MACBEATH'S CURVE AND THE MODULAR GROUP

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To Robert Rankin on the occasion of his 70th birthday

0. Introduction. The theory of algebraic curves associated with subgroups of finite index in the modular group Γ is highly developed for such subgroups of Γ as may be defined by means of congruences in the ring $\mathbb{Z}$ of rational integers. The situation in the case of non-congruence subgroups of Γ, on the other hand, is drastically different. It reduces to a few isolated examples, two of which may be found in [**12**]. Related research by A. O. L. Atkin and H. P. F. Swinnerton-Dyer began in [**1**].

Observing that Macbeath's curve [**4**] affords another pertinent example made us look at it more closely. Its automorphism group is isomorphic with the simple group PSL(2, 8) of order 504. Accordingly, the associated subgroup Δ of Γ is a maximal normal subgroup of index 504. We shall prove, in passing, that there are exactly two such subgroups in Γ, neither of them a congruence group.

The view above renders Macbeath's curve as a covering of the projective line with Galois group G isomorphic to PSL(2, 8). Corresponding to any of its Sylow 7-subgroups and its normalizer in G we find two intermediate curves **B** and **A**, respectively elliptic and rational, of which the former covers the latter 2-fold and with 4 branch points. In this classical situation the 4 points of **A** under the branch points may, moreover, be made explicit through Macbeath's model. Their cross ratio, or Legendre's modulus λ, and in turn the absolute invariant J of **B** could then be calculated.

There is, however, more to gain with less effort. We find the said 4 points on **A**, after a Möbius transformation, to satisfy an algebraic 4th-degree equation $P(X)=0$ with integral rational coefficients. Thus $Y^2=P(X)$ describes **B** as a curve over $\mathbb{Q}$. Its Weierstrass normal form yields the invariants $g_2=196$, $g_3=-196$, and $J=(14/13)^2$. The Mordell–Weil rank of $\mathbf{B}(\mathbb{Q})$ may then be seen to be greater than 1, by reduction modulo 29.

The genus $g=7$ of Macbeath's curve and the order $h=504$ of its automorphism group are related by $h=84(g-1)$. G is then called a *Hurwitz group*. Many instances of such groups were recently constructed by J. M. Cohen [**2**, **3**], as abelian extensions of PSL(2, 7). We should like to point out that all such extensions occur in [**10**]; they correspond to the ideals I of algebraic integers in $\mathbb{Q}(\sqrt{-7})$, and their orders are $h=168(N(I))^3$. This also accounts for the orders of some groups in Sinkov [**7**].

1. Notation and generalities. We take $\Gamma=\mathrm{PSL}(2,\mathbb{Z})$ for the modular group, and we represent its elements by matrices $L=(a\quad b\ \mid\ c\quad d)\in\mathrm{SL}(2,\mathbb{Z})$; L and $-L$ are identified under this representation. We write

$$U=(1\quad 1\ \mid\ 0\quad 1),\qquad T=(0\quad -1\ \mid\ 1\quad 0),\qquad S=TU=(0\quad -1\ \mid\ 1\quad 1).$$

Γ is then the free product of two cyclic groups of orders 2 and 3, respectively, generated by (Möbius transformations induced by) T and S.

Glasgow Math. J. **27** (1985) 239–247.

https://doi.org/10.1017/S0017089500006212 Published online by Cambridge University Press

Let G be any finite group generated by two elements τ and σ, respectively of periods 2 and 3. Then $\pi(T)=\tau$ and $\pi(S)=\sigma$ determine a surjective homomorphism $\pi:\Gamma\to G$. Its kernel Δ is a normal subgroup of Γ with factor group isomorphic with G. Moreover, under π the set of all subgroups X of G corresponds bijectively to the set of all subgroups Φ of Γ which contain Δ, by $X=\pi(\Phi)$, and $[G:X]=[\Gamma:\Phi]$.

