp slope bound $47/6$.

The analogy with rational curves and these small-genus examples suggest that the moving curves in $\overline{M}_g$ described above are well worth studying. Unfortunately we do not know the intersection numbers of these curves with the classes λ and δ in general. We pose calculating these numbers as an interesting open problem.

3 The Effective Cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$

In this section we prove Theorem 1.5. Every effective divisor class in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ is a positive linear combination of D_{deg} and the boundary divisors.

Since D_{deg} and the boundary divisors are effective, any positive linear combination also is a class in the effective cone. In order to prove Theorem 1.5 we have to show that we can write the class of every effective divisor as

$$\alpha D_{\mathrm{deg}} + \sum_{k=1}^{\lfloor d/2 \rfloor} \beta_{k,d-k} \quad \Delta_{k,d-k},$$

where α and $\beta_{k,d-k}$ are non-negative.

First we observe that if D is an effective divisor on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ and D has the class

$$a\mathcal{H} + \sum_{k=1}^{\lfloor d/2 \rfloor} b_{k,d-k}\Delta_{k,d-k},$$

then $a \geq 0$. Furthermore, if $a = 0$, then $b_{k,d-k} \geq 0$. Consider a general projection of the d-th Veronese embedding of $\mathbb{P}^2$ to $\mathbb{P}^d$. Consider the image of a pencil of lines in $\mathbb{P}^2$. By Lemma 1.8, this is a moving one-parameter family C of degree d rational curves that has intersection number zero with the boundary divisors. It follows from the inequality $C \cdot D \geq 0$ that $a \geq 0$.

Furthermore, suppose that $a = 0$. Consider a general pencil of $(1, 1)$ curves on $\mathbb{P}^1 \times \mathbb{P}^1$. Take a general projection to $\mathbb{P}^d$ of the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ by the linear system $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(i, d-i)$. By Lemma 1.8, the image of the pencil gives a moving one-parameter family C of degree d curves whose intersection with $\Delta_{k,d-k}$ is zero unless $k = i$. The relation $C \cdot D \geq 0$ implies that if $a = 0$, then $b_{i,d-i} \geq 0$. We conclude that Theorem 1.5 is true if $a = 0$. We can, therefore, assume that $a > 0$.

Suppose that for every $1 \leq i \leq \lfloor d/2 \rfloor$, we could construct a moving curve C_i in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ with the property that $C_i \cdot \Delta_{k,d-k} = 0$ for $k \neq i$ and that the ratio of $C_i \cdot \Delta_{i,d-i}$ to $C_i \cdot \mathcal{H}$ is given by

$$\frac{C_i \cdot \Delta_{i,d-i}}{C_i \cdot \mathcal{H}} = \frac{d+1}{i(d-i)}. \tag{3.1}$$

Observe that given these intersection numbers, Lemma 2.1 implies that $C_i \cdot D_{\deg} = 0$. Theorem 1.5 follows from the inequalities $C_i \cdot D \geq 0$.

In the rest of this section we will first give a construction of one-parameter families of C_i with these properties. However, our construction will depend on the Harbourne–Hirschowitz conjecture. We will then modify the construction to get a sequence of curves (not depending on any conjectures) that "approximate" these intersection numbers. These curves will suffice to conclude Theorem 1.5.

3.1 Construction 1: Depending on the Harbourne–Hirschowitz Conjecture

Let F_1 and F_2 denote the two fiber classes on $\mathbb{P}^1 \times \mathbb{P}^1$. We will abuse notation and denote the proper transform of the fibers in any blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ also by F_1 and F_2. Let d, j and k be positive integers subject to the condition that $2k \leq d$. Consider S the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in $j(d+1)$ general points $p_1, \dots, p_{j(d+1)}$. Let E_i denote the i-th exceptional divisor lying over p_i. Let $L(j)$ be the following linear system on S:

$$L(j) = dF_1 + \frac{jk(k+1)}{2}F_2 - \sum_{i=1}^{j(d+1)} kE_i.$$

Suppose M is a linear system on S and that $M - F_2$ is non-special, that is,

$$h^1(S, \mathcal{O}_S(M - F_2)) = 0.$$

Consider the exact sequence

$$0 \to \mathcal{O}_S(M - F_2) \to \mathcal{O}_S(M) \to \mathcal{O}_{F_2}(M) \to 0.$$

The long exact sequence of cohomology implies that taking the one-parameter family of proper transforms of the fiber class F_2 under the image of the linear system $|M|$ gives a one-parameter family of rational curves of degree $M \cdot F_2$ spanning $\mathbb{P}^{M \cdot F_2}$.

