Canad. Math. Bull. Vol. 15 (3), 1972.

DISK PACKINGS WHICH HAVE NON-EXTREME EXPONENTS

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1. **Introduction.** Let U be an open set in the Euclidean plane which has finite area. A complete (or solid) packing of U is a sequence of pairwise disjoint open disks $C=\{D_n\}$, each contained in U and whose total area equals that of U. A simple osculatory packing of U is one in which the disk D_n has, for each n, the largest radius of disks contained in $U\backslash(D_1^- \cup \cdots \cup D_{n-1}^-)$. ($S^-$ denotes the closure of the set U.) If r_n is the radius of D_n, then the exponent of the packing, $e(C, U)$ is the infimum of real numbers t for which $\sum r_n^t < \infty$. In the sequel we refer to a complete packing simply as a packing.

We shall be concerned here with packings of the unit disk D, and of curvilinear triangles T with mutually tangent circular sides. Let C_0 be the simple osculatory packing of T. Melzak [3] showed that $1{\cdot}035 < e(C_0, T) < 1{\cdot}999971$. Wilker [5] showed that $e(C_0, T)$ is a constant S independent of the radii of the sides of T. In [4], Melzak has given numerical evidence to suggest that $S \approx 1{\cdot}306951$. By results of [1] and [2], it is known that $1{\cdot}2846 < S < 1{\cdot}3500$(2).

Clearly there are packings of D with exponent S, and Melzak [3] gives examples of packings of D with exponent 2. In [3], [4] and [5], the question is raised as to whether there are packings of D with exponents other than S or 2, and it is conjectured that the set of possible exponents is the interval $[S, 2]$. In this paper we lend support to this conjecture by showing that the set of possible exponents *contains* the interval $[S, 2]$. We in fact show that if μ is the infimum of possible exponents for packings of T then the set of such exponents is either $[\mu, 2]$ or $[\mu, 2]$.

After this paper was submitted, it was pointed out to the author that J. B. Wilker in his Ph.D. thesis, University of Toronto, 1968, had proved that the set of packing exponents contains the interval $[S, 2]$ (Theorem 5.15). Since the construction which he gives is somewhat more complicated than that given here, and since it is still not known whether $S=\mu$, we hope that the publication of our result will be of some interest.

2. **Preliminaries.** We first discuss some well-known facts concerning inversion. A similar discussion is found in [5]. Let $T(a, b, c)$ be a curvilinear triangle with

Received by the editors October 1, 1970 and, in revised form, February 19, 1971.

(1) Supported in part by NSF Grant GP-14133.

(2) (Added in proof: In a paper by the auther to appear in Aequationes Mathematicae, this has been improved to $1{\cdot}300197 < S < 1{\cdot}314534$.)

https://doi.org/10.4153/CMB-1972-061-8 Published online by Cambridge University Press

sides of radii $|a| \geq b \geq c$, where $a = +|a|$ or $-|a|$ depending on whether the circle of radius a touches the other circles externally or internally; in case of internal tangency, $T(a, b, c)$ is the smaller of the two possible triangles. We allow $a = \infty$; in this case one of the sides of $T(a, b, c)$ is a straight line; but we insist that b and c be finite.

If $T = T(a, b, c)$ and $T' = T(a', b', c')$ are any two such triangles, there is an inversion f which maps T onto T' in such a way that a, b, c correspond respectively to a', b', c'. There is a constant $k \geq 1$ such that, if K is a circle of radius ρ contained in T, and $K' = f(K)$ has radius ρ', then

$$k^{-1} \leq \rho'/\rho \leq k. \tag{1}$$

To see this, we take as our standard triangle, $T_0 = T(\infty, 1, 1)$, and show how to invert $T(a, b, c)$ into $T(\infty, 1, 1)$ by applying at most three inversions. For each inversion, (1) will be shown to hold by applying the Corollary to Lemma 3.6 of [5], then (1) will hold for the composition of the inversions. We first map $T(a, b, c)$ into $T(a, c, c)$ by an inversion f_1 which has centre on the common tangent line to the sides of $T(a, b, c)$ which have radii a and c. The circle of inversion passes through the point of tangency. The triangle $T(a, b, c)$ lies outside the circle of inversion, hence we can apply the result from [5] to which we have just referred. Now we dilate $T(a, c, c)$ to $T(a/c, 1, 1)$, which presents no difficulty for (1).

We shall describe in more detail the inversion of a triangle $T(r, 1, 1)$ into $T(\infty, 1, 1)$ since we shall be using certain facts concerning this in Theorem 1. Set up coordinates so that the centre of the circle of radius r is at the origin and so that the point of tangency of the other two sides is on the negative real axis. The inverting circle then has centre I with coordinates $(|r|, 0)$ and radius $\gamma = |r| + \sqrt{(1+r)^2 - 1}$ so that it is orthogonal to the two circles of radius 1. If $K \subset T(r, 1, 1)$ is any circle with radius ρ, and d is the length of the tangent from I to K, then ρ inverts into a circle of radius $\rho' = \rho\gamma^2/d^2$. However, the following inequalities hold: if $r > 0$, then $\gamma^2 \geq d^2 \geq \gamma^2 r/(r+1)$ (see [2, Lemma 1]), and if $r < 0$ then $\gamma^2 \leq d^2 \leq (2r)^2$. Thus, (1) holds with a constant $k(r) > 1$ such that $k(r) \to 1$ as $|r| \to \infty$.