If G is a transitive permutation group on, say, symbols $0, 1, \ldots, m-1$ then π is a permutation representation of Γ, the stabilizers X_j of j $(0\leq j\leq m)$ form a set of conjugate subgroups of index m in G, and their intersection is the trivial subgroup. Their inverse images $\Phi_j=\pi^{-1}(X_j)$ under π constitute a set of conjugage subgroups of index m in Γ with Δ as their intersection. The level of each of the Φ_j, in the sense of [**9**], equals the level of Δ, and the Φ_j are congruent subgroups of the Ψ if and only if Δ is such a subgroup.

Conversely, if Φ is any subgroup of finite index m in Γ, a transitive permutation representation π of degree m of Γ is induced by right-hand multiplication of the cosets ΦL, where $L\in\Gamma$ with elements $M\in\Gamma$. Its image $G=\pi(\Gamma)$ is generated by $\tau=\pi(T)$ and $\sigma=\pi(S)$. Also $X=\pi(\Phi)$ is the stabilizer of the coset Φ under π. The intersection of the conjugates of Φ in Γ is again the kernel Δ of π. Such representations were introduced in [**5**], [**6**], and [**11**].

Let $\pi:\Gamma\to G$ be any transitive permutation representation, as above, and X a subgroup of G. To $\Phi=\pi^{-1}(X)$ as a subgroup of Γ there canonically corresponds a modular curve, or compact Riemann surface, which we shall denote by $\mathbf{X}$. If Y is another subgroup of G and $Y<X$, then $\Psi=\pi^{-1}(Y)$ is a subgroup of Φ, and there is a natural projection turning $\mathbf{Y}$ into a covering of $\mathbf{X}$: each Ψ-orbit of a point z in the extended upper half-plane is mapped on the Φ-orbit of z. The index of Y in X equals the degree of the covering.

2. Two normal subgroups of Γ

PROPOSITION. *There are exactly two normal subgroups Δ of Γ with factor group Γ/Δ isomorphic to the simple group* PSL(2, 8) *of order* 504. *They are both non-congruence groups, and their levels are* 7 *and* 9.

Proof. The group PSL(2, 8) may (in various ways) be realized as a transitive subgroup G of the symmetric group S_9, acting on the set $N=\{0, 1, \ldots, 8\}$ (see [**11**]). G contains elements of periods 1, 2, 3, 7, and 9. If $\tau, \sigma\in G$ respectively have periods 2 and 3 then they generate G if and only if their product $\omega=\tau\sigma$ is of period 7 or 9. To prove the proposition we have to show that there are essentially, i. e. up to a permutation, or a renumbering of N, two different ways G may be generated by such a pair of elements τ and σ.

As all 63 elements of period 2 are conjugate in G we may fix $\tau=(14)(26)(37)(58)$. There is then exactly one Sylow 2-subgroup D of G, abelian of type (2, 2, 2), containing τ. Also D has index 7 in its normalizer $C=N_G(D)$, and we may take $\eta=(2456873)$ to be an element of C.

Again, all 56 elements of period 3 are conjugate in G, and if $\sigma=(015)(274)(386)$ is one of them, then the others are of the form $\alpha^{-1}\sigma\alpha$ with $\alpha\in C$. But $\alpha\in D$ implies $\alpha^{-1}\tau\alpha=\tau$, and so we may confine ourselves to the 7 pairs (τ,σ_j), $\sigma_j=\eta^{-j}\sigma\eta^j$ (j mod 7). A short calculation and suitable renumbering then yields essentially two ways to generate G: either

$$\tau=(14)(26)(37)(58),\qquad \sigma=(015)(274)(386),\qquad \omega=(012345678);$$

or

$$\tau=(13)(28)(46)(57),\qquad \sigma=(014)(283)(576),\qquad \omega=(0123456).$$

The permutation representations π of Γ induced in those two cases have kernels Δ as required, and their respective levels are 7 and 9. As the index 504 of Δ in Γ is not a divisor of $[\Gamma:{}_7\Gamma]=168$ or of $[\Gamma:{}_9\Gamma]=324$, neither of the principal congruence groups ${}_l\Gamma$ of level $l=7$ or $l=9$ is a subgroup of Δ. Therefore, Δ is not a congruence group, and the proposition is proved.

