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ON ONE-DIMENSIONAL LOCAL RINGS AND BERGER'S CONJECTURE

CLETO B. MIRANDA-NETO

Departamento de Matemática, Universidade Federal da Paraíba, João Pessoa, PB, Brazil (cleto@mat.ufpb.br)

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Abstract Let k be a field of characteristic zero and let $\Omega_{A/k}$ be the universally finite differential module of a k-algebra A, which is the local ring of a closed point of an algebraic or algebroid curve over k. A notorious open problem, known as Berger's Conjecture, predicts that A must be regular if $\Omega_{A/k}$ is torsion-free. In this paper, assuming the hypotheses of the conjecture and observing that the module $\mathrm{Hom}_A(\Omega_{A/k}, \Omega_{A/k})$ is then isomorphic to an ideal of A, say $\mathfrak{h}$, we show that A is regular whenever the ring $A/a\mathfrak{h}$ is Gorenstein for some parameter a (and conversely). In addition, we provide various characterizations for the regularity of A in the context of the conjecture.

Keywords: module of differentials; one-dimensional regular local ring; Berger's Conjecture; algebraic and algebroid curves; homological characterizations of regularity

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1. Introduction

A long-standing conjecture posed by R. W. Berger [3] states the following:
Berger's Conjecture. Let k be a field of characteristic zero and let A be a one-dimensional reduced local ring which is either

$$k[X_1, \ldots, X_n]_{(X_1,\ldots,X_n)}/\mathfrak{A} \quad \text{or} \quad k[[X_1, \ldots, X_n]]/\mathfrak{A},$$

where the X_i's are $n \geq 2$ indeterminates over k and $\mathfrak{A}$ stands for an ideal. If the (universally finite) module $\Omega_{A/k}$ of k-differentials of A is torsion-free, then A is regular.

Example 1. The standard and easiest illustration is the cuspidal plane cubic, that is, the local domain $A = \mathbb{C}[[X, Y]]/(Y^2 - X^3) = \mathbb{C}[[x, y]]$. We have $\Omega_{A/\mathbb{C}} \cong A^2/A(-3x^2, 2y)$, which, as expected, has non-trivial torsion. For instance, if $\omega \in \Omega_{A/\mathbb{C}} \setminus \{0\}$ is the differential corresponding to the image of the vector $(-3xy, 2x^2) \in A^2$, then $x\omega = 0$.

Remark 2. In Berger's problem, the local k-algebra A is not required to be an integral domain (even though this constitutes a fundamental case), and moreover, in the algebroid case, its defining ideal is not required to be generated by polynomials. Also, the original statement assumes the field k to be perfect only, but as in Berger's survey [5] we restrict ourselves to the characteristic zero case. The conjecture has been confirmed in a number of special situations; see, for instance, Bassein [2], Berger [3, 4], Buchweitz and Greuel [7], Cortiñas et al. [11], Cortiñas and Krongold [10], Güttes [15], Herzog [17], Herzog and Waldi [21, 22], Isogawa [26], Pohl [28], Scheja [29] and Ulrich [31]. We also refer to Herzog's survey [19] about this and other differential conjectures, for instance, the so-called rigidity conjecture (cf. also Herzog [17, 18] and Ulrich [31]), which is closely related to the problem considered here (see Remark 13).

In this note, we start from the fact that the key hypothesis of the conjecture allows us to realize the endomorphism module $\mathrm{Hom}_A(\Omega_{A/k}, \Omega_{A/k})$ as an ideal $\mathfrak{h}_{A/k}$ of A (Lemma 2). Then, our main result (Theorem 4) characterizes when A is regular and shows that, in order to solve Berger's Conjecture, it suffices to prove that $A/a\mathfrak{h}_{A/k}$ can be taken as Gorenstein for some parameter a of A.

Several other characterizations for the regularity of $(A, \mathfrak{m})$ in the context of Berger's Conjecture (hence, assuming $\Omega_{A/k}$ to be torsion-free) are given. Later, we mention some of them.

Proposition 6 makes use, in particular, of the vanishing of (co)homology modules and shows, among other equivalences, that A is regular if and only if

$$\mathrm{Ext}^i_A(\Omega_{A/k}, \mathfrak{m}^\nu) = 0 \quad \text{for some}\, i, \nu \geq 1.$$

In Proposition 8 and Corollary 9, we apply a similar approach but eventually exploring the finiteness of the injective dimension of certain modules; for instance, in the latter, it is observed that A is regular if and only if $\mathrm{injdim}\,\mathrm{Hom}_A(\Omega_{A/k}, \Omega_{A/k}) < \infty$.

