VERBAL WREATH PRODUCTS AND CERTAIN PRODUCT VARIETIES OF GROUPS

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1. Introduction

Recently A. L. Šmel'kin [14] proved that a product variety[1] $\mathfrak{U}\mathfrak{V}$ is generated by a finite group if and only if $\mathfrak{U}$ is nilpotent, $\mathfrak{V}$ is abelian, and the exponents of $\mathfrak{U}$ and $\mathfrak{V}$ are coprime. Alternatively, by the theorem of Oates and Powell [13], we may say that a Cross variety is decomposable if and only if it is of the above form.

Throughout this paper such a variety will be denoted by $\mathfrak{N}\mathfrak{A}_n$, the class of $\mathfrak{N}$ by c, and the exponent of $\mathfrak{N}$ by m : n is the exponent of the abelian variety $\mathfrak{A}_n$.

In this paper we find the least number $l = l(\mathfrak{N}\mathfrak{A}_n)$ such that $\mathfrak{N}\mathfrak{A}_n$ is generated by its l-generator groups but not by its $(l-1)$-generator groups. For $c = 1$, that is $\mathfrak{N}$ abelian, C. H. Houghton proved, generalizing a result of Graham Higman [6], that $\mathfrak{A}_m\mathfrak{A}_n$ is generated by its 2-generator groups. In fact he showed that $C_m wr C_n$ generates the product variety (unpublished). (Here C_m, C_n denote cycles of orders m, n.) As a generalization of this it can be shown easily from the structure of the critical groups in $\mathfrak{N}\mathfrak{A}_n$, obtained in § 2, that the verbal wreath product $W_c = F_c(\mathfrak{N}) wr_{\mathfrak{N}} F_c(\mathfrak{A}_n)$ generates $\mathfrak{N}\mathfrak{A}_n$, where $F_c(\mathfrak{N})$ and $F_c(\mathfrak{A}_n)$ are the free groups of rank c of $\mathfrak{N}$ and $\mathfrak{A}_n$ respectively. This yields $l \leqq 2c$ immediately. However the following theorem is proved.

THEOREM. *The variety $\mathfrak{N}\mathfrak{A}_n$, where $\mathfrak{N}$ is any nilpotent variety of class $c > 1$ and exponent m, with $(m, n) = 1$, is generated by its c-generator groups but not by its $(c-1)$-generator groups. That is $l(\mathfrak{N}\mathfrak{A}_n) = c$.*

The precision of this result contrasts with the result of Higman [6], that any nilpotent variety of class c is generated by its c-generator groups, and the result of Gilbert Baumslag, B. H. Neumann, Hanna Neumann and Peter M. Neumann [1], that even for the variety $\mathfrak{B}$ say, of all nilpotent groups of class $\leqq c$, we have

$$[c/2] \leqq l(\mathfrak{B}) \leqq c,$$

[1] This and other terms used here seem to be standard now. Full definitions may be found in [6] and [13].

with no more precise bounds yet known.

In § 2, $l(\mathfrak{N}\mathfrak{A}_n) \leqq c$ is proved (this is due to L. G. Kovács) and in § 5 the properties of W_c are exploited to give the reverse inequality.

It is also shown in § 2 that every critical group in $\mathfrak{N}\mathfrak{A}_n$ needs at most $c+1$ generators and in § 3 critical groups actually requiring $c+1$ generators are constructed for $c = 2$ and certain pairs of exponents m, n. (Cf. the result of Paul M. Weichsel [15] that critical groups of class $\leqq c$ are c-generator.)

This, together with the result of Hanna Neumann (Theorem 3.1) that a variety generated by a single critical group which is strictly k-generator, cannot be generated by its $(k-1)$-generator groups, prompts the following question. Can one construct, for any pair of integers $k, l > 0$, a variety generated by its k-generator groups and also by a set S of critical groups some of which are strictly $(k+l)$-generator? This question is answered affirmatively for $k = 2$ and arbitrary l, mainly by reference to [2]. However, if we insist that in the above question not all of the critical groups in S requiring $> k$ generators be redundant; that is, there exist $G \in S$ requiring $> k$ generators such that $\operatorname{var}(S) \neq \operatorname{var}(S \setminus \{G\})$, then the question remains completely unanswered.

In § 4 a characterization of the normal closure of the top group of an arbitrary verbal wreath product is obtained. This generalizes a corresponding result of Peter M. Neumann [12] on the standard restricted wreath product.

In § 5, a special case of the main result of § 4 (Corollary 4.2) and an embedding theorem for a more restricted class of verbal wreath products, are used to prove $l(\mathfrak{N}\mathfrak{A}_n) \geqq c$. The embedding theorem is also contained in A. L. Šmel'kin's paper [14] but the proof given here is short and more direct.

We shall denote the least number of generators required to generate a group G by $d(G)$. The definition and some properties of the verbal wreath product may be found in [14] from which also the notation has been taken over.

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2. Upper bounds

In this section we prove that:

2.1 $d(G) \leqq c+1$ *for any critical group* G *in* $\mathfrak{N}\mathfrak{A}_n$ *and for all* $c \geqq 1$;

2.2. $l(\mathfrak{N}\mathfrak{A}_n) \leqq c$ *for all* $c > 1$.

PROOF OF 2.1. Firstly, for all $c \geqq 1$ we obtain the bound $2c$ for $d(G)$ where G is any critical group in $\mathfrak{N}\mathfrak{A}_n$. The following argument is due to L. G. Kovács and is derived in part from an argument in the paper [7] of Kovács and Newman.

Lemma 2.4.2 of Oates and Powell [13] is used:

2.3. *If a group G has a set of normal subgroups $M_1, \cdots, M_s$ and a subgroup L such that*

2.3.1 $G = LM_1 \cdots M_s$;

2.3.2 *G is not generated by L together with any proper subset of the set* $\{M_1, \cdots, M_s\}$;

2.3.3 $[M_{\pi(1)}, \cdots, M_{\pi(s)}] = E$, *the unit subgroup, for every permutation π of the integers* $1, \cdots, s$;

then G is not critical.

Now return to the critical group G in $\mathfrak{N}\mathfrak{A}_n$. We may assume G is non-abelian since otherwise G is cyclic and $d(G) = 1$. By definition, a critical group is finite. Let $F = F(G)$ be the greatest normal nilpotent subgroup (the Fitting subgroup) and $\Phi = \Phi(G)$ the Frattini subgroup of G. Since Φ is nilpotent and normal, $\Phi \leqq F$. F is a p-group for some prime p, otherwise G would not be monolithic (i.e. it would not possess a unique minimal normal subgroup). Furthermore $p \mid m$ where m is the exponent of $\mathfrak{N}$. For, by the definition of a product variety and since G has been assumed non-abelian, G is the extension of a non-trivial subgroup S in $\mathfrak{N}$ by a group in $\mathfrak{A}_n$, and we must have $S \leqq F$. Since $(m, n) = 1$ there is, by the Schur-Zassenhaus Theorem, a complement L of F in G. It follows that $L\Phi/\Phi$ ($\cong L$) is a complement of F/Φ in G/Φ. By Theorems 2, 5 and 9 of Gashütz [4], F/Φ is an elementary abelian p-group (Clearly $\Phi \neq F$.) Write

$$F/\Phi = M_1/\Phi \times \cdots \times M_s/\Phi,$$

where M_i/Φ, $i = 1, \cdots, s$, is an elementary abelian minimal normal subgroup of G/Φ. That such a decomposition exists is a consequence of Maschke's Theorem (see for example [5] p. 253): F/Φ may be regarded as an L/Φ-module over $GF(p)$. Then $G = LM_1 \cdots M_s$ and conditions 2.3.1 and 2.3.2 are satisfied. If $s > c$, condition 2.3.3 will also be satisfied, contrary to the criticality of G. Thus $s \leqq c$.