In particular, suppose that $L(j) - F_2$ is non-special. Then by the discussion in the previous paragraph, the linear system $L(j)$ embeds the general curve in the linear system $|F_2|$ on S as a rational normal curve of degree d in $\mathbb{P}^d$. We thus obtain a moving curve $C_k(j)$ that has intersection number zero with all the boundary classes except for $\Delta_{k,d-k}$. Clearly, $C_k(j) \cdot \Delta_{k,d-k} = j(d+1)$. The degree of the surface that these curves span is given by $L(j)^2 = jk(d-k)$. Hence, $C_k(j) \cdot \mathcal{H} = jk(d-k)$. It follows from Lemma 2.1 that $C_k(j) \cdot D_{\deg} = 0$. Hence, Theorem 1.5 would immediately follow if $L(j) - F_2$ were non-special for at least one value of j.

We recall that the celebrated conjecture due to Harbourne and Hirschowitz characterizes the linear systems that are special on a general blow-up of $\mathbb{P}^2$ as those linear systems that have a multiple (-1)-curve in their base locus. Here we will need a weaker form of the conjecture (see [CM]).

Conjecture 3.1 **(Harbourne–Hirschowitz)** *Let M be a complete linear system on a general blow-up S of $\mathbb{P}^2$. If $E \cdot M$ is non-negative for every (-1)-curve E on S, then M is non-special.*

Since the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at a point is isomorphic to the blow-up of $\mathbb{P}^2$ at two points, the Harbourne–Hirschowitz conjecture applies to the linear systems $L(j)$. The class of any (-1)-curve on S may be expressed as $\alpha F_1 + \beta F_2 - \sum_{i=1}^{j(d+1)} \gamma_i E_i$, where α and β are non-negative integers and $\gamma_i \geq -1$ is an integer. Since for a (-1)-curve E we have $E \cdot K = -1$, it follows that

$$-\sum_{i=1}^{j(d+1)} \gamma_i = 1 - 2\alpha - 2\beta.$$

The intersection of the (-1)-curve with $L(j) - F_2$ is

$$d\beta + \alpha\Big(\frac{jk(k+1)}{2} - 1\Big) - k\sum_{i=1}^{j(d+1)} \gamma_i = (d-2k)\beta + \alpha\Big(\frac{jk(k+1)}{2} - 2k - 1\Big) + k.$$

When $k > 3$ and $j \geq 1$, or $k = 2, 3$ and $j > 1$, or $k = 1$ and $j \geq 3$, the intersection is non-negative. We conclude the following.

Proposition 3.2 *Suppose Conjecture* 3.1 *holds for $L(j) - F_2$ for some $j \geq 1$. Then the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ is spanned by the classes of $D_{\deg}$ and the boundary divisors. Furthermore, every codimension* 1 *face of the effective cone is the null-locus of a moving curve.*

Remark 3.3. It is easy to prove that $L(j) - F_2$ is non-special for small values of d and k and to deduce Proposition 3.2 without any conditions. However, we could not see how to prove the non-specialty of $L(j) - F_2$ in general.

3.2 Construction 2: Completing the Proof

We modify the previous construction by imposing fewer k-fold points on the linear system $dF_1 + \frac{jk(k+1)}{2}F_2$ on $\mathbb{P}^1 \times \mathbb{P}^1$. If we do not impose too many k-fold points on the linear system, we can prove the non-specialty of the desired linear system. The following proposition makes this precise.

Proposition 3.4 *Let k, j and d be positive integers subject to the condition that $2k \leq d$. There exists an integer $n(k,d)$ depending only on k and d such that the linear system*

$$L'(j) = d\,F_1 + \Big(\frac{jk(k+1)}{2} - 1\Big)F_2 - \sum_{i=1}^{j(d+1)-n(k,d)} kE_i - \sum_{i=j(d+1)-n(k,d)+1}^{j(d+1)+n(k,d)\frac{(k-1)(k+2)}{2}} E_i$$

on the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at $j(d+1) + n(k,d)\frac{(k-1)(k+2)}{2}$ general points is non-special for every $j \gg 0$. The integer $n(k,d)$ may be taken to be $n(k,d) = \lceil 2(d+1)/k \rceil$.