In Lemma 12, we use a suitable Artinian derivation module to provide a numerical characterization of the vanishing of $\mathrm{Ext}^1_A(\Omega_{A/k}, A)$, and this turns out to be one of the key ingredients to Corollary 14, which also characterizes when A is regular.

Another result, Corollary 16, has as a consequence the fact that if in addition A is a Gorenstein domain, then A is regular, provided that the tensor product module $\mathfrak{h}_{A/k} \otimes_A \mathrm{Hom}_A(\mathfrak{h}_{A/k}, A)$ is torsion-free and

$$\bigcup_{j \geq 1} \mathfrak{h}^j_{A/k} :_{A'} \mathfrak{h}^j_{A/k} = \mathfrak{h}_{A/k} :_{A'} \mathfrak{h}_{A/k},$$

where $A^{'}$ is the integral closure of A in its fraction field.

Finally, Corollary 19 shows that A is regular if and only if the quotient ring

$$\mathfrak{R}_A(\mathfrak{h}_{A/k})/(t^2 + at + b)A[t] \cap \mathfrak{R}_A(\mathfrak{h}_{A/k})$$

is Gorenstein for some – equivalently, all – $a, b \in A$, where $\mathfrak{R}_A(\mathfrak{h}_{A/k}) = \bigoplus_{i \geq 0} \mathfrak{h}^i_{A/k} t^i$ (hence, an A-subalgebra of $A[t]$) is the Rees algebra of the ideal $\mathfrak{h}_{A/k}$.

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2. Preliminaries

2.1. Conventions and basics

Throughout the paper, by *ring*, we tacitly mean Noetherian, commutative, unital ring. The set of zero divisors of a ring S is equal to $\bigcup_{\mathfrak{p}\in \operatorname{Ass} S}\mathfrak{p}$, where as usual $\operatorname{Ass} S$ denotes the (finite) set of associated primes of S. We say that an S-module M is *finite*, simply, if it is finitely generated over S. The S-torsion submodule of M is formed by the elements that are killed by some non-zero-divisor of S. If such submodule is trivial, then M is said to be torsion-free (over S). We say that a finite S-module M has a (generic, constant) rank, say $r \geq 0$, if the $S_{\mathfrak{p}}$-module $M_{\mathfrak{p}} = M \otimes_S S_{\mathfrak{p}}$ is free of rank r for every $\mathfrak{p} \in \operatorname{Ass} S$. For instance, an ideal containing a non-zero-divisor has rank 1, and any finite module over a domain has a rank.

Now let $(S, \mathfrak{n})$ be a local ring. We denote the $\mathfrak{n}$-depth of a finite S-module M by $\operatorname{depth} M$. Thus, M is maximal Cohen-Macaulay over S if $\operatorname{depth} M = \dim S$. If S is a Cohen–Macaulay local ring with a canonical module, say ω_S (which is unique up to isomorphism), then it is well-known that M is maximal Cohen–Macaulay if and only if $\operatorname{Ext}^i_S(M, \omega_S) = 0$ for all $i \geq 1$. For further information, see [6].

2.2. Differentials

Let k be a ring and S be a k-algebra. Consider the tensor product algebra $S \otimes_k S$ and the 'diagonal' ideal $\mathfrak{D} \subset S \otimes_k S$ generated by all elements of the form $x \otimes 1 - 1 \otimes x$, with $x \in S$. It is easy to see that $\mathfrak{D}$ is the kernel of the natural multiplication map $S \otimes_k S \to S$, so that $(S \otimes_k S)/\mathfrak{D} \cong S$. This endows $\mathfrak{D}/\mathfrak{D}^2$ with a natural structure of an S-module, which is denoted

$$\mathrm{D}_{S/k} := \mathfrak{D}/\mathfrak{D}^2,$$

the so-called *module of Kähler differentials of S over k*. This S-module is finite if, for example, the k-algebra S is essentially of finite type, but it is important to recall that $\mathrm{D}_{S/k}$ may not be finite in general and such a pathology can occur even if S is a regular local ring; for instance, if $S = k[[X_1, \ldots, X_n]]$ is a formal power series ring over a field k of characteristic zero, then $\mathrm{D}_{S/k}$ is not finite (see [27, Example 5.5(a)]).

For some classes of k-algebras S, there exists, however, a suitable modification of $\mathrm{D}_{S/k}$, typically written $\Omega_{S/k}$ and called *universally finite differential module of S over k*, which turns out to be a finite S-module. For instance, if S is either essentially of finite type over k or a complete local ring $(S, \mathfrak{n})$ such that $S/\mathfrak{n}$ is a finite extension of a field k, then $\Omega_{S/k}$ does exist; in the former case, we simply put $\Omega_{S/k} := \mathrm{D}_{S/k}$, and, in the latter,

$$\Omega_{S/k} := \mathrm{D}_{S/k}/\bigcap_{i\geq 0}\mathfrak{n}^i\mathrm{D}_{S/k}.$$

If, for example, k is a field and $S = k[[X_1, \ldots, X_n]]$, then $\Omega_{S/k}$ is free of rank n. Finally, it should be mentioned that the formation of the universally finite differential module does not commute with localization in general. For the theory, see [27], also [19].