3. Elliptic and parabolic fixed points. Let Φ be a subgroup of finite index m in Γ and $\mathbf{X}$ the associated Riemann surface of genus g and Hurwitz characteristic $\chi=2g-2$. Then

$$6\chi=m-6h-3e_2-4e_3,$$

where h denotes the number of classes of parabolic, and e_k the number of classes of elliptic fixed points, of order k, of Φ.

e_2 and e_3 may be determined as numbers of cosets of Φ in Γ, respectively fixed under $\Phi L\mapsto\Phi LT$ and $\Phi L\mapsto\Phi LS$. h is less simple to find: this is the number of orbits under $\Phi L\mapsto\Phi LU$, the length of each orbit equalling the respective cusp amplitude. Hence the level l of Φ is equal to the period of $\Phi L\mapsto\Phi LU$ as a permutation of cosets. Let $\kappa(t)$ denote the number of fixed elements of the tth power of that permutation, and $h(t)$ the number of cusps of Φ of amplitude t. Then obviously $\kappa(d)=\sum_{t\mid d}th(t)$. Möbius inversion gives

$$h(d)=d^{-1}\sum_{t\mid d}\mu(d/t)\kappa(t)$$

for each divisor d of the level l of Φ.

We shall later use a lemma whose proof is straightforward.

LEMMA. *Let X be a subgroup of finite index in a group G and let b be an element of G. Then the number of cosets Xa of X, fixed under $Xa\mapsto Xab$, equals the number of conjugates of X in G containing b, multiplied by the index of X in its normalizer $N_G(X)$ with respect to G.*

The lemma often obviates the need to construct sets of coset representatives when numbers (of classes) of fixed points are to be determined. Besides, it may directly be applied to $X=\pi(\Phi)$ and G instead of to Φ and Γ, with $\pi:\Gamma\to G$ any surjective homomorphism.

4. Branch schemes. In the diagram below G denotes the simple group of order 504, H its trivial subgroup, and the arrows represent injections; a number m near an arrow

https://doi.org/10.1017/S0017089500006212 Published online by Cambridge University Press

gives the corresponding (relative) group index. B, D, and F are Sylow subgroups of G, and A, C, and E are their respective normalizers in G.

Again, if each of the letters A through H is boldfaced the same diagram may be read as one of covering maps of algebraic curves, together with their degrees.

Or, finally, the letters in the original diagram may be interpreted as the Galois groups X of the field $K(\mathbf{H})$ of functions on $\mathbf{H}$ over $K(\mathbf{X})$; here, of course, $K(\mathbf{G})$ is the rational function field.

$$\begin{tikzcd}
 & G & \\
A \arrow[ur, "36"] & C \arrow[u, "9"] & E \arrow[ul, "28"'] \\
B \arrow[u, "2"] & D \arrow[u, "7"] & F \arrow[u, "2"] \\
 & H \arrow[ul, "7"] \arrow[u, "8"] \arrow[ur, "9"'] &
\end{tikzcd}$$

In regard to a permutation representation $\pi : \Gamma \to G$ the diagram refers to either the level $l = 7$ or $l = 9$. In the latter case $\mathbf{C}$ has genus 1 and known absolute invariant [**12**]. $\mathbf{C}$ and $\mathbf{D}$ are isogenous as elliptic curves. In the former case, as we shall see, $\mathbf{H}$ has genus 7 and G is its automorphism group. Macbeath [**4**] has constructed a canonical model of $\mathbf{H}$ and explicit generators of G. We shall use his results to determine the curve $\mathbf{B}$.

