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ON TRANSPORTABLE FORMS

BY
TAPIO KLEMOLA

1. **Introduction.** In the theory of deformations of compact complex manifolds, the hypothesis of constancy of the dimension of diverse structural cohomology groups pertaining to a fibre plays an important role (see, for instance, [3, Propositions 2.5 and 2.7, Theorems 2.2 and 2.3, and Definition 6.1]). This paper is the first of two devoted to the investigation of conditions under which constancy of the dimension of given cohomology groups is assured, and more generally, to the study of the variation of that dimension.

In [2] Griffiths introduces extendible forms in a holomorphic deformation. We consider in this paper a differentiable family, and besides extendible, also co-extendible and transportable forms (see §5), and deduce from their existence conclusions about the variation of the dimension of the corresponding structural cohomology groups. It is left to a subsequent paper(1) to give more explicit conditions by means of cohomology operations, and to deal with some applications.

§2 recalls a few central facts from the theory of linear elliptic operators depending on a parameter, and §3 gives more specific results, needed in the sequel, concerning "Laplacian" operators. In §4 transportable forms are defined and regular and singular points of deformed cohomology groups are introduced. §5 introduces extendible and co-extendible forms and contains the main results of the paper. In §6, we introduce an example in terms of a finite chain complex, which, even if not a genuine counter-example, suggests that in general an extendible and co-extendible form need not be transportable(2).

A general reference for §§2 and 3 is [4](3).

2. **Preliminaries.** Let $M=\{t\in R^m \mid |t|<\varepsilon\}$, V a compact differentiable manifold of dimension n (differentiable always in the sense C^∞) and suppose $\mathscr{A}$ is a differentiable complex vector bundle of finite rank d on $V\times M$. By Lemma 1, reference [4, p. 49], we may suppose that $\mathscr{A}=A\times M$ where $A=\mathscr{A}\mid V\times\{0\}$ with projection $p: A\to V$. Let $\{U_i\}$ be a finite open covering of V by local charts $x_i: U_i\to R^n$ and fibre charts (for A)

$$A/U_i \ni b\mapsto (x_i(p(b)),\quad \phi_i(b))\in x_i(U_i)\times C^d.$$

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(1) Added in proof: A second part, *On the stability of the cohomology of complex structures*, was published in Trans. Amer. Soc. **157** (1971), 87–97.

(2) I wish to thank Professor M. Kuranishi for a suggestion that led to the example of §6.

(3) For the general background, see also J. Morrow and K. Kodaira, *Complex manifolds*, Holt, New York, 1971.

An element ψ of the space $L(A)$ of differentiable sections of A can then be represented, on U_i, by a differentiable function

$$\psi_i : x_i(U_i) \to C^d.$$

Define now, for each nonnegative integer k, the k-norm $\|\psi\|_k$ of $\psi \in L(A)$ by

$$\|\psi\|_k^2 = \sum_i \sum_\nu \sum_{0 \le |\alpha| \le k} \int_{x_i(U_i)} |D^\alpha \psi_i^\nu(x)|^2 \, dm, \tag{2.1}$$

where dm is the Lebesgue measure on R^n and D^α denotes the operator

$$\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \quad \text{for } \alpha = (\alpha_1, \dots, \alpha_n).$$

The three propositions of this section will be given without proof.

PROPOSITION 2.1. (Sobolev's lemma, see [4, p. 50, Lemma 2].) *For each pair (k, l) of integers $k > n/2$, $l \ge 0$ there exist constants $c_{k,l}$ such that for any α with $|\alpha| \le l$*

$$|D^\alpha \psi_i^\nu(x)| < c_{k,l} \|\psi\|_{k+l}$$

where $\psi \in L(A)$ and $x \in x_i(U_i)$.

Consider now a family $\{E_t \mid t \in M\}$ of linear strongly elliptic formally selfadjoint operators E_t of even order m acting on $L(A)$.