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3. One-dimensional regular local rings

3.1. Main result

Before stating our main result, we introduce some notation as well as a basic lemma.

Notation 1. Let k be a field of characteristic zero and let A be a local ring which is a quotient of either $k[X_1, \ldots, X_n]_{(X_1,\ldots,X_n)}$ or $k[[X_1, \ldots, X_n]]$; in particular, $\Omega_{A/k}$ exists. Throughout the paper, we set

$$\mathrm{H}_{A/k} := \mathrm{Hom}_A(\Omega_{A/k}, \Omega_{A/k}).$$

Lemma 2. *Let* k *be a field of characteristic zero and let* A *be a one-dimensional reduced local ring which is a quotient of either* $k[X_1, \ldots, X_n]_{(X_1,\ldots,X_n)}$ *or* $k[[X_1, \ldots, X_n]]$. *The following assertions hold:*

(i) $\Omega_{A/k}$ *has rank* 1 *as an* A*-module.*
(ii) *If* $\Omega_{A/k}$ *is torsion-free, then* $\mathrm{H}_{A/k}$ *can be identified with an ideal of* A*.*

Proof. First, note that, being a reduced one-dimensional local ring, A must be Cohen–Macaulay. The assertion (i) is well known and standard notably in the algebraic case. In the algebroid case, it was mentioned, for example, in [28, proof of Lemma 1], and moreover proofs can be found in the literature under the hypothesis that A is an integral domain. Below we give a proof without assuming this condition.

Write $A = B/\mathfrak{A}$, where $B = k[[X_1, \ldots, X_n]]$ is a formal power series ring over k and $\mathfrak{A} \subset B$ is a radical proper ideal. By [27, Corollary 11.10], the A-module $\Omega_{A/k}$ fits into the conormal sequence

$$\mathfrak{A}/\mathfrak{A}^2 \longrightarrow A^n \longrightarrow \Omega_{A/k} \longrightarrow 0.$$

Explicitly, the map $\mathfrak{A}/\mathfrak{A}^2 \to A^n$ sends the class (modulo $\mathfrak{A}^2$) of any given $f \in \mathfrak{A}$ to $(\partial f/\partial x_1, \ldots, \partial f/\partial x_n) \in A^n$, where $\partial f/\partial x_j$ stands for the image in A of the formal partial derivative $\partial f/\partial X_j$, for every $j = 1, \ldots, n$. Hence, this map has kernel $\mathfrak{A}^{\langle 2\rangle}/\mathfrak{A}^2$, where $\mathfrak{A}^{\langle 2\rangle}$ is the so-called second differential power of $\mathfrak{A}$, that is, the subideal formed by the $g \in \mathfrak{A}$ such that $\partial g/\partial X_j \in \mathfrak{A}$ for every $j = 1, \ldots, n$. Now, consider the second symbolic power $\mathfrak{A}^{(2)} = \bigcap_{\mathfrak{P}}(\mathfrak{A}^2 B_{\mathfrak{P}} \cap B)$, where $\mathfrak{P} \subset B$ ranges over the associated primes of the B-module $B/\mathfrak{A}$. Since $\mathfrak{A}$ is radical and $\operatorname{char} k = 0$, we have $\mathfrak{A}^{(2)} = \mathfrak{A}^{\langle 2\rangle}$ by the general versions of the Zariski–Nagata theorem obtained in [12, § 2.1] (which, as pointed out therein, hold as well in the present case of power series rings). It follows a short exact sequence

$$0 \longrightarrow \mathfrak{A}/\mathfrak{A}^{(2)} \longrightarrow A^n \longrightarrow \Omega_{A/k} \longrightarrow 0.$$

Therefore, in order to prove that $\Omega_{A/k}$ has rank 1, it suffices to show that $\mathfrak{A}/\mathfrak{A}^{(2)}$ has rank $n-1$. As is well-known, the torsion of the conormal module $\mathfrak{A}/\mathfrak{A}^2$ is precisely $\mathfrak{A}^{(2)}/\mathfrak{A}^2$, which then has rank zero. Thus, by the short exact sequence of A-modules $0 \to \mathfrak{A}^{(2)}/\mathfrak{A}^2 \to \mathfrak{A}/\mathfrak{A}^2 \to \mathfrak{A}/\mathfrak{A}^{(2)} \to 0$, it suffices to check that $\mathfrak{A}/\mathfrak{A}^2$ has rank $n-1$.