Since Φ is the (finite) set of non-generators of G, $d(G/\Phi) = d(G)$. Therefore we can restrict our attention to G/Φ. Write $G/\Phi = G_1$, $L\Phi/\Phi = L_1$, $F/\Phi = F_1$, and $M_i/\Phi = N_i$, $i = 1, \cdots, s$. Let x_i be any non-trivial element from N_i, $i = 1, \cdots, s$. Then

2.4 $$G_1 = sgp\{L_1, x_1, \cdots, x_s\},$$

since the conjugates of x_i under L_1 must together generate the whole of the minimal normal subgroup N_i. We now bound $d(L_1)$. Let K_i be the kernel of the representation of L_1 on N_i, $i = 1, \cdots, s$: that is $K_i = C_{L_1}(N_i) \lhd L_1$. By Theorem 10 of [4], $F(G/\Phi) = F/\Phi$. This, together with the abelianness of L_1, implies that $C_{G_1}(F_1) \leqq F_1$: that is, L_1 is faithfully represented on $F_1 = N_1 \times \cdots \times N_s$. Therefore $\bigcap_{i=1}^s K_i = E$. Now L_1/K_i is abelian and is represented faithfully and irreducibly on N_i. By a classical theorem of representation theory, this implies that L_1/K_i is cyclic. Thus L_1 contains s normal subgroups intersecting trivially and with cyclic factor groups. It follows that L_1 is embeddable in the direct product $L_1/K_1 \times \cdots \times L_1/K_s$ of s cycles and that $d(L) \leqq s$. We have $d(G) = d(G_1) \leqq 2s \leqq 2c$.

Secondly, we reduce this bound to $c+1$ with the help of the following lemma.

2.5 Lemma. *Let B be a finite abelian group with $\leqq s$ generators and let $B_1, \cdots, B_{s-1}$ be $s-1$ subgroups such that B/B_i is cyclic, $i = 1, \cdots, s-1$. Then there exists a set $\{g_1, \cdots, g_s\}$ of generators of B such that $g_i \in B_i$ for $i = 1, \cdots, s-1$.*

Proof. If B is trivial then so also is the lemma. Assume $B \neq E$.

Write B as the direct product of its Sylow subgroups: $B = S_{p_1} \times \cdots \times S_{p_k}$ say. Then, for each j, $1 \leqq j \leqq k$, S_{p_j} and $B_1 \cap S_{p_j}, \cdots, B_{s-1} \cap S_{p_j}$ satisfy the conditions of the lemma. If there exists a set $\{g_{ip_j}, \cdots, g_{sp_j}\}$ of generators of S_{p_j} such that $g_{ip_j} \in B_i \cap S_{p_j}$, $i = 1, \cdots, s-1$, put

$$g_i = \prod_{j=1}^{k} g_{ip_j}, \qquad i = 1, \cdots, s.$$

Then $\{g_1, \cdots, g_s\}$ satisfies the requirements of the lemma.

Hence we may assume B is a p-group. We use induction on s. For $s = 1$ the lemma holds vacuously. Suppose $s > 1$ and assume the lemma true for $s-1$. Write $B = C_{p^{n_1}} \times \cdots \times C_{p^{n_s}}$ where $n_i \geqq 0$, $i = 1, \cdots, s$, and $n_s \geqq n_j$, $j < s$, and let y_i be a generator of $C_{p^{n_i}}$. Consider the case $B_1 < B$. In this case there exists some element $y = y_1^{\alpha_1} \cdots y_{s-1}^{\alpha_{s-1}} y_s \notin B_1$ since elements of this form generate B. Hence $B = C_{p^{n_1}} \times \cdots \times C_{p^{n_{s-1}}} \times sgp\{y\} = A \times sgp\{y\}$ say. For this decomposition of B, the projection of B_1 on A must be the whole of A since, if $a \neq 1$ were not in this projection then, modulo B_1, the set $\{a, y\}$ would generate a non-cyclic group. Thus for $B_1 \leqq B$ there is a generating set $\{c_1, \cdots, c_{s-1}, g_s\}$ for B such that $c_1, \cdots, c_{s-1} \in B_1$.

Write $H = sgp\{c_1, \cdots, c_{s-1}\} \leqq B_1$ and consider the cyclic group $HB_i/B_i \cong H/H \cap B_i$, $i = 2, \cdots, s-1$. The groups H and $H \cap B_2, \cdots, H \cap B_{s-1}$ satisfy the conditions of the lemma and so, by the inductive hypothesis, there exist generators $g_1, \cdots, g_{s-1}$ for H such that

$g_i \in H \cap B_i \leqq B_i$, for $i = 2, \cdots, s-1$. Since $g_1 \in B_1$, the set $\{g_1, \cdots, g_s\}$ satisfies the requirements of the lemma. This completes the proof of the inductive step and the proof of the lemma.

For the application of this we return to 2.4: $G_1 = sgp\{L_1, x_1, \cdots, x_s\}$. L_1 and $K_1, \cdots, K_{s-1}$ satisfy the conditions of 2.5 and hence there is a generating set $\{l_1, \cdots, l_s\}$ for L_1 with $l_i \in K_i$ for $i = 1, \cdots, s-1$. Since K_i centralizes x_i and they have coprime orders, the set $\{l_1 x_1, \cdots, l_{s-1} x_{s-1}, l_s, x_s\}$ generates G_1. This completes the proof of 2.1.

One further lemma will complete the proof of 2.2. Since $\mathfrak{N}\mathfrak{A}_n$ is generated by its critical groups we have immediately from 2.1 that $l(\mathfrak{N}\mathfrak{A}_n) \leqq c+1$. In § 3, for $c = 2$ and certain m, n, critical groups in $\mathfrak{N}\mathfrak{A}_n$ having not fewer than 3 generators are constructed. Thus we cannot hope to obtain the upper bound c for $l(\mathfrak{N}\mathfrak{A}_n)$ by considering the critical groups alone. However, L. G. Kovács has proved the following result.