PROPOSITION 2.2. (Inequality of Friedrichs, see [4, p. 51, Lemma 3].) *Let U be a subdomain of M such that $\bar{U}$ is compact and $\bar{U} \subset M$. For each nonnegative integer k we have*

$$\|\psi\|_{k+m}^2 \le c_k(\|E_t\psi\|_k^2 + \|\psi\|_0^2)$$

for $\psi \in L(A)$, $t \in U$, where c_k is a constant independent of $t \in U$.

Suppose now that there is a family g_t of hermitian metrics on $\mathscr{A}$ depending differentiably on $t \in M$. This gives a hermitian scalar product $\langle h, k \rangle_t$ in the fibre $A_t(x) = A(x) \times \{t\}$ and the product

$$(\phi, \psi)_t = \int_{V_t} \langle \phi(x), \psi(x) \rangle_t \, dV_t, \quad \phi, \psi \text{ in } L(A)$$

where $V_t = V \times \{t\}$ and dV_t is the volume element defined on V_t by a Riemannian metric G_t depending differentiably on t.

Set $\sqrt{(\phi, \phi)_t} = |\phi|_t$ for $\phi \in L(A)$. Then there exist constants K_1, K_2 such that

$$0 < K_1 < \frac{|\phi|_t}{\|\phi\|_0} < K_2$$

for $\phi \in L(A)$, $t \in U$, where U is chosen as in Proposition 2.2.

PROPOSITION 2.3. [4, p. 47, Theorems 1, 2]. *The operator E_t has a complete orthonormal set (with respect to the product $(\ ,\ )_t$) of eigenfunctions $e_h(t) \in L(A_t)$,*

$h=1, 2, \ldots$. *The eigenvalues* $\lambda_1(t) \leq \lambda_2(t) \leq \cdots \leq \lambda_h(t) \leq \cdots$ *satisfy* $\lambda_h(t) \to \infty$ *as* $h \to \infty$. *Furthermore,* $\lambda_h(t)$ *is a continuous function of* t, *for each* $h=1, 2, \ldots$.

Finally, let us specify what is meant by dependency of class C^k on the parameter t. Consider a family $\{\psi_t \mid t \in M\}$ of differentiable sections of the bundle over V and denote by $\psi_i(x, t)=(\psi_i^1(x, t), \ldots, \psi_i^d(x, t))$ the fibre coordinates on ψ_t on U_i. Then we say that ψ_t is of class C^k in t if and only if all the partial derivatives $D^\alpha \psi_i^s(x, t)$, $s=1, \ldots, d$, (with respect to $x^1, \ldots, x^n$) are of class C^k as a function of (x, t).

3. **Harmonic theory.** Let $\mathscr{V} \xrightarrow{\pi} M$ be a differentiable family of compact complex manifolds over $M=\{t \in R^m \mid |t| < \varepsilon\}$ and $\mathscr{B} \longrightarrow \mathscr{V} \xrightarrow{\pi} M$ a differentiable family of holomorphic vector bundles (see [4, p. 58, Definitions 1, 2]). Let $L_t^{r,s}$ be the vector space of differentiable sections of scalar forms of type (r, s) over $V_t=\pi^{-1}(t)$, and $L^{r,s}$ the space of differentiable forms on $\mathscr{V}$ whose restriction to V_t gives a differentiable form of type (r, s) on V_t, for all $t \in M$. The elements of $L^{r,s}$ will be called differentiable forms along the fibres of $\mathscr{V}$, of type (r, s).