But this is true (see, e.g., [6, Exercise 4.7.17]) since A is reduced – hence, generically a complete intersection – and Cohen–Macaulay, and since $n-1 = \operatorname{height}\mathfrak{A}$.

Finally, to prove (ii), set $\mathrm{H} := \mathrm{H}_{A/k}$ for simplicity. Having shown that $\Omega_{A/k}$ has rank 1 as an A-module, we get that H has rank 1 as well; indeed, we can write

$$\mathrm{H}_{\mathfrak{p}} \cong \operatorname{Hom}_{A_{\mathfrak{p}}}((\Omega_{A/k})_{\mathfrak{p}}, (\Omega_{A/k})_{\mathfrak{p}}) \cong \operatorname{Hom}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}, A_{\mathfrak{p}}) \cong A_{\mathfrak{p}}$$

for any minimal prime $\mathfrak{p}$ of A. It is also clear that H is torsion-free since so is $\Omega_{A/k}$ by assumption. It follows that H affords an embedding into A and hence is isomorphic to the image, which is a non-zero ideal of A. □

Notation 3. Consider the setting and hypotheses of Lemma 2(ii). As we have seen, we can fix an A-module isomorphism between $\mathrm{H}_{A/k}$ and an ideal of A; throughout the paper, such an ideal will be denoted $\mathfrak{h}_{A/k}$, or simply $\mathfrak{h}$. Theoretically, its generators can be seen in the usual way. Consider generators $\mathrm{H}_{A/k} = \sum_{j=1}^{\mu} A\xi_j$ of $\mathrm{H}_{A/k}$ as an A-module. According to the fixed embedding $\mathrm{H}_{A/k} \subset A$, each ξ_j corresponds to a uniquely determined $\alpha_j \in A$, so that $\mathfrak{h}_{A/k} = (\alpha_1, \ldots, \alpha_\mu)$.

Our result is as follows.

Theorem 4. *Assume the setting and hypotheses of Berger's Conjecture. The following assertions are equivalent:*

(i) A *is regular.*
(ii) $A/a\mathfrak{h}_{A/k}$ *is a hypersurface ring for some (any) parameter* $a \in \mathfrak{m}$.
(iii) $A/a\mathfrak{h}_{A/k}$ *is Gorenstein for some (any) parameter* $a \in \mathfrak{m}$.

Proof. It is a well-known fact that $\Omega_{A/k}$ is free if and only if A is regular (see [27, Theorem 14.1], where the hypothesis $\operatorname{char} k = 0$ is required). Consequently, if the one-dimensional local ring A is regular, then $\mathrm{H}_{A/k}$ is free, that is, the ideal $\mathfrak{h}$ is principally generated by a non-zero-divisor, say b. Hence, if $a \in \mathfrak{m}$ is a parameter, the ring $A/a\mathfrak{h} = A/(ab)$ is a hypersurface ring. Also recall that hypersurface rings are Gorenstein. This proves (i)⇒(ii)⇒(iii).

It remains to show that (iii)⇒(i), which is the core of the proof. Since $\Omega_{A/k}$ is torsion-free of rank 1, we can fix a proper ideal $\mathfrak{o}$ of A such that

$$\Omega_{A/k} \cong \mathfrak{o}.$$

Since A is one-dimensional and $\Omega_{A/k}$ is torsion-free, $\mathfrak{o}$ is a maximal Cohen–Macaulay A-module and then, as an ideal, it must be $\mathfrak{m}$-primary. Hence, as, in addition, the residue field of A is infinite, there exists a non-zero-divisor $x \in \mathfrak{m}$ such that the principal ideal (x) is a (minimal) reduction of $\mathfrak{o}$ (this follows, e.g., from [24, Theorem 8.6.3]). Let T denote the total quotient ring of A, that is, $T = W^{-1}A$, where W is the multiplicative set formed by the non-zero-divisors of A. Consider the A-submodule $\mathfrak{F} \subset T$ formed by the fractions y/x with $y \in \mathfrak{o}$. Clearly, $\mathfrak{F} \cong \mathfrak{o}$ as A-modules. Let $\mathfrak{F}\colon \mathfrak{F} = \{\sigma \in T \mid \sigma\mathfrak{F} \subset \mathfrak{F}\}$. Each $a \in A$ satisfies $ay/x \in \mathfrak{F}$ for all $y \in \mathfrak{o}$, and moreover, $1 = x/x \in \mathfrak{F}$. Hence, $A \subset \mathfrak{F}\colon \mathfrak{F} \subset \mathfrak{F}$. Observe that the co-kernel $\overline{\mathfrak{F}}$ of the latter inclusion is of finite length; the