2.6 Lemma. *For each critical group G in $\mathfrak{N}\mathfrak{A}_n$ with $s > 1$ defined as previously, there exists an s-generator permutational verbal wreath product lying in $\mathfrak{N}\mathfrak{A}_n$ and having G as a factor.*

Proof. We retain the notation of the proof of 2.1 for the relevant subgroups etc. of the critical group G. Identify L_1 with L under the mapping $l\Phi \to l, l \in L$. Then $L/K_i, i = 1, \cdots, s$, is a cyclic group of order dividing n. Let Z_i be a group isomorphic to L/K_i, $i = 1, \cdots, s$, such that $Z_j \cap Z_k = \emptyset$ for $j \neq k$. Let $\theta_i : L/K_i \to Z_i$ be an isomorphism, $i = 1, \cdots, s$. Form the set-theoretical union $\bigcup_{i=1}^{s} Z_i, = Z$ say. Then for each $z_i \in Z_i$, $i = 1, \cdots, s$, a permutation ζ_i of Z is defined as follows:

$$z\zeta_i = zz_i \text{ if } z \in Z_i;$$
$$z\zeta_i = z \quad \text{if } z \notin Z_i.$$

The group Q generated by all such permutations is isomorphic to $Z_1 \times \cdots \times Z_s$ and the restriction to $Z_i \subsetneqq Z$ of this group of permutations is the right regular representation of Z_i. Take $|Z|$ distinct isomorphic copies of the p^α-cycle C_{p^α} where p^α is the exponent of $F(G)$, and denote them by $C_{p^\alpha}(z)$, $z \in Z$. We form the verbal $\mathfrak{N}$-product (see S. Moran, [9])

$$B = \prod_{z \in Z}^{\mathfrak{N}} C_{p^\alpha}(z),$$

and split-extend B by Q in the usual way for permutational wreath products: that is, the action of $\zeta \in Q$ on $C_{p^\alpha}(z)$ is defined by

$$(a(z))^\zeta = a(z\zeta),$$

for $a \in C_{p^\alpha}$. Then BQ is the permutational verbal wreath product mentioned in the lemma. Obviously $BQ \in \mathfrak{N}\mathfrak{A}_n$.

We now choose a subgroup of BQ and find an epimorphism from this subgroup onto G. Let ψ be the isomorphism $Z_1 \times \cdots \times Z_s \to Q$ defined by

$$z_i \psi = \zeta_i, \quad z_i \in Z_i, \quad i = 1, \cdots, s.$$

Then the mapping

$$\theta : l \to (lK_1)\theta_1\psi \cdot (lK_2)\theta_2\psi \cdots (lK_s)\theta_s\psi, \; l \in L,$$

is a monomorphism of L into Q. Write $L\theta = L_2 \leqq Q$. Let $b_i \in M_i \diagdown \Phi(G)$, e_i be the identity of Z_i, $i = 1, \cdots, s$, and let a generate C_{p^α}. Then the mapping

$$\begin{aligned} a(e_i) &\to b_i, \quad i = 1, \cdots, s; \\ l\theta &\to l, \quad l \in L, \end{aligned}$$

can be extended to an epimorphism of BL_2 onto G. This follows from the structure of G obtained in the proof of 2.1, from the freeness of B in $\mathfrak{N}$ and the well-known von Dyck's Theorem ([8], Vol. I, p. 130).

Finally we find the number of generators sufficient to generate BQ. If $s = 1$, BQ is 2-generator. If $s > 1$, BQ is s-generator: for, if $a(e_i)$ generates $C_{p^\alpha}(e_i)$ and z_i generates Z_i, $i = 1, \cdots, s$, then BQ is generated by the set $\{a(e_i), z_i\psi \mid i = 1, \cdots, s\}$; but $a(e_j)$ and $z_k\psi$ commute if $j \neq k$ and have comprime orders and hence $\{a(e_i) \cdot (z_{(i+1) \bmod s}\psi) \mid i = 1, \cdots, s\}$ also generates BQ. This completes the proof of 2.6 and hence 2.2.

3. Critical groups

Firstly we construct the critical groups promised in § 1, to show that the bound $c+1$ for the least number of generators of the critical groups in $\mathfrak{N}\mathfrak{A}_n$ is best possible, at least in some cases.

We know from 2.2 that, in particular, for $c = 2$, $\mathfrak{N}\mathfrak{A}_n$ is generated by its 2-generator groups. It is also generated by its critical groups.

For a pair of primes p, q related in a way to be described, we shall construct a critical group $G = G(p, q)$ of exponent pq which is strictly 3-generator and is an extension of a nilpotent group of class 2 and exponent p, by an abelian group of exponent q.

Suppose a prime $p > 2$ is given arbitrarily and let q be any prime dividing $p-1$. Then there exists an r, $1 < r < p$, such that $r^q \equiv 1 (\mathrm{mod}\, p)$ since the non-zero integers modulo p form the multiplicative group of $GF(p)$. Suppose N is the reduced free group of exponent p and class 2 on two free generators a, b, and let x generate a cycle of order q. G is the splitting extension of N by $sgp\{x\}$ obtained by defining $a^x = a^r$, $b^x = b^r$. Thus

$$G = gp\{N, x \mid x^q = 1, a^x = a^r, b^x = b^r\}.$$

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G is critical because any proper factor of N is abelian whence it follows easily that any proper factor of G is metabelian, whereas $[a, x] = a^{r-1}$ and $[b, x] = b^{r-1}$ generate the non-abelian group N, so that G itself is not metabelian.

For the verification of $d(G) = 3$ we proceed as follows. If G is 2-generator, some pair $\{xu, v\}$, $u, v \in N$, will do to generate it. There exists $w \in N$ such that $w^{r-1} = u$ since $(r-1, p) = 1$. Thus $xu = xw^r w^{-1} = xw^x w^{-1} = wxw^{-1}$ and $G = sgp\{x, v'\}$ where $v' = w^{-1}vw$. However, modulo $sgp\{[a, b]\}$, the monolith of G, $sgp\{x, v'\}$ is an extension of C_p by C_q, whereas mod $sgp\{[a, b]\}$, G is an extension of $C_p \times C_p$ by C_q. Thus we have reached a contradiction and $d(G) = 3$.

Let $\mathfrak{V}$ be a variety generated by its critical groups and also by its k-generator groups, for some finite k. We now touch briefly on the question whether or not there exists a connexion between $l(\mathfrak{V})$ and the numbers of generators required by the critical groups in $\mathfrak{V}$.

We prove firstly the following theorem.

3.1 Theorem. (Hanna Neumann) *The variety generated by a single critical group G, with $d(G) = k$, is not generated by its $(k-1)$-generator groups.*

Proof. Suppose the theorem false and that G is a counterexample, G critical, $d(G) = k$. Denote by F_{k-1} the reduced free group of rank $k-1$ of $\mathrm{var}(G)$. Then $\mathrm{var}(G) = \mathrm{var}(F_{k-1})$ by the supposition. Hence F_{k-1} is isomorphic to a factor of a cartesian power G^I where $I \neq \emptyset$ is some index set. Since F_{k-1} is free in $\mathrm{var}(G)$, it is in fact embeddable in $G^I : F_{k-1} \cong A \leqq G^I$, say. Let θ_i, $i \in I$, be the projection of G^I on its i-th co-ordinate. If $A_i \cong G$ for some $i \in I$, G would be isomorphic to a factor group of $A \cong F_{k-1}$ and would then have fewer than k generators. Thus $A\theta_i$ is isomorphic to a proper subgroup of G for all $i \in I$. Hence F_{k-1} is in the variety generated by all $A\theta_i$ which is in turn in the variety generated by the proper subgroups of G. This gives us a contradiction and completes the proof.

This gives rise to the question asked in § 1.