If $T_t' \oplus T_t''$ is the decomposition of the complexified tangent bundle of V_t into the direct sum of the holomorphic tangent bundle T_t' and its conjugate bundle T_t'', and $\mathscr{F}_t$ the dual bundle of T_t', then we have the corresponding unique bundles T', T'' and $\mathscr{F}$ defined on $\mathscr{V}$ such that $T' \mid V_t=T_t'$, etc. The vector space $L_t^{r,s}$ is the space of sections of

$$\mathscr{F}_t(r, s) = (\Lambda^r \mathscr{F}_t) \wedge (\Lambda^s \overline{\mathscr{F}}_t)$$

which is equal to the restriction of

$$\mathscr{F}(r, s) = (\Lambda^r \mathscr{F}) \wedge (\Lambda^s \overline{\mathscr{F}})$$

to V_t. Denote by $L^{r,s}(\mathscr{B})$ the vector space of differentiable sections of $\mathscr{B} \otimes \mathscr{F}(r, s)$ over $\mathscr{V}$ and by $L^{r,s}(B_t)$ the vector space of differentiable sections of $B_t \otimes \mathscr{F}_t(r, s)$, where $B_t = B/V_t$.

We define the operators $\bar\partial_t : L^{r,s}(B_t) \to L^{r,s+1}(B_t)$ and $\bar\partial : L^{r,s}(\mathscr{B}) \to L^{r,s+1}(\mathscr{B})$ as in [4, p. 61], and denoting by $Z_{\bar\partial_t}^{r,s}$ and $Z_{\bar\partial}^{r,s}$ the respective kernels we set

$$H_{\bar\partial_t}^{r,s}(B_t) = \frac{Z_{\bar\partial_t}^{r,s}(B_t)}{\bar\partial_t L^{r,s-1}(B_t)},$$

$$H_{\bar\partial}^{r,s}(\mathscr{B}) = \frac{Z_{\bar\partial}^{r,s}(\mathscr{B})}{\bar\partial L^{r,s-1}(\mathscr{B})}.$$

The inclusion $B_t \to \mathscr{B}$ induces the restriction map

$$r_t : H_{\bar\partial}^{r,s}(\mathscr{B}) \longrightarrow H_{\bar\partial_t}^{r,s}(B_t)$$

which is surjective if dim $H_{\bar\partial_t}^{r,s}(B_t)$ is independent of $t \in M$ ([4, p. 66, Theorem 9]).

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Suppose there is given a Riemannian metric on $\mathscr{V}$ whose restriction G_t to V_t is hermitian, for each $t \in M$ (this is called a quasi-hermitian metric). Similarly we suppose a hermitian metric along fibres given on $\mathscr{B}$ (this amounts to a reduction of the structure group of $\mathscr{B}$ from $GL(d, C)$ to $U(d)$). We can then introduce the norms $|\psi|_t$ for elements $\psi \in L^{r,s}(B_t)$. Let θ_t be the adjoint operator of $\bar{\partial}_t$ defined by

$$(\theta_t\psi_t, \phi_t)_t = (\psi_t, \bar{\partial}_t\phi_t)_t$$

and set

$$\Box_t = \theta_t\bar{\partial}_t + \bar{\partial}_t\theta_t.$$

Then $\{\Box_t\}_{t\in M}$ gives a family of strongly elliptic formally selfadjoint linear differential operators acting on the spaces $L^{r,s}(B_t)$. Defining θ and $\Box$ by the formulas

$$r_t(\theta\psi) = \theta_t r_t(\psi), \; r_t(\Box\psi) = \Box_t r_t(\psi)$$

for $\psi \in L^{r,s}(\mathscr{B})$, we have

$$\Box = \theta\bar{\partial} + \bar{\partial}\theta$$

as a map of $L^{r,s}(\mathscr{B})$ into itself.

As our investigations are local in nature, and $\mathscr{V} \to M$ is differentiably locally trivial, we may suppose $\mathscr{V} = X \times M$ as a differentiable manifold. Hence $\mathscr{B} \otimes \mathscr{F}(r, s)$ forms a differentiable family composed of differentiable vector bundles $B_t \otimes \mathscr{F}_t(r, s)$ over X, and the strongly elliptic formally selfadjoint linear differential operator $\Box_t$ acting on $L^{r,s}(B_t)$ depends differentiably on t. Hence the results of §2 apply.