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argument is standard, but for completeness, we provide it here. First, we may identify $A \cong \{a/x \mid a \in (x)\} \subset \mathfrak{F}$, which is the kernel of the natural epimorphism $\mathfrak{F} \to \mathfrak{o}/(x)$ given by $y/x \mapsto y \bmod(x)$ for $y \in \mathfrak{o}$. In the present setting, the A-module $\mathfrak{o}/(x)$ is of finite length, and so is $\mathfrak{F}/A$, and the induced epimorphism

$$\mathfrak{F}/A \longrightarrow \mathfrak{F}/(\mathfrak{F}\colon \mathfrak{F}) = \overline{\mathfrak{F}}$$

forces $\overline{\mathfrak{F}}$ to be of finite length as well. It will be also useful to notice that $1 \notin \mathfrak{m}\mathfrak{F}$. Indeed, assuming otherwise and using the fact that the integral closure $A^{'}$ of A in T contains $\mathfrak{F}$ (to see this, recall (x) is a reduction of $\mathfrak{o}$ and apply [24, Corollary 1.2.5]), we would get $1 \in \mathfrak{m}A'$, hence $1 \in \mathfrak{m}A' \cap A = \mathfrak{m}$, a contradiction.

Now we claim that, even more, the hypothesis (iii) forces the equality $\mathfrak{F}\colon \mathfrak{F} = \mathfrak{F}$ to hold. To prove this, suppose by way of contradiction that the finite length A-module $\overline{\mathfrak{F}}$ is non-zero. Then its socle contains a non-zero element, say u. Note there exists $F_u \in \mathrm{Hom}_A(\mathfrak{F}, \overline{\mathfrak{F}})$, which sends 1 to u. To obtain this map, first notice that the natural A-module isomorphism

$$\mathrm{Soc}_A(\overline{\mathfrak{F}}) \cong \mathrm{Hom}_A(k, \overline{\mathfrak{F}})$$

gives the well-defined non-zero A-linear map $k \to \overline{\mathfrak{F}}$ given by $1 + \mathfrak{m} \mapsto u$. Moreover, as shown earlier, 1 is part of a minimal generating set of $\mathfrak{F}$, so that $1 + \mathfrak{m}\mathfrak{F}$ is part of a basis for the k-vector space $\mathfrak{F}/\mathfrak{m}\mathfrak{F}$. Hence, there is a k-linear map (which is also an A-module homomorphism) $\mathfrak{F}/\mathfrak{m}\mathfrak{F} \to k$ such that $1 + \mathfrak{m}\mathfrak{F} \mapsto 1 + \mathfrak{m}$. Finally, we define F_u simply by the composite $\mathfrak{F} \to \mathfrak{F}/\mathfrak{m}\mathfrak{F} \to k \to \overline{\mathfrak{F}}$. Now, let $\pi \in \mathrm{Hom}_A(\mathfrak{F}, \overline{\mathfrak{F}})$ denote the natural projection. We have an exact sequence

$$\mathrm{Hom}_A(\mathfrak{F}, \mathfrak{F}) \longrightarrow \mathrm{Hom}_A(\mathfrak{F}, \overline{\mathfrak{F}}) \longrightarrow \mathrm{Ext}^1_A(\mathfrak{F}, \mathfrak{F}\colon \mathfrak{F}).$$

Because A is reduced and T is a direct product of finitely many fields (namely, the residue class fields corresponding to the minimal primes of A), the natural A-module homomorphism $\mathfrak{F}\colon \mathfrak{F} \to \mathrm{Hom}_A(\mathfrak{F}, \mathfrak{F})$ is surjective and has kernel $0\colon \mathfrak{F} \subset T$ (see [24, Lemma 2.4.2]). But $0\colon \mathfrak{F} = 0$ since $1 \in \mathfrak{F}$. Therefore, $\mathfrak{F}\colon \mathfrak{F} \cong \mathrm{Hom}_A(\mathfrak{F}, \mathfrak{F})$. Furthermore, as $\mathfrak{F} \cong \mathfrak{o}$, we obtain

$$\mathrm{Ext}^1_A(\mathfrak{F}, \mathfrak{F}\colon \mathfrak{F}) \cong \mathrm{Ext}^1_A(\mathfrak{o}, \mathrm{Hom}_A(\mathfrak{o}, \mathfrak{o})) \cong \mathrm{Ext}^1_A(\mathfrak{o}, \mathfrak{h}).$$