The branch behaviour of the various covering maps depends on the numbers of classes of elliptic and parabolic fixed points of the relevant groups. They are found on applying the lemma in Section 3, using appropriate information on the structure of G. For any subgroup X of G the number of conjugates of X in G which contain some element b of G only depends on the period of b. We collect those numbers in the following table and, besides, note in its last column the indices of the various subgroups X of G in their normalizers.

per (b)	1	2	3	7	9	*index*
A	36	4	0	1	0	1
B	36	0	0	1	0	2
C	9	1	0	2	0	1
D	9	1	0	0	0	7
E	28	4	1	0	1	1
F	28	0	1	0	1	2
G	1	1	1	1	1	1
H	1	0	0	0	0	504

Restricting attention now to the left-hand chain of maps in the diagram and using the

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lemma we find the following numbers of classes of fixed points.

	e_2	e_3	$h(1)$ $l=7$	$h(1)$ $l=9$	$h(l)$ $l=7$	$h(l)$ $l=9$
H	0	0	0	0	72	56
B	0	0	2	0	10	8
A	4	0	1	0	5	4
G	1	1	1	1	0	0

Let now $\lambda : \mathbf{Y} \to \mathbf{X}$ be one of the covering maps considered. Its branch points are over points of $\mathbf{X}$ which, with reference to a fundamental domain of $\Phi = \pi^{-1}(X)$ in the extended upper half-plane $\mathbb{H}$, are represented by points in the Γ-orbit of $i = \exp(\pi i/2)$, $\rho = \exp(\pi i/3)$, or ∞. The fixed point numbers given above then allow for the relevant genera and branch schemes to be evaluated. If over some point of $\mathbf{X}$ corresponding to $z \in \mathbb{H}$ there are r branch points in $\mathbf{Y}$, each of multiplicity k, we write $\frac{r*k}{z}$; if $r=1$ we abbreviate to $\frac{k}{z}$. Then we obtain the following table.

	level 7 genus	level 7 branch scheme	level 9 genus	level 9 branch scheme
$\mathbf{H}$	7		15	
$\downarrow 7$		$\frac{7 \quad 7}{\infty_1 \quad \infty_2}$		$\frac{\text{none}}{\text{none}}$
$\mathbf{B}$	1		3	
$\downarrow 2$		$\frac{2 \quad 2 \quad 2 \quad 2}{i_1 \quad i_2 \quad i_3 \quad i_4}$		$\frac{2 \quad 2 \quad 2 \quad 2}{i_1 \quad i_2 \quad i_3 \quad i_4}$
$\mathbf{A}$	0		1	
$\downarrow 36$		$\frac{16*2 \quad 12*3 \quad 5*7}{i \quad \rho \quad \infty}$		$\frac{16*2 \quad 12*3 \quad 4*9}{i \quad \rho \quad \infty}$
$\mathbf{G}$	0		0	

We note that in the level 7 case the mapping $\mathbf{B} \to \mathbf{A}$ is a 2-fold covering with 4 branch points of a complex line $\mathbf{A}$ by an elliptic curve $\mathbf{B}$. Thus given a univalent function $f : \mathbf{A} \to \mathbb{P}$, the cross ratio of $f(i_1)$ through $f(i_4)$ equals Legendre's modulus λ, and so the absolute invariant

$$J = 4(1-\lambda\lambda')^3/27(\lambda\lambda')^2$$

of $\mathbf{B}$ becomes calculable. We shall use Macbeath's model [**4**] to construct such a function f.

As it will appear, the numbers $f(i_j)$ satisfy a 4th-degree equation over $\mathbb{Z}$. A description of **B** as a curve over $\mathbb{Q}$ is hence immediate. The Weierstrass invariants g_2, g_3 and therefore J are then much more easily determined than via the above cross ratio.