Let H_t be the harmonic operator and G_t the Green's operator associated with $\Box_t$ operating on $L^{r,s}(B_t)$. We have

$$\psi_t = \Box_t G_t\psi_t + H_t\psi_t, \quad \psi_t \in L^{(r,s)}(B_t)$$
$$G_t H_t = H_t G_t = 0$$

and G_t commutes with $\bar{\partial}_t$ and θ_t. The orthogonal projections $\pi_{\bar{\partial}_t} = \bar{\partial}_t\theta_t G_t$, $\pi_{\theta_t} = \theta_t\bar{\partial}_t G_t$ and H_t are mutually orthogonal; obviously

$$I = \pi_{\bar{\partial}_t} + \pi_{\theta_t} + H_t.$$

Further $\psi_t = \bar{\partial}_t\theta_t G_t\psi_t + H_t\psi_t$ for $\psi_t \in Z^{r,s}_{\bar{\partial}_t}(B_t)$ which gives the isomorphism

$$H^{r,s}_{\bar{\partial}_t}(B_t) = \mathbf{H}^{r,s}(B_t) = \operatorname{Ker} \Box_t \cap L^{r,s}(B_t).$$

By Proposition 2.3, $\mathbf{H}^{r,s}(B_t)$ is finite dimensional.

Let us finish this section by two fundamental results from [4, p. 65]:

THEOREM 3.1. (*Upper semi-continuity*) $\dim H^{r,s}_{\bar{\partial}_t}(B_t)$ *is an upper semi-continuous function of* t.

THEOREM 3.2. *If* $\dim H^{r,s}_{\bar{\partial}_t}(B_t)$ *is independent of* $t \in M$, *then the linear operators* G_t *and* H_t *acting on* $L^{r,s}(B_t)$ *depend differentiably on* $t \in M$.

4. **Transportable forms.** Consider a differentiable family $\mathscr{B} \longrightarrow \mathscr{V} \xrightarrow{\pi} M$ of holomorphic vector bundles, and suppose that each bundle $B_t \to V_t$ is equipped with a hermitian metric depending differentiably on t, and that $\mathscr{V}$ has a quasi-hermitian metric.

DEFINITION 4.1. For $t_0 \in M$ let $\gamma \in \mathbf{H}^{r,s}(B_{t_0}) = \mathbf{H}^{r,s}_{t_0}$. We say that the form γ is *transportable* at $t_0 \in M$ in the family $\mathscr{B} \longrightarrow \mathscr{V}$ with the given metrics, if there exists a neighborhood U of t_0 in M and a differentiable form $\gamma_t \in L^{r,s}(B_t)$ for each $t \in U$, depending differentiably on t in U and such that $\Box_t \gamma_t = 0$ for all t in U, and that $\gamma_{t_0} = \gamma$. The form γ_t depending on the parameter t is called a *transportation* of γ.

Evidently the set of transportable forms at t_0 forms a subspace $\mathscr{T}^{r,s}(t_0)$ of the vector space $\mathbf{H}^{r,s}(B_{t_0})$ of all harmonic forms of type (r, s) at $t_0 \in M$.

PROPOSITION 4.1. *The function* $\dim \mathbf{H}^{r,s}_t$ $(=\dim H^{r,s}_{\bar\partial_t}(B_t))$ *is constant in a neighborhood of* t_0 *if and only if* $\mathscr{T}^{r,s}(t_0) = \mathbf{H}^{r,s}(B_{t_0})$.