Now, by [13, Proposition 2.2], the hypothesis (iii) gives that $\mathfrak{h}$ is isomorphic to a canonical module of A, so that $\mathrm{Ext}^1_A(\mathfrak{o}, \mathfrak{h}) = 0$ as $\mathfrak{o}$ is maximal Cohen–Macaulay (see 2.1). It follows that the map $\mathrm{Hom}_A(\mathfrak{F}, \mathfrak{F}) \to \mathrm{Hom}_A(\mathfrak{F}, \overline{\mathfrak{F}})$ induced by π is surjective. In particular, we can pick $G \in \mathrm{Hom}_A(\mathfrak{F}, \mathfrak{F})$ satisfying $F_u = \pi \circ G$. According to the isomorphism $\mathfrak{F}\colon \mathfrak{F} \cong \mathrm{Hom}_A(\mathfrak{F}, \mathfrak{F})$, there exists $\alpha \in \mathfrak{F}\colon \mathfrak{F}$ such that G is given by multiplication by α. Hence, $G(1) = \alpha$, and thus $F_u(1) = \pi(\alpha) = 0$, which is a contradiction because $F_u(1) = u \neq 0$. This proves the claim.

To finish the proof, the equality $\mathfrak{F}\colon \mathfrak{F} = \mathfrak{F}$ can be expressed as an isomorphism $\mathrm{Hom}_A(\mathfrak{o}, \mathfrak{o}) \cong \mathfrak{o}$, that is, $\mathfrak{h} \cong \mathfrak{o}$. It follows that $\mathfrak{o}$ is a canonical module of A because so is $\mathfrak{h}$. But then $\mathrm{Hom}_A(\mathfrak{o}, \mathfrak{o}) \cong A$, so that $\mathfrak{o} \cong A$. Therefore, $\Omega_{A/k}$ is free, as needed. □

Remark 5. (a) The final part of the above proof (from the use of [13, Proposition 2.2] on) admits a more general statement, to wit, if $\mathfrak{b}$ is an ideal of height 1 in A such that $\mathrm{Hom}_A(\mathfrak{b},\mathfrak{b}) \cong \omega_A$, then $\mathfrak{b} \cong A$. To see this, let $B = \mathrm{Hom}_A(\mathfrak{b},\mathfrak{b}) = \mathfrak{b} :_T \mathfrak{b}$, which we can choose to be ω_A. Consequently,

$$B :_T B = \omega_A :_T \omega_A = A.$$

But $B \subset B :_T B = A \subset B$, that is, $B = A$, which shows that A is Gorenstein. Now, because $\mathrm{Hom}_A(\mathfrak{b},\mathfrak{b}) \cong A$, we can apply [33, Theorem 3.1] to conclude that $\mathfrak{b} \cong A$, as claimed. In the same spirit, by replacing $\Omega_{A/k}$ with any torsion-free A-module M of rank 1, our theorem (as well as related results given in this note, such as Proposition 6, Proposition 8 and Corollary 9) admits an easy adaptation, which in fact yields $M \cong A$ instead of A being regular.

(b) A comment on the condition of $A/a\mathfrak{h}_{A/k}$ being Gorenstein. It was proved in [23] that if B is an unramified regular local ring and $\mathfrak{b}$ is an ideal such that $B/\mathfrak{b}$ is Gorenstein, then $\mathfrak{b}$ cannot be a product *if* height $\mathfrak{b} \geq 2$ (in particular, $\dim B \geq 2$). Also note B is already assumed to be regular. So, there is no conflict with our setting here.

3.2. Other characterizations

Proposition 6 describes (co)homological characterizations of the regularity of $(A,\mathfrak{m})$ in the presence of the torsion-freeness of $\Omega_{A/k}$ – see also Proposition 8 and, in the next subsection, Corollary 9 and Corollary 14. Wherever they appear throughout the paper, $\mathfrak{E}$ and $\mathrm{projdim}_A$ denote, respectively, the injective hull of $A/\mathfrak{m}$ and projective dimension over A.

Proposition 6. *Assume the setting and hypotheses of Berger's Conjecture. The following assertions are equivalent:*

(i) A *is regular.*
(ii) $\mathrm{Tor}_i^A(\Omega_{A/k}, A/\mathfrak{m}^\nu) = 0$ *for some* $i,\nu \geq 1$.
(iii) $\mathrm{Ext}_A^i(\Omega_{A/k}, \mathrm{Hom}_A(A/\mathfrak{m}^\nu, \mathfrak{E})) = 0$ *for some* $i,\nu \geq 1$.
(iv) $\mathrm{Ext}_A^i(\Omega_{A/k}, \mathfrak{m}^\nu) = 0$ *for some* $i,\nu \geq 1$.
(v) $\mathfrak{m}^\nu \otimes_A \Omega_{A/k}$ *is torsion-free for some* $\nu \geq 1$.
(vi) $\mathfrak{m}^\nu \otimes_A \mathrm{Hom}_A(\mathfrak{m}^\nu, A)$ *is torsion-free for some* $\nu \geq 1$.