Proposition 3.4 implies Theorem 1.5. As in the previous subsection, we consider the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in

$$j(d+1) + \frac{n(k,d)(k-1)(k+2)}{2}$$

general points. The proper transform of the fibers F_2 under the linear system

$$dF_1 + \frac{jk(k+1)}{2}F_2 - \sum_{i=1}^{j(d+1)-n(k,d)} kE_i - \sum_{i=j(d+1)-n(k,d)+1}^{j(d+1)+n(k,d)\frac{(k-1)(k+2)}{2}} E_i$$

gives a one-parameter family $C_k(j)$ of rational curves of degree d that has intersection number zero with $D_{\deg}$. Letting j tend to infinity, we obtain a sequence of moving curves $C_k(j)$ in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ that has intersection zero with all the boundary divisors but $\Delta_{1,d-1}$ and $\Delta_{k,d-k}$. Unfortunately, the intersection of $C_k(j)$ with $\Delta_{1,d-1}$ is not zero and the ratio of $C_k(j) \cdot \mathcal{H}$ to $C_k(j) \cdot \Delta_{k,d-k}$ is not the one required by (3.1). However, as j tends to infinity, the ratio of the intersection numbers $C_k(j) \cdot \Delta_{1,d-1}$ to $C_k(j) \cdot \mathcal{H}$ tends to zero and the ratio of $C_k(j) \cdot \Delta_{k,d-k}$ to $C_k(j) \cdot \mathcal{H}$ tends to the desired ratio $\frac{d+1}{k(d-k)}$. Theorem 1.5 follows.

Proof of Proposition 3.4 The proof of this proposition is an application of the standard degeneration techniques used to study the Harbourne–Hirschowitz conjecture. The global sections of a linear system

$$aF_1 + bF_2 - \sum_i r_iE_i$$

on a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at points p_i correspond to proper transforms of curves of type (a,b) on $\mathbb{P}^1 \times \mathbb{P}^1$ that have multiplicity r_i at p_i. In general, it is hard to obtain an upper bound on the dimension of global sections directly. However, if the points

p_i are in a special position, it might be possible to estimate the dimension of global sections using the special geometry. By the upper-semi-continuity, the same estimate holds also when the points are in general position.

A specialization that works well is to set the points p_i one at a time to a fixed point q. More precisely, for our linear system $L(j)'$ we specialize the k-fold points as follows: we begin with a k-fold point p in general position. We first specialize p along a fiber f_1 (in the class F_1) onto the fiber f_2 in (the fiber class F_2) containing the point q. We then specialize the point p onto q along f_2. The limiting linear systems that result from this specialization are well known. For example, they have been elegantly described by a checker game [Ya, §2]. We will use Yang's description to complete the proof. We note that Yang works on $\mathbb{P}^2$, but since the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at one point is isomorphic to the blow-up of $\mathbb{P}^2$ at two points, the description carries over to $\mathbb{P}^1 \times \mathbb{P}^1$ with little modification.

We now recall Yang's description (phrased for $\mathbb{P}^1 \times \mathbb{P}^1$). The global sections of the linear system $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b)$ are bi-homogeneous polynomials of bi-degree a and b in the variables x, y and z, w, respectively. A basis for the space of global sections is given by the monomials $x^i y^{a-i} z^j w^{b-j}$, where $0 \leq i \leq a$ and $0 \leq j \leq b$. We can record the coefficients of these monomials in a rectangular $(a + 1) \times (b + 1)$ grid. In this grid, the box in the i-th row and the j-th column corresponds to the coefficient of the monomial $x^i y^{a-i} z^j w^{b-j}$.

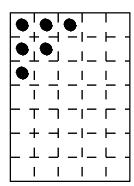

Figure 1: Imposing a triple point on $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4, 6)$.

A point p is a k-fold point of a curve if the first $k - 1$ derivatives of the defining equation of the curve vanish at p. If we impose a k-fold point on the linear system at the point $([x:y], [z:w]) = ([0:1], [0:1])$, then the coefficients of the monomials

$$y^a w^b, xy^{a-1} w^b, \ldots, x^{k-1} y^{a-k+1} z^{k-1} w^{b-k+1}$$

must vanish. We depict this condition by filling the $k \times k$ triangle of boxes corresponding to the coefficients of these monomials with checkers. See Figure 1 for an example. In general, an $(a + 1) \times (b + 1)$ checker diagram with checkers filled in at some boxes will denote the subspace of $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b))$ spanned by the monomials corresponding to the boxes that do not contain checkers.