It can be proved (see [2], chapter 6) that if $\mathfrak{V}$ (as above) is also Cross, then it is generated by its k-generator critical groups. Assuming that $l(\mathfrak{N}\mathfrak{A}_n) = c$ (the proof of this is completed in § 5) it follows that the Cross variety $\mathfrak{N}\mathfrak{A}_n$ must contain at least one critical group G with $d(G) = c$. Hence there exists for each $c \geqq 1$, a critical group G_c such that $d(G_c) = c$. The group G_c can be embedded in some finite symmetric group $S(c)$ say, which has a set of two generators. Thus $\mathrm{var}(S(c))$ is generated by its 2-generator groups (and its critical groups since it is Cross) and contains at least one critical group requiring c generators. This answers the question asked in § 1, for $k = 2$.

However the question of the existence of a set of critical groups some of which require $> k$ generators, generating irredundantly (in the sense given in § 1) a variety generated by its k-generator groups, remains unanswered.

An an example of what may happen, let us return for the moment to the variety $\mathfrak{N}\mathfrak{A}_q$, where $\mathfrak{N}$ is assumed to be of class 2 and exponent $p > 2$, and the primes p, q are such that $q \mid p-1$. (There is only one such variety $\mathfrak{N}$.) In [2], chapter 6, the complete subvariety lattice of $\mathfrak{N}\mathfrak{A}_q$ is determined by means of its critical groups. It turns out that the strictly 3-generator group $G = G(p, q)$ constructed above is the only such in $\mathfrak{N}\mathfrak{A}_q$, and that if $\mathfrak{V}$ is any subvariety of $\mathfrak{N}\mathfrak{A}_q$ containing the critical group G, and such that $l(\mathfrak{V}) = 2$, then G can be omitted from any set S of critical groups generating $\mathfrak{V}$. That is, if $\mathfrak{V} = \operatorname{var}(S)$ then $\mathfrak{V} = \operatorname{var}(S \setminus \{G\})$.

4. The verbal wreath product

In this section we prove a theorem on the standard verbal wreath product $W = A \, wr_{\mathfrak{V}} B$ where A and B are arbitrary non-trivial groups and $\mathfrak{V}$ is an arbitrary variety. It is a generalization of Theorem 4.1 of [12] and part of the proof is a generalization of the proof in [12].

Since our only concern from now on will be with the standard verbal wreath product, the adjective "standard" will, for brevity, be dropped.

The theorem is applied in § 5 in the special case $V(A) = E$ (Corollary 4.2) to obtain a lower bound for $l(\mathfrak{N}\mathfrak{A}_n)$. For this special case a simpler proof is possible. However the full theorem is of independent interest and not much is saved by proving only the weaker version.

Before we can state the theorem, some definitions are needed. Suppose F is a free group on a countably infinite set $\{x_1, x_2, \cdots\}$ of free generators. Let V be any fully invariant subgroup of F and D the derived group of F. For any group G let $V(G)$ denote the verbal subgroup of G determined by V, and $\mathfrak{V}$ the variety of all groups G for which $V(G) = E$. V and D are verbal subgroups of F and therefore $V \cap D$ is fully invariant and hence also a verbal subgroup. It is easy to verify that, for any group G, $(V \cap D)(G) \leqq V(G) \cap D(G)$. In view of subsequent arguments it is worth while pointing out that B. H. Neumann [11] has found an example where the inequality is strict. (Cf. also S. Moran [10].). Finally we shall use without comment that $V(G\theta) = V(G)\theta$ for every homomorphism θ.

Let K denote the base group of W. Then we have the following result.

4.1 THEOREM. *Let $B_1 > E$ be normal in B, T_1 be any transversal for B_1 in B, and P denote the verbal product $\prod^{\mathfrak{V}}_{t \in T_1} A(t) < K$. Then*

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$$B_1^w \cap K = [B_1, K] = \{\alpha_1^{b_1}\alpha_2^{b_2}\cdots\alpha_r^{b_r} \mid r \geqq 1;\ b_i \in B_1,\ \alpha_i \in P,\ \textit{for}\ i = 1, \cdots, r;\ b_j \neq b_{j+1}\ \textit{for}\ j = 1, \cdots, r-1;\ \alpha_1\alpha_2\cdots\alpha_r \in (V \cap D)(P)\}.$$

Here B_1^w denotes the normal closure of B_1 in W.

The proof consists of a string of lemmas. The first two, 4.1.1 and 4.1.2, reduce the problem to the case $B_1 = B$. Lemma 4.1.1 is well known and the proof is omitted.

4.1.1 LEMMA. *If B_2 is any subgroup of B and T_2 is any left transversal for B_2 in B, then*

$$B_2 \cdot K \cong (\prod_{t\in T_2}^{\mathfrak{V}} A(t))\, wr_{\mathfrak{V}} B_2.$$

If, in the right hand side, we interpret the action of B_2 on $\prod_{t\in T_2}^{\mathfrak{V}} A(t)$ as in W, we may (and shall in future) replace $\cong$ by $=$.

4.1.2 LEMMA. *The normal closure of B_1 in W is $B_1[B_1, K]$.*

PROOF. The subgroup $[B_1, K]$ of W is normalized by K: for, if $b_1 \in B_1$; $k, k_1 \in K$, then

$$[b_1, k]^{k_1} = [b_1, k_1]^{-1}[b_1, kk_1] \in [B_1, K].$$

Also, $[B_1, K]$ is normalized by B since $B_1 \trianglelefteq B$ and $K \triangleleft W$. Hence $[B_1, K]$ is normal in $BK = W$.

The subgroup $B_1[B_1, K]$ is obviously normalized by B. Also, if $k \in K$, $b_1 \in B_1$, then

$$b_1^k = b_1[b_1, k] \in B_1[B_1, K].$$

Hence $B_1[B_1, K]$ is normal in W and must be the normal closure of B_1 in W.

4.1.3 COROLLARY. *The normal closure of B_1 in W is the same as its normal closure in B_1K.*

This follows at once from Lemma 4.1.2. This corollary, together with Lemma 4.1.1, allows us to assume that $B_1 = B$. The following few lemmas are concerned with this case and with a particular verbal wreath product.

We consider in particular the verbal wreath product $W^* = Awr_*B$ (which is in fact isomorphic to the free product $A*B$ — but we shall not need this) since the more general wreath product W is a factor group of it. The base group of W^* is the free product $\prod_{b\in B}^* A(b) = \prod^* A(b)$, say. Now by the definition of the verbal product,

$$K = \prod_{b\in B}^{\mathfrak{V}} A(b) = \prod{}^* A(b)/(V(\prod{}^* A(b)) \cap C),$$

where C is the cartesian subgroup of the free product $\prod^* A(b)$. We put for brevity, $\Lambda = V(\prod^* A(b)) \cap C$, so that

$$K = \prod{}^* A(b)/\Lambda; \quad W = W^*/\Lambda$$

Any element of $\prod^* A(b)$ can be written uniquely in the normal form $a_1(b_1) \cdots a_r(b_r)$ where $a_i \in A$, $b_i \in B$, $i = 1, \cdots, r$; $b_j \neq b_{j+1}$, $j = 1, \cdots, r-1$. Write

$$\begin{aligned} X &= \{a(1)a^{-1}(b) \mid a \in A,\, b \in B\} \\ &= \{[a^{-1}(1), b] \mid a \in A,\, b \in B\} \subset \prod{}^* A(b). \end{aligned}$$

Then the following is true.