Proof. Suppose all harmonic forms of type (r, s) are transportable and let $\{\gamma^1, \ldots, \gamma^k\}$ be a basis of $\mathbf{H}^{r,s}_{t_0}$ with transportations $\{\gamma^1_t, \ldots, \gamma^k_t\}$, defined in a neighborhood of $t_0 \in M$. To show that these transportations are linearly independent in a neighborhood of t_0, suppose the contrary. Then there exist a sequence of points $t_\nu \in M$ such that $t_\nu \to t_0$ as $\nu \to \infty$, and complex numbers $\alpha^1_\nu, \ldots, \alpha^k_\nu$ with $\sum_{i=1}^k |\alpha^i_\nu|^2 = 1$ such that $\sum_{i=1}^k \alpha^i_\nu \gamma^i_{t_\nu} = 0$ for $\nu = 1, 2, 3, \ldots$. As $a_\nu = (\alpha^1_\nu, \ldots, \alpha^k_\nu) \in S^{2k-1}$ which is compact, there is a cluster point $a_0 = (\alpha^1_0, \ldots, \alpha^k_0) \in S^{2k-1}$ of the sequence $\{a_\nu\}$. By restricting to a subsequence we may suppose $a_\nu \to a_0$. Then

$$|\textstyle\sum \alpha^i_0 \gamma^i_{t_\nu}|_{t_\nu} = |\sum \alpha^i_\nu \gamma^i_{t_\nu} - \sum \alpha^i_0 \gamma^i_{t_\nu}|_{t_\nu} \leq \sum |\alpha^i_\nu - \alpha^i_0|\, |\gamma^i_{t_\nu}|_{t_\nu} \to 0$$

as $\nu \to \infty$. But as on the other hand

$$|\textstyle\sum \alpha^i_0 \gamma^i_{t_\nu}|_{t_\nu} \to |\sum \alpha^i_0 \gamma^i_{t_0}|_{t_0} \neq 0,$$

we have a contradiction. Hence $\dim \mathbf{H}^{r,s}_t \geq \dim \mathbf{H}^{r,s}_{t_0}$ in a neighborhood of $t_0 \in M$. The principle of upper semi-continuity (Theorem 3.1) gives the reverse inequality near t_0, hence $\dim \mathbf{H}^{r,s}_t = \text{constant}$ in a neighborhood of that point.

If, on the other hand, $\dim \mathbf{H}^{r,s}_t$ stays constant in a neighborhood of t_0, then by Theorem 3.2 the projection operator $H^{r,s}_t = H_t$ depends differentiably on t near t_0 and hence for any $\gamma \in \mathbf{H}^{r,s}_{t_0}$ the form $H_t\gamma$ gives a transportation of γ in a neighborhood of t_0. This completes the proof.

By Proposition 2.3 the operator $\Box_t$ acting on $L^{r,s}(B_t)$ has eigenvalues $\lambda_1(t) \leq \lambda_2(t) \leq \cdots \leq \lambda_k(t) \leq \cdots$ with $\lambda_k(t) \to \infty$, as $k \to \infty$, and $\lambda_k(t)$ depends continuously on t. By, for instance, [4, p. 67, Theorem 11] all eigenvalues of $\Box_t$ are non-negative. Hence, if $\lambda_{k+1}(t)$ is the first positive eigenvalue at $t \in M$, then $\dim \mathbf{H}^{r,s}_t = k$.

DEFINITION 4.2. Let $\lambda_{k+1}(t_0)$ be the first positive eigenvalue of $\Box_{t_0}$. If $\lambda_1(t) = \cdots = \lambda_k(t) = 0$ for $t \in U$, where U is a neighborhood of $t_0 \in M$, then we say that t_0 is

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a *regular point* of $H^{r,s}_{\bar{\partial}_t}(B_t)$. If there is no such neighborhood then t_0 is a *singular point* of $H^{r,s}_{\bar{\partial}_t}(B_t)$.

Obviously t_0 is an isolated singular point if and only if there exists an integer $h \le k$ and a neighborhood U of t_0 such that $\lambda_h(t) > 0$ for $t \in U - \{t_0\}$ and that $\lambda_1(t) = \cdots = \lambda_{h-1}(t) = 0$ for $t \in U$.