5. Macbeath's model. We write $\zeta = \exp(2\pi i/7)$. The index j, wherever it occurs, runs through a complete set of residues of $\mathbb{Z}$ modulo 7. The points $y = (y_0 : y_1 : \ldots : y_6)$ of Macbeath's curve [**4**] are in complex projective space $\mathbb{P}^6_{\mathbb{C}}$ of six dimensions and characterized by 10 equations:

$$\sum y_j^2 = \sum \zeta^j y_j^2 = \sum \zeta^{-j} y_j^2 = 0,$$
$$(\zeta^4 - \zeta^{-4}) y_{j+4} y_{j-2} + (\zeta^2 - \zeta^{-2}) y_{j+2} y_{j-1} + (\zeta^1 - \zeta^{-1}) y_{j+1} y_{j-4} = 0.$$

The automorphism group of the curve may be generated by substitutions u, v, and w, as follows:

$$uy_j = \tfrac{1}{2}(y_{-j} - y_{1-j} - y_{2-j} - y_{4-j}),$$
$$vy_j = \varepsilon_j y_j, \qquad \varepsilon_j = 1 \quad (j = 1, 2, 4) \text{ or } = -1 \text{ (otherwise)},$$
$$wy_j = y_{j+1}.$$

They satisfy

$$w^7 = v^2 = u^2 = 1. \qquad uwu^{-1} = w^{-1}, \qquad (vu)^3 = 1.$$

The 4 fixed points of v are characterized by

$$y_0^2 = 1, \qquad y_1 = y_2 = y_4 = 0, \qquad y_0 y_3 y_5 y_6 = 1,$$
$$y_3^2 = \zeta^2 + \zeta^{-2}, \qquad y_5^2 = \zeta^1 + \zeta^{-1}, \qquad y_6^2 = \zeta^4 + \zeta^{-4}.$$

Coordinates of the 4 fixed points of u may be derived thereof. w has 2 fixed points, $(1 : \zeta : \ldots : \zeta^6)$ and $(1 : \zeta^{-1} : \ldots : \zeta^{-6})$.

Macbeath's curve, because of its uniqueness property, is isomorphic to **H**. The coordinates y_j of his model may, therefore, serve to describe the points of **H**. Also, u, v and w may now be viewed as elements of G. Then, as u normalizes the subgroup of G generated by w, we may choose A among its conjugates in G such that **A** is invariant under both u and w. In the branch scheme of the covering map $\mathbf{H} \to \mathbf{A}$, composed via **B**,

$$\frac{7*2 \qquad 7*2 \qquad 7*2 \qquad 7*2 \qquad 2*7}{i_1 \qquad\;\; i_2 \qquad\;\; i_3 \qquad\;\; i_4 \qquad\;\; \infty},$$

the 4 fixed points of u lie over i_1 through i_4, while the 2 fixed points of w are the points over ∞.

We now introduce homogeneous linear forms

$$F_k(y) = \sum \zeta^{jk} y_j \quad (k \bmod 7),$$

which transform according to

$$F_k(uy) = \kappa_k F_{-k}(y), \qquad \kappa_k \kappa_{-k} = 1,$$
$$F_k(wy) = \zeta^{-k} F_k(y).$$

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On Macbeath's curve, F_1 vanishes in one of the 2 fixed points of w, F_{-1} in the other, and F_0 in both of them.

Taking local parameters t and remembering that Macbeath's model is canonical, we have abelian differentials $F_k(y)\,dt$ of the first kind. Hence the sum of the orders of the zeros, in particular, of F_0 on $\mathbf{H}$ is $2g-2=12$. But F_0 is invariant under w and thus vanishes in each of the fixed points of w, of period 7, to an order congruent to 6 modulo 7. The 2 fixed points of w then account for all the zeros of F_0 on $\mathbf{H}$.

In view of the transformation formulae, $f=F_1F_{-1}/F_0^2$ is invariant under u and w; hence f is a function on $\mathbf{A}$, holomorphic but for a simple pole at ∞, and therefore univalent. We shall now study the values $f(i_j)$ $(j=1,2,3,4)$, or, equivalently, viewing f as a function on $\mathbf{H}$, its values at the 4 fixed points of the automorphism u.