4.1.4 Lemma. *If $a_1(b_1) \cdots a_r(b_r)$, $a_i \in A$, $b_i \in B$, $b_j \neq b_{j+1}$, is any element of $\prod^* A(b)$, then the element*

$$g = (a_1 \cdots a_r)^{-1}(b_1)a_1(b_1) \cdots a_r(b_r)$$

lies in $sgp(X)$.

Proof. Any element of the form $a^{-1}(b')a(b'')$, $a \in A$; $b', b'' \in B$, belongs to $sgp(X)$ since

$$a^{-1}(b')a(b'') = \big(a^{-1}(1)a(b')\big)^{-1}a^{-1}(1)a(b'') \in sgp(X).$$

Write

$$\begin{aligned} x_1 &= a_r^{-1}(b)_r a_r(b_{r-1}), \\ x_2 &= (a_{r-1}a_r)^{-1}(b_{r-1})(a_{r-1}a_r)(b_{r-2}), \\ x_{r-1} &= (a_2 a_3 \cdots a_r)^{-1}(b_2)(a_2 a_3 \cdots a_r)(b_1). \end{aligned}$$

Then

$$x_1 x_2 \cdots x_{r-1} = g^{-1},$$

and since $x_1 x_2 \cdots x_{r-1} \in sgp(X)$ by the preceding remark, the required result is obtained.

4.1.5 Corollary. *In W^*, $[B, \prod^* A(b)] = sgp(X)$.*

Proof. Obviously

$$[B, \prod{}^* A(b)] \geqq sgp(X).$$

To prove the reverse inclusion, we show that every generator $[b', k]$ of $[B, \prod^* A(b)]$, $b' \in B$, $k \in \prod^* A(b)$, lies in $sgp(X)$. Suppose $k = a_1(b_1) \cdots a_r(b_r)$ is in the normal form. Then

$$\begin{aligned} [b', k] &= \big(a_1(b_1 b') \cdots a_r(b_r b')\big)^{-1} a_1(b_1) \cdots a_r(b_r) \\ &= a_r^{-1}(b_r b') \cdots a_1^{-1}(b_1 b')a_1(b_1) \cdots a_r(b_r). \end{aligned}$$

Thus, by Lemma 4.1.4, $(a_r^{-1} \cdots a_1^{-1} a_1 \cdots a_r)(b_r b')[b', k] \in sgp(X)$; that is

$$[b', k] \in sgp(X),$$

and the proof is complete.

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The next lemma is concerned with the free product $\prod^* A(b)$ only.

4.1.6 LEMMA. *The epimorphism* $\theta : \prod^* A(b) \to A$ *defined by* $a(b)\theta = a$, $a \in A$, $b \in B$, (*that is,* θ *amalgamates the* $A(b)$) *maps* $\Lambda = V(\prod^* A(b)) \cap C$ *onto* $(V \cap D)(A)$.

This is slightly surprising as one might expect on the face of it that $\Lambda\theta = V(A) \cap D(A)$ which, as remarked above, it not always the same as $(V \cap D)(A)$.

Before proving this lemma we need the following result which for clarity we state and prove separately.

When we say that a word in F "involves" the variable x_i, we shall mean that the word contains x_i or x_i^{-1} when written in reduced form. A commutator with entries from $\{x_1^{\pm 1}, x_2^{\pm 1}, \cdots\}$ that involves x_i, takes the value 1 when x_i is replaced by 1. (See Higman [6], p. 169.)

4.1.7 LEMMA [2]. *The verbal subgroup* $V \cap D$ *is generated by the set of all those words* $w = w(x_1, \cdots, x_l)$ *in* $V \cap D$ *each of which can be written as a product* $c_1 \cdots c_s$ *say, of commutators of weight* > 1 *with entries from* $\{x_1^{\pm 1}, \cdots, x_l^{\pm 1}\}$, *such that there exist two distinct subscripts* j *and* k *with the property that each commutator* c_i, $1 \leqq i \leqq s$, *involves both* x_j *and* x_k.

PROOF. The proof is by induction on the number of variables a word in $V \cap D$ involves. If this number is 2, the word already has the required form. Suppose that words in $V \cap D$ involving fewer than l variables are products of words of the form w, and let $v = v(x_1, \cdots, x_l)$ be any element of $V \cap D$ which involves all of $x_1, \cdots, x_l$. Since $v \in D$, it can be written as a product of commutators of weight > 1 in $x_1^{\pm 1}, \cdots, x_l^{\pm 1}$. By using the identity $y_1 y_2 = y_2 y_1 [y_1, y_2]$ repeatedly, we see that we can write

$$v = v_1 v_2 v_3 v_4,$$

where the v_i, $i = 1, 2, 3, 4$, are products of commutators of weight > 1 in the x_i's and their inverses, and in v_1 all factors involve neither x_1 nor x_2; in v_2 all involve x_1 but not x_2; in v_3 all involve x_2 but not x_1; and the factors of v_4 all involve both x_1 and x_2. If we put successively $x_1 = x_2 = 1$; $x_1 = 1$; $x_2 = 1$, in v, we see that $v_1, v_2, v_3, v_4 \in V \cap D$. Now v_1, v_2 and v_3 involve at most $l-1$ variables and so, by the inductive hypothesis, they are products of words in $V \cap D$ of the right form. The element v_4 is already of the required form. Obviously no generality has been lost by working with the particular l variables $x_1, \cdots, x_l$, and the proof is complete.

PROOF OF 4.1.6. Let $w = w(x_1, \cdots, x_l)$ be as in the statement of Lemma 4.1.7: $w = c_1 \cdots c_s$ and the c_i simultaneously involve the variables

[2] This lemma was suggested by Professor Hanna Neumann as a correction of my original proof of 4.1.6.

x_j, x_k. Since $B \neq E$, there exist $b', b'' \in B$ for which $b' \neq b''$. Substitute in w any l elements $g_1, \cdots, g_l \in A$, for $x_1, \cdots, x_l$ respectively. The resulting element is in $(V \cap D)(A)$. On the other hand, if we substitute $g_i(b')$ for x_i, $i \neq k$, and $g_k(b'')$ for x_k, the form of w ensures that the resulting element is in Λ. Now

$$w(g_1(b'), \cdots, g_k(b''), \cdots, g_l(b'))\theta = w(g_1, \cdots, g_l),$$

and since, by 4.1.7, all such elements $w(g_1, \cdots, g_l)$ generate $(V \cap D)(A)$, we have proved that

$$\Lambda\theta \geqq (V \cap D)(A).$$

We now prove the reverse inclusion.