Observe that the notions introduced in Definition 4.2 do not depend on the hermitian metrics chosen.

COROLLARY 4.1. *The point $t_0 \in M$ is a regular point of $H^{r,s}_{\bar{\partial}_t}(B_t)$ if and only if all harmonic forms at t_0 are transportable.*

Proof. Follows from Proposition 4.1.

5. Extendible and co-extendible forms.

DEFINITION 5.1. A $\bar{\partial}_{t_0}$-closed form $\gamma \in L^{r,s}(B_{t_0})$ is extendible (see [2]) at $t_0 \in M$ if there is a neighborhood U of t_0 and a family $\{\eta_t\}_{t \in U}$ of forms $\eta_t \in L^{r,s}(B_t)$ depending differentiably on t and such that $\bar{\partial}_t \eta_t = 0$ and that $\eta_{t_0} = \gamma$. The form η_t depending on the parameter t is called an extension of γ. The vector space of all extendible forms of the given type is denoted by $E^{r,s}_{t_0}$.

Notice that this definition does not make use of the introduced metrics.

DEFINITION 5.2. A θ_{t_0}-closed form $\gamma \in L^{r,s}(B_{t_0})$ is co-extendible at $t_0 \in M$ if there is a neighborhood U of t_0 and a family $\{\sigma_t\}_{t \in U}$ of forms $\sigma_t \in L^{r,s}(B_t)$ depending differentiably on t and such that $\theta_t \sigma_t = 0$ and that $\sigma_{t_0} = \gamma$. The vector space of all co-extendible forms of the given type is denoted by $C^{r,s}_{t_0}$.

As $\Box_t \psi_t = 0$ is equivalent to the pair of equations

$$\bar{\partial}_t \psi_t = 0 \quad \text{and} \quad \theta_t \psi_t = 0,$$

it is clear that transportability implies extendability and co-extendability, for a form $\gamma \in H^{r,s}_{t_0}$. The example in §6 suggests that the reverse implication is not true without further assumptions (see Theorem 5.2).

Set $\mathscr{A} = \mathscr{B} \otimes (\Lambda^r \mathscr{F}) \wedge (\Lambda^s \bar{\mathscr{F}})$, $A_t = B_t \otimes (\Lambda^r \mathscr{F}_t) \wedge (\Lambda^s \bar{\mathscr{F}}_t)$ and $A_{t_0} = A$. By the results cited in the beginning of §2 we may suppose

$$\mathscr{A} = A \times M$$

for the differentiable structure, where then A is a bundle on V_0. Given a differentiable form $\psi_t \in L^{r,s}(B_t)$ for each $t \in U \subset M$, we can then consider it as a function $U \to L(A)$, and in particular, apply the norms $\|\psi_t\|_k$, $k = 0, 1, 2, \ldots$ defined in §2.

THEOREM 5.1. *Let η_t and σ_t be an extension and a co-extension, respectively, of a form $\gamma \in H^{r,s}_{t_0}$, in a neighborhood $U \subset M$ of t_0. Then the harmonic parts $H_t \eta_t$ and $H_t \sigma_t$ depend continuously on t at the point t_0.*

Proof. Suppose the neighborhood U of t_0 is such that $\bar{U}$ is compact with $\bar{U} \subset M$.