Proof. If we assume that A is regular, then $\Omega_{A/k}$ is a free A-module and hence (ii), (iii) and (iv) hold. In addition, $\mathfrak{m}$ is a principal generated by a non-zero-divisor; hence, $\mathfrak{m}^\nu \cong A$ as A-modules for any $\nu \geq 1$, so that (v) and (vi) hold as well.

The equivalence between (ii) and (iii) follows by Ext-Tor duality (see [30, 1.4.1]), which gives for each $i \geq 0$ an isomorphism

$$\mathrm{Ext}_A^i(\Omega_{A/k}, \mathrm{Hom}_A(A/\mathfrak{m}^\nu, \mathfrak{E})) \cong \mathrm{Hom}_A(\mathrm{Tor}_i^A(\Omega_{A/k}, A/\mathfrak{m}^\nu), \mathfrak{E}),$$

the latter being the Matlis dual of $\mathrm{Tor}_i^A(\Omega_{A/k}, A/\mathfrak{m}^\nu)$.

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Now we show that (ii)$\Rightarrow$(i). So, assume $\mathrm{Tor}_i^A(\Omega_{A/k}, A/\mathfrak{m}^\nu) = 0$ for some $i, \nu \geq 1$, and consider first the case $i = 1$. Then there is a short exact sequence

$$0 \longrightarrow \mathfrak{m}^\nu \otimes_A \Omega_{A/k} \longrightarrow \Omega_{A/k} \longrightarrow (A/\mathfrak{m}^\nu) \otimes_A \Omega_{A/k} \longrightarrow 0,$$

which yields $\mathfrak{m}^\nu \otimes_A \Omega_{A/k} \cong \mathfrak{m}^\nu \Omega_{A/k}$. This gives an embedding

$$\mathfrak{m}^\nu \otimes_A \Omega_{A/k} \subset \Omega_{A/k}.$$

On the other hand, we know that $\Omega_{A/k}$ possesses a rank as an A-module (see Lemma 2(i)), and being by hypothesis torsion-free, it embeds into a free A-module of finite rank. It follows that $\mathfrak{m}^\nu \otimes_A \Omega_{A/k}$ is torsion-free. Thus, since A is a reduced local ring of positive depth, we are in a position to apply [8, Corollary 2.7] in order to conclude that $\Omega_{A/k}$ is free as an A-module, that is, A is regular.

Next, we consider the case $i \geq 2$. Then (ii) gives

$$\mathrm{Tor}_{i-1}^A(\Omega_{A/k}, \mathfrak{m}^\nu) = 0,$$

and this in turn implies, by [8, Corollary 1.3], that $\mathrm{projdim}_A \Omega_{A/k} < \infty$. Since $\Omega_{A/k}$ is torsion-free and $\dim A = 1$, we have $\mathrm{depth}\, \Omega_{A/k} = 1$ and then the well-known Auslander–Buchsbaum formula forces $\Omega_{A/k}$ to be free, as needed.

Now let us prove that (iv)$\Rightarrow$(i). By [8, Corollary 1.3], the module $\mathfrak{m}^\nu$ has for any $\nu \geq 1$ the so-called *strongly rigid* property (i.e., $\mathrm{projdim}_A N < \infty$ whenever N is a finite A-module satisfying $\mathrm{Tor}_s^A(N, \mathfrak{m}^\nu) = 0$ for some $s \geq 1$). Therefore, since $\mathrm{depth}\, A = 1$ and assuming (iv), we are in a position to apply [34, Theorem 5.8(2)] (with $r = 0$) to get

$$\mathrm{projdim}_A \Omega_{A/k} \leq i - 1 < \infty.$$

By the Auslander–Buchsbaum formula once again, we obtain that $\Omega_{A/k}$ is free.

Finally, the implication (v)$\Rightarrow$(i) (respectively, (vi)$\Rightarrow$(i)) follows by [8, Corollary 2.7] (respectively, [8, Corollary 1.5]). $\square$

We provide some remarks about Proposition 6.