Let R be the kernel of an epimorphism ψ from a free group F_1 of suitable rank, onto $A : A \cong F_1/R$. Take $|B|$ isomorphic copies of F_1, denoted by $F_1(b)$, $b \in B$. Then corresponding to ψ we have, for each $b \in B$, the obvious epimorphism $\psi(b) : F_1(b) \to A(b)$. Consider the free product $\prod^*_{b \in B} F_1(b) = \prod^* F_1(b)$, say. Let ϕ be the epimorphism $\prod^* F_1(b) \to \prod^* A(b)$, whose restriction to $F_1(b)$ is $\psi(b)$. It then suffices to prove that

4.1.8 $$(V \cap D)(\textstyle\prod^* F_1(b))\phi \geqq \Lambda.$$

For,

$$(V \cap D)(\textstyle\prod^* F_1(b))\phi\theta = (V \cap D)(A),$$

and so, from 4.1.8,

$$(V \cap D)(A) \geqq \Lambda\theta.$$

4.1.8 is proved as follows. Let $v(\boldsymbol{x}) = v(x_1, \cdots, x_l) \in V$ be such that for some $k_1, \cdots, k_l \in \prod^* A(b)$, we have

$$v(k_1, \cdots, k_l) \in C.$$

Every element g of $\Lambda = V(\prod^* A(b)) \cap C$ is obtainable in this way from some $v(\boldsymbol{x}) \in V$. Suppose $f_i\phi = k_i$, where $f_i \in \prod^* F_1(b)$, $i = 1, \cdots, l$. We now consider the element $v(\boldsymbol{f}) = v(f_1, \cdots, f_l)$. Modulo its cartesian, $\prod^* F_1(b)$ is the direct product of the $F_1(b)$. From this, together with the fact that the complete inverse image of C under ϕ is the product of the cartesian of $\prod^* F_1(b)$ and the normal closure in $\prod^* F_1(b)$ of all $R(b)$, we see that we may write

$$v(\boldsymbol{f}) = d \cdot h_1(b_1) \cdots h_t(b_t),$$

where d belongs to the cartesian of $\prod^* F_1(b)$; $h_i \in R$, $b_i \in B$, $i = 1, \cdots, t$, and $b_j \neq b_k$ for $j \neq k$.

Next suppose $\{y_\gamma | \gamma \in \Gamma\}$ (Γ some index set) is a set of free generators of F_1. Then $\{y_\gamma(b) | \gamma \in \Gamma, b \in B\}$ generates $\prod^* F_1(b)$ freely and the f_i may be regarded as reduced words in these generators. Consider the subset

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$S = \{y_\gamma(b) \mid y_\gamma(b)$ occurs in f_i for some $i = 1, \cdots, l\}$, and let $\mu : gp(S) \to F$ be the monomorphism extending any (1, 1) mapping from S into the free generators $\{x_1, x_2, \cdots\}$ of F. Then $v(f)\mu \in V$. If, for some fixed j, $1 \leqq j \leqq t$, we set $x_i = 1$ in the word $v(f)$ whenever $x_i\mu^{-1} \notin F_1(b_j)$, then we find that also

$$h_j(b_j)\mu \in V, \quad j = 1, \cdots, t.$$

Hence $d\mu \in V$; obviously $d\mu \in D$. Write $d\mu = v'$.

Thus, for each element $g \in A$, we have found a word $v' \in V \cap D$ such that the substitution $\mu^{-1}\phi$ gives $g : v'\mu^{-1}\phi = g$. Now $v'\mu^{-1} \in (V \cap D)$ $(\prod^* F_1(b))$ and hence $(V \cap D)(\prod^* F_1(b))\phi \geqq A$ as required. This completes the proof of 4.1.8 and thence of Lemma 4.1.6.

4.1.9 Corollary. *For all $b \in B$, $(V \cap D)(A(b))$ is contained in $sgp(X)A$.*

Proof. Corollary 4.1.5 implies that $sgp(X)$ is normal in W^*. Let g be any element in $(V \cap D)(A)$. Then by Lemma 4.1.6 there exists an element $h = a_1(b_1) \cdots a_r(b_r)$ in A such that $a_1 \cdots a_r = g$. By Lemma 4.1.4 $g^{-1}(b_1)h \in sgp(X)$, whence $g(b_1) \in sgp(X)A$. The normality of $sgp(X)$ in W^* then gives the stated result.

Finally we return to W for which we obtain the following corollary.

4.1.10 Corollary. $[B, K] = sgp(X)A/A = M$, *where M is the set of cosets*

$$\{a_1(b_1) \cdots a_r)b_r)A \mid r \geqq 1; b_1 \in B, a_i \in A, i = 1, \cdots, r;$$
$$b_j \neq b_{j+1}, j = 1, \cdots, r-1; a_1 \cdots a_r \in (V \cap D)(A)\}.$$

Proof. It is easy to see from 4.1.6 that the set M forms a subgroup of W. Thus, since $XA/A \subset M$, we have $sgp(X)A/A \leqq M$. To prove the reverse inclusion write $h = a_1(b_1) \cdots a_r(b_r)$, where $a_1 \cdots a_r \in (V \cap D)(A)$. Then, by 4.1.4,

$$(a_1 \cdots a_r)^{-1}(b_1)a_1(b_1) \cdots a_r(b_r) \in sgp(X).$$

Therefore

$$h \in (V \cap D)(A(b_1)), sgp(X)$$

which by 4.1.9 is contained in $sgp(X) \cdot A$.

Proof of Theorem 4.1. The proof is immediate from Lemmas 4.1.1, 4.1.2 and Corollary 4.1.10).

For the application of Theorem 4.1 in § 5, we require it only in the case $V(A) = E$. In this case, for convenience, we restate it as a corollary.

4.2 Corollary. *With the same notation as in Theorem 4.1, suppose further that $V(A) = E$. Then the normal closure of the normal subgroup $B_1 > E$ of B (with a transversal T_1 in B) in $W = A wr_{\mathfrak{B}} B$, is $B_1 M_1$ where*

$$M_1 = \{\alpha_1^{b_1} \cdots \alpha_r^{b_r} \mid r \geqq 1;\, b_i \in B_1,\, \alpha_i \in \prod_{t \in T_1}^{\mathfrak{V}} A(t),$$
$$i = 1, \cdots, r;\, b_j \neq b_{j+1},\, j = 1, \cdots, r-1;\, \alpha_1\alpha_2 \cdots \alpha_r = 1\}.$$

5. Lower bound for $l(\mathfrak{N}\mathfrak{A}_n)$

In this section we aim at proving that

5.1 $l(\mathfrak{N}\mathfrak{A}_n) \geqq c$ *for all* $c \geqq 1$.

This, together with 2.2, yields $l(\mathfrak{N}\mathfrak{A}_n) = c$ for $c > 1$. (For $c = 1$, $l(\mathfrak{N}\mathfrak{A}_n) = 2$.)

For any variety $\mathfrak{V}$ we shall denote by $F_k(\mathfrak{V})$ the reduced free group of rank k of $\mathfrak{V}$. As mentioned in § 1, the method of proving 5.1 uses certain properties of the verbal wreath product

$$W_c = F_c(\mathfrak{N}) wr_{\mathfrak{N}} C_n^c,$$

where C_n^c is the direct product of c isomorphic copies of C_n. Clearly $W_c \in \mathfrak{N}\mathfrak{A}_n$. We show that if $W_c \in var(F_{c-1}(\mathfrak{N}\mathfrak{A}_n))$ then W_c is not only a factor of, but can be embedded in, a (finite) direct power of $F_{c-1}(\mathfrak{N}\mathfrak{A}_n)$. Hence there must be a set of normal subgroups of W_c, with trivial intersection, giving rise to factor groups embeddable in $F_{c-1}(\mathfrak{N}\mathfrak{A}_n)$. We prove that, for all possible such sets, the factor group of at least one normal subgroup is not so embeddable.