Then (§2) there exist constants K_1, K_2 such that for $\psi_t \in L^{r,s}(B_t)$ and $t \in U$ we have

$$0 < K_1 < \frac{|\psi_t|_t}{\|\psi_t\|_0} < K_2. \tag{5.1}$$

As $\pi_{\square_t} = \pi_{\theta_t} + \pi_{\bar{\partial}_t}$ is an orthogonal projection in the pre-Hilbert space $L^{r,s}(B_t)$, we have

$$\begin{aligned} |\sigma_t - \eta_t|_t \geq |(\pi_{\theta_t} + \pi_{\bar{\partial}_t})(\sigma_t - \eta_t)|_t &= |\pi_{\theta_t}\sigma_t - \pi_{\bar{\partial}_t}\eta_t|_t \\ &\geq |\pi_{\theta_t}\sigma_t|_t \end{aligned} \tag{5.2}$$

because $\pi_{\theta_t}\eta_t = \pi_{\bar{\partial}_t}\sigma_t = 0$ and $\pi_{\theta_t}\sigma_t \perp \pi_{\bar{\partial}_t}\eta_t$. Similarly

$$|\sigma_t - \eta_t|_t \geq |\pi_{\bar{\partial}_t}\eta_t|_t. \tag{5.3}$$

By (5.1) $\|\sigma_t - \gamma\|_0 \to 0$ and $\|\eta_t - \gamma\|_0 \to 0$ imply $|\sigma_t - \gamma|_t \to 0$ and $|\eta_t - \gamma|_t \to 0$, respectively, as $t \to t_0$. Because

$$|\sigma_t - \eta_t|_t \leq |\sigma_t - \gamma|_t + |\eta_t - \gamma|_t$$

we get from (5.2) and (5.3)

$$|\pi_{\theta_t}\sigma_t|_t \to 0 \quad \text{and} \quad |\pi_{\bar{\partial}_t}\eta_t|_t \to 0 \quad (t \to t_0).$$

Now

$$|\gamma - H_t\eta_t|_t \leq |\gamma - \eta_t|_t + |\pi_{\bar{\partial}_t}\eta_t|_t.$$

Hence $|\gamma - H_t\eta_t|_t \to 0$ as $t \to t_0$, and similarly $|\gamma - H_t\sigma_t|_t \to 0$. In view of (5.1), this implies

$$\|\gamma - H_t\eta_t\|_0 \to 0 \quad \text{and} \quad \|\gamma - H_t\sigma_t\|_0 \to 0 \quad (t \to t_0). \tag{5.4}$$

Applying now the Friedrichs inequality (Proposition 2.2) to the form

$$\psi_t = \gamma - H_t\eta_t$$

and the operator $\square_t$ we get

$$\|\gamma - H_t\eta_t\|_{k+m}^2 \leq c_k(\|\square_t\gamma\|_k^2 + \|\gamma - H_t\eta_t\|_0^2) \tag{5.5}$$

for all nonnegative integers k, m. As $\square_t$ depends differentiably on t, $\|\square_t\gamma\|_k$ is a continuous function of t; hence the fact that $\square_{t_0}\gamma = 0$ and (5.4) imply that the right hand side of the inequality (5.5) approaches zero as $t \to t_0$. Hence

$$\|\gamma - H_t\eta_t\|_k \to 0 \quad (t \to t_0). \tag{5.6}$$

Sobolev's lemma (Proposition 2.1) gives for the partial derivatives of the components of $\psi_t = \gamma - H_t\eta_t \in L(A)$ in some fibre coordinates

$$D^\alpha\psi_i^\nu(x_i(p), t) \to 0$$

uniformly on $U_i \subset V_0$ as $t \to t_0$ and for all α, $|\alpha| = 0, 1, 2, \ldots$. This means, then, that the partial derivatives of $H_t\eta_t$ approach those of γ as a limit uniformly in x when

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$t \to t_0$, in other words $H_t\eta_t$ depends continuously on t at t_0 (see the end of §2). Similarly for $H_t\sigma_t$.