6. Equations for an elliptic curve. Let y be a fixed point of v, $vy=-y$, normalized by $y_0=1$. Then

$$y_3y_5y_6=1, \qquad y_3^2=\zeta^2+\zeta^{-2}, \qquad y_5^2=\zeta^1+\zeta^{-1}, \qquad y_6^2=\zeta^4+\zeta^{-4},$$

and hence

$$y_3^2+y_5^2+y_6^2=-1, \qquad 1/y_3^2+1/y_5^2+1/y_6^2=-2.$$

Writing $R=y_3+y_5+y_6$, $r=1/y_3+1/y_5+1/y_6$, we find that

$$R^2=2r-1, \qquad r^2=2R-2, \qquad r^4+4r^2-8r+8=0.$$

Because of $(uv)^3=1$, the point $x=-2vuy$ is fixed under u, $ux=-x$. Its coordinates are related to those of y by formulae given earlier. We wish to express the value of the function f at x in terms of the coordinates of y. First we have

$$F_0(x)=2(2-R), \qquad F_0(x)^2=4(3-4R+2r).$$

Next

$$F_1(x)=(1+\zeta+\zeta^2+\zeta^4)y_0+(\zeta-\zeta^4-\zeta^5-\zeta^6)y_3+(\zeta^4-\zeta^6-\zeta^2-\zeta^3)y_5+(\zeta^2-\zeta^3-\zeta^5-\zeta)y_6,$$

whence $F_{-1}(x)$ results on replacing ζ with ζ^{-1}. After some calculation, we find that

$$F_1(x)F_{-1}(x)=-2(1-2R+2r), \qquad -2f(x)=(1-2R+2r)/(3-4R+2r).$$

We may replace f with any Möbius transform, and in particular with $g=4(1+4f)/(1+2f)$. Then finally

$$g(x)=2(2r-1)/(R-1)=z^2, \qquad z=2R/r,$$

for any one of the 4 fixed points of u.

The quadratic relations between r and R, and the biquadratic equation satisfied by r, now yield an algebraic equation for z,

$$X^4-2X^3-2X^2-4X+18=0.$$

Replacing X by $-X$ and multiplying leads to an equation for z^2,

$$X^4-8X^3+24X^2-88X+324=0,$$

https://doi.org/10.1017/S0017089500006212 Published online by Cambridge University Press

whose solutions are now just the 4 values $g(x)$, or those points of **A** over which **B** is branched. Hence **B** is defined by

$$Y^2 = X^4 - 8X^3 + 24X^2 - 88X + 324$$

as an algebraic curve over $\mathbb{Q}$.

In order to obtain its Weierstrass normal form we first substitute $X+2$ for X; next apply the birational transformation

$$X = \bar{Y}/2\bar{X}, \qquad Y = \bar{Y}^2/4\bar{X}^2 - 2\bar{X},$$
$$2\bar{X} = X^2 - Y, \qquad \bar{Y} = X(X^2 - Y);$$

then write X and Y again for $\bar{X}$ and $\bar{Y}$, and finally substitute $Y+14$ for Y. This gives

$$Y^2 = 4X^3 - 196X + 196,$$

and invariants

$$g_2 = 196, \qquad g_3 = -196, \qquad \Delta = g_2^3 - 27g_3^2 = 14^4 13^2;$$

hence

$$J = g_2^3/\Delta = (14/13)^2.$$

Changing coordinates once more we shall now use this minimal equation for **B**:

$$Y^2 = X^3 - 49X + 49.$$

By Tate's algorithm [**8**], the curve has conductor $2^2 7^2 13$. There are at least 30 points on **B** whose coordinates are rational integers, none of which has finite period in the group of rational points on **B**. Indeed, that group has zero torsion.

Two of the points with integer coordinates are $P = (-7, 7)$ and $Q = (21, 91)$; the others appear as $mP + nQ$, with integers m and n. Thus the Mordell–Weil rank of **B** may well equal 2. It is, at any rate, not less than 2, for, let $L = \langle P, Q \rangle$ be the subgroup of $\mathbf{B}(\mathbb{Q})$ generated by P and Q. Then the reduction homomorphism $\lambda : L \to \mathbf{B}(\mathbb{F}_{29})$ modulo the prime $p = 29$ turns out to be surjective and with non-cyclic image. Therefore, neither L nor $\mathbf{B}(\mathbb{Q})$ is cyclic, which proves our claim.

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