In addition to Corollary 4.2, a few lemmas are needed. The following lemma has also been proved by A. L. Šmel'kin [14]. His proof relies on the main theorem of that paper. We give a short, more direct proof.

5.2 Lemma. *Let* $\mathfrak{U}$ *and* $\mathfrak{V}$ *be locally finite varieties of coprime exponents* m *and* n *respectively. Then the verbal wreath product*

$$W(k) = F_k(\mathfrak{U}) wr_{\mathfrak{U}} F_k(\mathfrak{V})$$

can be embedded in $F_{2k}(\mathfrak{U}\mathfrak{V})$, *the free group of rank* $2k$ *of* $\mathfrak{U}\mathfrak{V}$, *for all* $k \geqq 1$.

Proof. Let F_{2k} be absolutely free on free generators $x_1, \cdots, x_{2k}$. Then, by the Schur-Zassenhaus Theorem and the conditions of the lemma, $F_{2k}(\mathfrak{U}\mathfrak{V}) = F_{2k}/U(V(F_{2k}))$ is a splitting extension of $V(F_{2k})/U(V(F_{2k}))$ by $F_{2k}(\mathfrak{V}) = F_{2k}/V(F_{2k})$.

If we write $F_k = sgp\{x_1, \cdots, x_k\}$, the same remark applied to $F_k < F_{2k}$ shows that there exists a set $T \subset F_k$ which is a transversal for $V(F_k)$ in F_k and is also, modulo $U(V(F_{2k}))$, a complement of $V(F_k)$ in F_k. Let T_1 be a right Schreier transversal for $V(F_k)$ in F_k. Since $V(F_k) = F_k \cap V(F_{2k})$, T and T_1 are subsets of some transversals for $V(F_{2k})$ in F_{2k}. For each $t_1 \in T_1$ there exists a unique $t \in T$ such that

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5.2.1 $t_1 = a_t t$,

where a_t depends on t and $a_t \in V(F_k)$. The mapping $T_1 \to T$ so defined is (1, 1) and onto.

We now show briefly that one can choose a right Schreier transversal T_2 for $V(F_{2k})$ in F_{2k} which contains the set

$$S = \{t_1 x_i^{\alpha_i} \mid t_1 \in T_1,\ 0 \leqq \alpha_i \leqq n-1,\ i = k+1, \cdots, 2k\},$$

by the following simple modification of the usual argument. It can easily be seen that the elements of S lie in distinct cosets and have the Schreier property. Choose the elements of S as representatives of their cosets. Let Q be the set of cosets not so represented. To find suitable representatives for the cosets in Q we use induction on the smallest length of elements in each coset in Q (as in [8], vol. II, p. 33). Thus we obtain the required T_2. (See also M. J. Dunwoody [3].)

A set of free generators of $V(F_{2k})$ is then

$$S_{2k} = \{t_2 x_i(\phi(t_2 x_i))^{-1} \mid i = 1, \cdots, 2k;\ t_2 \in T_2\} \setminus \{1\},$$

where $\phi(g)$, $g \in F_{2k}$, is the element of T_2 representing the coset $gV(F_{2k})$. The set

$$S_k = \{t_1 x_i(\phi(t_1 x_i))^{-1} \mid i = 1, \cdots, k,\ t_1 \in T_1\} \setminus \{1\}$$

is a set of free generators of $V(F_k)$. By the way T_2 was chosen, $S_k \subset S_{2k}$.

Write

$$\begin{aligned} X &= \{t_2 x_i(\phi(t_2 x_i))^{-1} \mid t_2 = t_1 x_i^{n-1},\ t_1 \in T_1,\ i = k+1, \cdots, 2k\} \\ &= \{(x_i^n)^{t_1^{-1}} \mid t_1 \in T_1,\ i = k+1, \cdots, 2k\} \subset S_{2k} \setminus S_k. \end{aligned}$$

By 5.2.1,

$$(x_i^n)^{t_1^{-1}} = \left((x_i^n)^{t^{-1}}\right)^{a_t^{-1}},\ a_t^{-1} \in sgp(S_k).$$

Write

$$Y = \{(x_i^n)^{t^{-1}} \mid t \in T,\ i = k+1, \cdots, 2k\}.$$

Since $a_t \in sgp(S_k)$ for all $t \in T$ and Y is obtained from $X \subset S_{2k} \setminus S_k$ by suitable conjugation by the a_t, it follows that the set $(S_{2k} \setminus X) \cup Y$ is an alternative set of free generators of $V(F_{2k})$.

Modulo $U(V(F_{2k}))$, $sgp\{x_i^n \mid i = k+1, \cdots, 2k\}$ is isomorphic to $F_k(U)$. Because of the way T was chosen, we have, modulo $U(V(F_{2k}))$,

$$\begin{aligned} & sgp\{T, x_i^n \mid i = k+1, \cdots, 2k\} \\ \cong\ & sgp\{x_i^n \mid i = k+1, \cdots, 2k\} wr_{\mathfrak{U}}\, sgp(T) \end{aligned}$$

which is isomorphic $\left(\operatorname{mod} U(V(F_{2k}))\right)$ to $W(k)$. This completes the proof.

5.3 Corollary. *If $F_\nu(\mathfrak{U}\mathfrak{V})$ generates $\mathfrak{U}\mathfrak{V}$ for some cardinal ν, then $W(k)$ can be embedded in some finite direct power of $F_\nu(\mathfrak{U}\mathfrak{V})$.*

PROOF. Any free group of $\mathfrak{U}\mathfrak{V}$ can be embedded in some suitably large finite direct power of $F_\nu(\mathfrak{U}\mathfrak{V})$. In particular, this applies to $F_{2k}(\mathfrak{U}\mathfrak{V})$ and thus to $W(k)$ by Lemma 5.2.

5.4 LEMMA. *If $n = p$, a prime, then every non-empty set of normal subgroups of $W_c = F_c(\mathfrak{N}) wr_{\mathfrak{N}} C_n^c$, such that none of the normal subgroups is wholly contained in the base group, has non-trivial intersection.*

For the proof a corollary of the following lemma is needed.

5.4.1 LEMMA. (Cf. Higman [6].) *Let F_k be (absolutely) free on $x_1, \cdots, x_k$, $k \geqq 2$, and let $\mathfrak{V}$ be a nilpotent variety of class c. Denote the subsets of $\{x_1, \cdots, x_k\}$ containing not less than 2 and not more than c elements by $S_1, S_2, \cdots, S_l$, and form $C(S_i) = sgp\{\gamma \mid \gamma$ a commutator of weight c with set of entries precisely $S_i\}$. Then, if θ is the natural homomorphism $F_k \to F_k/V(F_k)$, we have*

$$sgp\{C(S_i) \mid i = 1, \cdots, l\}\theta = C(S_1)\theta \times \cdots \times C(S_l)\theta.$$

PROOF. It suffices to prove that if

$$\gamma_1\gamma_2 \cdots \gamma_l \in V(F_k),$$

where $\gamma_i \in C(S_i)$, $i = 1, \cdots, l$, then

$$\gamma_i \in V(F_k) \text{ for each } i.$$

We use induction on the cardinal of S_i. Let ϕ_i be the endomorphism of F_k which fixes the elements of S_i and maps all other free generators in $\{x_1, \cdots, x_k\}$ onto 1. Then if $|S_i| = 2$,

$$(\gamma_1\gamma_2 \cdots \gamma_l)\phi_i = \gamma_1\phi_i\gamma_2\phi_i \cdots \gamma_l\phi_i = \gamma_i.$$

Therefore $\gamma_i \in V(F_k)$. Assume as inductive hypothesis that $\gamma_j \in V(F_k)$ for all S_j with $|S_j| \leqq s < k$. We may then omit from $\gamma_1 \cdots \gamma_l$ those γ_j's whose corresponding S_j's contain $\leqq s$ elements, and then have the remaining product still belonging to $V(F_k)$. If $|S_i| = s+1$, an application of ϕ_i to this smaller product completes the proof of the inductive step and thence the proof of the lemma.