If $C = \{D_n\}$ is a packing of T_0 with exponent $e(C, T_0)$, and if f inverts T_0 into $T(a, b, c)$ as above, then $f(C)$ is a packing of $T(a, b, c)$ with the same exponent. This follows from (1). By abuse of notation, we write $f(r_n)$ for the radius of $f(D_n)$. If $t > e(C, T_0)$, we write

$$M(a, b, c; t) = \sum_{n=1}^{\infty} f(r_n)^t.$$

Note that $M(\lambda a, \lambda b, \lambda c; t) = \lambda^t M(a, b, c; t)$ for any $\lambda > 0$, and if $|r| \to \infty$, $M(r, 1, 1; t) \to M(\infty, 1, 1; t)$ (since the constant $k(r)$ in (1) tends to 1 as $|r| \to \infty$).

3. The construction.

THEOREM 1. *Let μ be the infimum of $e(C, T_0)$ over all complete packings of T_0, and let x satisfy $\mu < x \leq 2$.*

Then, there is a complete packing C_x of D for which

$$e(C_x, D) = x.$$

Proof. We assume $x<2$, since $x=2$ is covered by the examples in [3].

Let square brackets denote the greatest integer function, let $s=(x-1)^{-1}>1$, let $\varepsilon_0=0$, and let

$$\varepsilon_n = \sin(\pi/4[n^s]) \quad \text{for } n = 1, 2, \ldots. \tag{2}$$

Define

$$c_n = \prod_{k=0}^{n-1} (1-\varepsilon_k)(1+\varepsilon_k)^{-1}, \quad \text{for } n = 1, 2, \ldots. \tag{3}$$

Then $c_1=1$ and $\{c_n\}$ is a strictly decreasing sequence with limit $c>0$ (the infinite product converges since $\sum \varepsilon_k < \infty$). Now, for $n \geq 1$, let $2\rho_n = c_n - c_{n+1}$, and $2b_n = c_n + c_{n+1}$. Then

$$2\rho_n = c_n(1-(1-\varepsilon_n)(1+\varepsilon_n)^{-1}) = 2c_n\varepsilon_n(1+\varepsilon_n)^{-1} = 2\varepsilon_n b_n. \tag{4}$$

We shall use complex numbers to denote points in the plane. Let A_n denote the annulus $\{z: c_{n+1} < |z| < c_n\}$, $n=1, 2, \ldots$ and B the disk $\{z: |z| < c\}$. Note that the largest disk which will fit into A_n has radius ρ_n and center at distance b_n from the origin. The angle subtended at the origin by this disk is θ_n, where

$$\theta_n = 2 \sin^{-1}(\rho_n/b_n) = 2 \sin^{-1}(\varepsilon_n) = (2\pi)/4[n^s],$$

using (4) and the definition of ε_n. Thus, exactly $4[n^s]$ disks of radius ρ_n will fit into A_n, each tangent to two others; removing these disks from A_n leaves $4[n^s]$ triangles $T_n = T(c_{n+1}, \rho_n, \rho_n)$ and $4[n^s]$ triangles $T'_n = T(-c_n, \rho_n, \rho_n)$.

Let C be a fixed packing of T_0 whose exponent satisfies $\mu \leq e(C, T_0) < x$. Let each of the triangles T_n and T'_n be packed with an inversive image of C, as described in §2. Our packing C_x consists of the disk B, the $4[n^s]$ disks of radius ρ_n for each n, and the disks in the packings of the T_n and T'_n for each n. This is clearly a complete packing of D, since D differs from $B \cup \bigcup_{n=1}^{\infty} A_n$ by a set of measure zero. Let $C_x = \{D_k: k=1, 2, \ldots\}$ and let r_k be the radius of D'_k. If $t > e(C, T_0)$, we have

$$\begin{aligned} \sum_{k=1}^{\infty} r_k^t &= c^t + \sum_{n=1}^{\infty} 4[n^s](\rho_n^t + M(c_{n+1}, \rho_n, \rho_n; t) + M(-c_n, \rho_n\rho_n; t)) \\ &= c^t + \sum_{n=1}^{\infty} 4[n^s]\rho_n^t(1 + M(c_{n+1}\rho_n^{-1}, 1, 1; t) + M(-c_n\rho_n^{-1}, 1, 1; t)). \end{aligned} \tag{5}$$

However, $\lim b_n = \lim c_n = c \neq 0$, so $\rho_n = b_n \varepsilon_n \sim (\pi c/4) n^{-s}$ as $n \to \infty$. Also,

$$M(c_{n+1}\rho_n^{-1}, 1, 1; t) \to M(\infty, 1, 1, t)$$

and $M(-c_n\rho_n^{-1}, 1, 1; t) \to M(\infty, 1, 1; t)$, so that the nth term in the sum in (5) is asymptotic to a constant multiple of n^{s-st}. The constant depends on t, but is finite for any $t > e(C, T_0)$. Hence the series in (5) diverges if $st - s \leq 1$ and converges if $st - s > 1$. Thus,

$$e(C_x, D) = (s+1)/s = x.$$

COROLLARY 1. *With the notation of the theorem, for each x in $\mu < x \leq 2$, there is a packing C_x' of T_0 with*

$$e(C_x', T_0) = x.$$

Proof. Pack the inscribed disk of T_0 with C_x (suitably scaled), and the remaining three curvilinear triangles with a packing whose exponent lies between μ and x.

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https://doi.org/10.4153/CMB-1972-061-8 Published online by Cambridge University Press