COROLLARY 5.1. *Let $k = \dim E_{t_0}^{r,s} \cap C_{t_0}^{r,s}$. Then there is a neighborhood U of t_0 in M such that for $t \in U$, $\dim \mathbf{H}_t^{r,s} \geq k$.*

Proof. Let $\gamma_1, \ldots, \gamma_k$ be a basis of the vector space $E_{t_0}^{r,s} \cap C_{t_0}^{r,s} \subset \mathbf{H}_{t_0}^{r,s}$. As by Theorem 5.5 the forms $H_t\gamma_1, \ldots, H_t\gamma_k$ depend continuously on t at t_0 (here continuity in the norm $\| \ \|_0$ would be sufficient), we conclude as in the proof of Proposition 4.1 that $H_t\gamma_1, \ldots, H_t\gamma_k$ are linearly independent in a neighborhood of $t_0 \in M$.

THEOREM 5.2. *If all harmonic forms $\gamma \in H_{t_0}^{r,s}$ are extendible and co-extendible, then $\dim H_t^{r,s}$ is constant in a neighborhood of t_0, and all harmonic forms are transportable at $t_0 \in M$.*

Proof. By Corollary 5.1, $\dim H_t^{r,s} \geq \dim H_{t_0}^{r,s}$ in a neighborhood of t_0. The principle of upper semi-continuity (Theorem 3.1) provides the reverse inequality. The last conclusion follows from Proposition 4.1.

6. **An example.** Let K^i, $i=0, 1, 2$, be three complex vector spaces with inner products, and let $\{a\}$, $\{b_1, b_2\}$ and $\{c\}$ be orthonormal bases for K^0, K^1 and K^2, respectively. Let $f: \mathbf{R} \to \mathbf{R}$ be such that $f(0)=0$ and that f has the same zeros and the same sign as the function $\sin(1/t)$ for $t \neq 0$, and with the property that the positive and negative parts $f^+ = \max(0, f)$ and $f^- = \min(0, f)$ are differentiable. Such a function can be constructed for instance by starting with $\exp(-t^{-2}) \sin(1/t)$ and smoothing at the zeros other than $t=0$. Define then $d_t^i: K^i \to K^{i+1}$ for $i=0, 1$ as follows:

$$d_t^0 a = f^-(t)b_2,$$
$$d_t^1 b_1 = -tf^+(t)c, \quad d_t^1 b_2 = f^+(t)c.$$

The adjoint maps (with respect to the given inner products) $\theta_t^i: K^{i+1} \to K^i$ are given by

$$\theta_t^0 b_1 = 0, \ \theta_t^0 b_2 = f^-(t)a,$$
$$\theta_t^1 c = f^+(t)b_2 - tf^+(t)b_1.$$

Let $[x_1, \ldots, x_k]$ denote the vector space spanned by some given vectors $x_1, \ldots x_k$. Then, setting $Z_t^i = \ker d_t^i$, $\Theta_t^i = \ker \theta_t^{i-1}$, we get

$$Z_t^1 = \begin{cases} K^1 & \text{for } f(t) \leq 0 \\ [b_1 + tb_2] & \text{for } f(t) > 0 \end{cases}$$

$$\Theta_t^1 = \begin{cases} [b_1] & \text{for } f(t) < 0 \\ K^1 & \text{for } f(t) \geq 0. \end{cases}$$

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Hence, if $\mathbf{H}_t^i = \ker d_t^i \cap \ker \theta_t^{i-1}$, then

$$H_t^1 = \begin{cases} [b_1] & \text{for } f(t) < 0 \\ [b_1, b_2] = K^1 & \text{for } f(t) = 0 \\ [b_1 + tb_2] & \text{for } f(t) > 0. \end{cases}$$

Now $b_1 \in H_0^1$ is "extendible" with an extension $\eta_t = b_1 + tb_2$, and "co-extendible" with a co-extension $\sigma_t = b_1$. The vector b_1 is not "transportable', however, as there obviously is no function γ_t of t, differentiable (or even continuous) in a neighborhood U of $t=0$ and such that $\gamma_0 = b_1$, $\gamma_t \in \mathbf{H}_t^1$ for $t \in U$.

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Université de Montréal,
Montréal, Quebéc