We apply this lemma to the base group K of W_c. Suppose $x_1, \cdots, x_c$ freely generate $F_c(\mathfrak{N})$, the bottom group of W_c. Then there is a commutator of weight c with its set of entries precisely $\{x_1, \cdots, x_c\}$ which does not reduce to 1. Otherwise, since $F_c(\mathfrak{N})$ is reduced free, all commutators of weight c would be 1 and $F_c(\mathfrak{N})$ would be nilpotent of class $< c$. Denote such a commutator by $\gamma(x_1, \cdots, x_c)$. An obvious set of free generators of K is

$$\{x_i(b) \mid i = 1, \cdot\cdot, c;\ b \in C_n^c\}$$

Then we have

5.4.2 Corollary. *The elements of the set*

$$\{\gamma(x_1(b_1), \cdots, x_c(b_c)) \mid b_i \in C_n^c\},$$

are all non-trivial of order m_1 where m_1 divides m, the exponent of $\mathfrak{N}$, and independent. If $m_1 = qm_2$ where q is a prime, then

$$S = \{(\gamma(x_1(b_1), \cdots, x_c(b_c)))^{m_2} \mid b_i \in C_n^c\}$$

is a basis for a direct product of n^{c^2} isomorphic copies of C_q, considered as a vector space of dimension n^{c^2} over $GF(q)$.

Proof of Lemma 5.4. We shall use P. Hall's well known theorem (see [5], p. 141) the relevant parts of which we shall state here for convenience.

5.4.3 *Let G be a soluble group of order rs where $(r, s) = 1$.*
Then

5.4.4 *G has at least one subgroup of order r;*

5.4.5 *any two subgroups of order r are conjugate;*

5.4.6 *any subgroup whose order divides r, is contained in a subgroup of order r.*

Since $\mathfrak{N}\mathfrak{A}_n$ is a soluble, locally finite variety, we may apply this theorem to any of its finitely generated groups.

We have $|W_c| = p^c t$ where $(p, t) = 1$. (We are now dealing with the case $W_c = F_c(\mathfrak{N}) wr_{\mathfrak{N}} C_p^c$ where p is a prime.) The base group K has order t and $K \lhd W_c$. By 5.4.5. K is then the only subgroup of order t and by 5.4.6 K contains every subgroup of order prime to p. Let N be any normal subgroup of W_c not wholly contained in K. Suppose $|N| = p^\alpha t_1$ where $(p, t_1) = 1$. Then by the above, $\alpha > 0$. By 5.4.4 N contains a subgroup of order t_1 and this must be $N \cap K$. Again by 5.4.4 $N \cap K$ is complemented in N by B, say, and by 5.4.5, 5.4.6 we may choose B to be a subgroup of the top group C_p^c of W_c.

Thus the truth of the lemma will follow if we show that the intersection of the normal closures in W_c of all p-cycles contained in C_p^c, is non-trivial. This we proceed to do.

Denote the elements of S (5.4.2) by $\gamma_1, \cdots, \gamma_s$ where $s = p^{c^2}$. We shall prove that an element of K of the form

$$x = \gamma_1^{l_1} \cdots \gamma_s^{l_s}$$

lies in the above-mentioned intersection for some $l_1, \cdots, l_s$ not all $\equiv 0 \pmod q$.

Let $\Gamma \cong C_p$ be any p-cycle in the top group C_p^c of W_c and let T be any transversal for Γ in C_p^c. Then $|T| = p^{c-1}$. Denote by ϕ_Γ the mapping of

C_p^c into itself which sends each element onto its representative in T. Then let $\phi'_\Gamma : sgp(S) \to sgp(S)$ be the homomorphism obtained by extending the mapping defined by

$$\gamma(x_1(b_1), \cdots, x_c(b_c))\phi'_\Gamma = \gamma(x_1(b_1\phi_\Gamma), \cdots, x_c(b_c\phi_\Gamma)).$$

By Corollary 4.2, $\gamma_1^{l_1} \cdots \gamma_s^{l_s} \in \Gamma^{W_c}$ if and only if

5.4.7 $$(\gamma_1^{l_1} \cdots \gamma_s^{l_s})\phi'_\Gamma = 1.$$

If we collect the γ_i's which have become identified under ϕ'_Γ, then in the collected expression we must have powers of distinct γ_i's congruent to $0 \pmod q$ for 5.4.7 to be satisfied. In this way we are led to $|T|^c = p^{c(c-1)}$ linear homogeneous equations in $l_1, \cdots, l_s$ over $GF(q)$. Thus $\gamma_1^{l_1} \cdots \gamma_s^{l_s} \in \Gamma^{W_c}$ if and only if $l_1, \cdots, l_s$ is a solution of this system of equations. It follows that if x is to lie in the intersection of the normal closures of all p-cycles contained in C_p^c, $l_1, \cdots, l_s$ must be a solution simultaneously of the corresponding systems of linear equations.

Since there are $(p^c-1)/(p-1)$ distinct p-cycles in C_p^c, we get in all $(p^c-1)/(p-1) \cdot p^{c(c-1)}$ (not necessarily independent) equations whose solutions are precisely the admissible values for $l_1, \cdots, l_s$. Now

$$s = p^{c^2} > \frac{p^c-1}{p-1} \cdot p^{c(c-1)}$$

for all p and therefore there exist non-trivial solutions. This completes the proof.

Proof of 5.1 : $l(\mathfrak{N}\mathfrak{A}_n) \geqq c$. Suppose $F_{c-1}(\mathfrak{N}\mathfrak{A}_n)$ generates $\mathfrak{N}\mathfrak{A}_n$. Then, by Corollary 5.3, we can find a (finite) set Σ of normal subgroups of W_c with trivial intersection and with factor groups embeddable in $F_{c-1}(\mathfrak{N}\mathfrak{A}_n)$. Let p be any prime dividing n, and consider the unique subgroup H of the top group C_n^c of W_c such that $H \cong C_p^c$. At least one normal subgroup, say $M \in \Sigma$, must intersect H trivially. For otherwise Lemma 5.4 together with Lemma 4.1.1 would tell us that $\bigcap_{M \in \Sigma} M \neq E$. Thus W_c/M contains a subgroup isomorphic to C_p^c and therefore, by our supposition, so does $F_{c-1}(\mathfrak{N}\mathfrak{A}_n)$. If $F_{c-1}(\mathfrak{N}\mathfrak{A}_n)$ contains a subgroup $G \cong C_p^c$, then, by 5.4.3, G is contained in a subgroup isomorphic to C_n^{c-1}, which is impossible. We have reached a contradiction and the proof is complete.

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