Suppose we begin with a linear subseries of $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b))$ defined by a checker diagram. If initially all the boxes in the $k \times k$ upper left triangle are empty, then the linear series obtained by imposing a new k-fold point at p corresponds to

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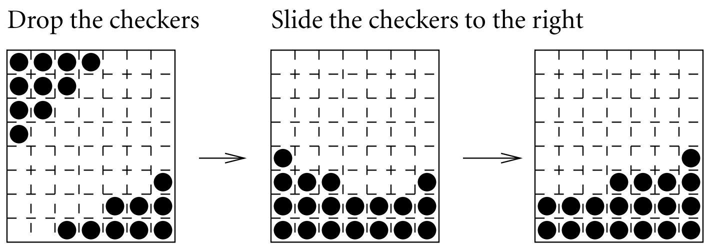

Figure 2: Depicting the degenerations by checkers.

the checker diagram where in addition the $k \times k$ upper left triangle is filled with checkers. The first panel of Figure 2 depicts an example. We first slide the k-fold point along the fiber f_1 onto the point $([x:y],[z:w]) = ([1:0],[0:1])$. This corresponds to the degeneration $([x:y],[z:w]) \mapsto ([x:ty],[z:w])$. The flat limit of the linear series under this degeneration is again a linear series which corresponds to a checker diagram. The new checker diagram is obtained from the old one by dropping the checkers vertically down until they reach a box that already contains a checker. The second panel in Figure 2 depicts the result of such a degeneration.

We then follow this degeneration with a degeneration that specializes the k-fold point to q by sliding along the fiber f_2. This degeneration is explicitly given by

$$([x:y],[z:w]) \mapsto ([x:y],[z:tw]).$$

The flat limit of the linear system is another linear system that corresponds to a checker diagram. The new checker diagram is obtained from the old one by sliding all the checkers as far right as possible. The third panel in Figure 2 gives an example. The proof consists of writing down a general member of the linear series, factoring out the lowest power of t from the expression and setting $t = 0$. We refer the reader to the proof of [Ya, Lemma 2] for the computation.

If one can carry out the checker degeneration with all the multiple points that one imposes on a linear system without any checkers falling off the grid, one can conclude that the linear system is non-special. The dimension of a linear system corresponding to a checker diagram is the number of empty boxes. If none of the checkers fall off the grid, then each point of multiplicity m imposes $m(m+1)/2$ conditions. Hence, the expected dimension is equal to the actual dimension. Riemann–Roch then implies that the linear system is non-special. The obstruction to proving the non-specialty is that there might not be enough empty boxes to impose a k-fold point. Specifically, in the initial diagram some of the boxes in the upper left $k \times k$ triangle may be full. In that case we cannot impose a k-fold point at p without losing a checker. See [Ya] for examples.

In order to conclude the proposition we need to show that if we impose at most $j(d+1) - n(k,d)$ points of multiplicity k on the linear system $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(d, jk(k+1)/2)$ where $2k \le d$, we do not lose any checkers when we specialize all the k-fold points by the degeneration just described. This suffices to conclude the proposition because general simple points always impose independent conditions.

The main observation is that if there is a safety net of empty boxes at the top of the rectangle, then the checkers will not fall out of the box. If we perform the checker degeneration using k-fold points, then, after specializing each of the k-fold points to q, the diagram clearly satisfies the following two properties:

- The uppermost row that contains any checkers is at most k rows higher than the uppermost row completely filled with checkers.
- The leftmost checker of a row is to the lower left of the leftmost checker of any row above it.

If there are at least $(k+1)(d+1)$ empty boxes in our rectangle, then by the above two observations we do not lose any of the checkers when we specialize a k-fold point. As long as $n(k,d) \geq \lceil 2(d+1)/k \rceil$, there are always at least $(k+1)(d+1)$ boxes empty. Hence we can specialize without losing any conditions. ■

Remark 3.5. While the asymptotic approach gives a proof of Theorem 1.5 independent of the Harbourne–Hirschowitz conjecture, it does not construct a moving curve that is dual to the codimension one faces of the effective cone of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$. However, the moving curves we have constructed approximate arbitrarily well the duals to the codimension one faces and suffice for the applications.

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