Remark 7. (a) Regarding the assertion (ii) (also (iii)), it is clear that the case of interest is $\nu \geq 2$, since for $\nu = 1$, the condition $\mathrm{Tor}_i^A(\Omega_{A/k}, A/\mathfrak{m}) = 0$ forces $\mathrm{projdim}_A \Omega_{A/k} < \infty$ (and hence $\Omega_{A/k}$ must be free). Indeed, the $A/\mathfrak{m}$-vector space dimension of $\mathrm{Tor}_i^A(\Omega_{A/k}, A/\mathfrak{m})$ is equal to the ith Betti number of $\Omega_{A/k}$ (see [6, Proposition 1.3.1]).

(b) The implications (v)$\Rightarrow$(i) and (vi)$\Rightarrow$(i) do not use the torsion-freeness of $\Omega_{A/k}$. So it seems natural to ask whether, for some $\nu \geq 1$, at least one of the A-modules $\mathfrak{m}^\nu \otimes_A \Omega_{A/k}$ and $\mathfrak{m}^\nu \otimes_A \mathrm{Hom}_A(\mathfrak{m}^\nu, A)$ must be torsion-free if so is $\Omega_{A/k}$. According to Proposition 6, an affirmative answer would imply the validity of Berger's Conjecture.

(c) It is also worth noticing that the A-module $\mathrm{Hom}_A(A/\mathfrak{m}^\nu, \mathfrak{E})$ (which appears as an ingredient of (iii)) is finite in the complete case, that is, the situation where A is a quotient of $k[[X_1, \ldots, X_n]]$. This follows from [6, Theorem 3.2.13(b)]. Such a Matlis dual module will also play a role in Proposition 8.

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We close the subsection with further characterizations for the regularity of A, involving in particular the finiteness of the injective dimension over A – which we denote injdim_A – of suitable modules (see also Corollary 9 in the next subsection). For item (v) below, we recall for completeness the notion of integral closure of ideals. Given an ideal $\mathfrak{a}$ of A, an element $a \in A$ is said to be *integral over* $\mathfrak{a}$ if there exists an equation $a^n + b_1 a^{n-1} + \cdots + b_n = 0$, with $b_i \in \mathfrak{a}^i$, $i = 1, \ldots, n$. The elements of A that are integral over $\mathfrak{a}$ form an ideal, denoted $\bar{\mathfrak{a}}$. Clearly, $\mathfrak{a} \subset \bar{\mathfrak{a}}$, and $\mathfrak{a}$ is *integrally closed* if $\mathfrak{a} = \bar{\mathfrak{a}}$. In part (vii), $A^{'}$ stands for the integral closure of A in its total quotient ring.

Proposition 8. *Assume the setting and hypotheses of Berger's Conjecture. Let* ω_{A} *denote the canonical module of* A. *The following assertions are equivalent:*

(i) A *is regular.*
(ii) $\mathrm{injdim}_A \mathrm{Hom}_A(A/\mathfrak{m}^\nu, \mathfrak{E}) < \infty$ *for some* $\nu \geq 1$.
(iii) $\mathrm{Ext}^i_A(\mathfrak{m}^\nu, \mathfrak{m}^\nu \otimes_A \omega_A) = 0$ *for some* $i \geq 2$ *and* $\nu \geq 1$.
(iv) $\mathrm{Ext}^i_A(\mathfrak{m}^\nu, \Omega_{A/k}) = 0$ *for some* $i \geq 2$ *and* $\nu \geq 1$, *and* $\mathrm{Ext}^j_A(\mathfrak{m}^\mu, A) = 0$ *for all* $j \gg 0$ *and some* $\mu \geq 1$.
(v) $\mathrm{Ext}^i_A(\mathfrak{m}^\nu, \Omega_{A/k}) = 0$ *for some* $i \geq 2$ *and* $\nu \geq 1$, *and* $\mathrm{Ext}^j_A(\mathfrak{a}, A) = 0$ *for some* $j \geq 1$ *and some integrally closed* $\mathfrak{m}$*-primary ideal* $\mathfrak{a}$ *of* A.
(vi) $\mathrm{Ext}^i_A(\mathfrak{m}^\nu, \Omega_{A/k}) = 0$ *for some* $i \geq 2$ *and* $\nu \geq 1$, *and* $\mathrm{Ext}^1_A(\omega_A, A) = 0$.
(vii) $\mathrm{Ext}^i_A(\mathfrak{m}^\nu, \Omega_{A/k}) = 0$ *for some* $i \geq 2$ *and* $\nu \geq 1$, *and* $\mathrm{Ext}^1_A(A', A) = 0$.

Proof. If (i) takes place, then all A-modules have finite injective dimension and the A-modules $\mathfrak{m}^\nu$ and ω_A are free, so that all items hold. Notice that the second part of (v) holds by taking $\mathfrak{a} = \mathfrak{m} \cong A$, and the second part of (vii) follows because a regular ring is normal, that is, $A' = A$.

Now suppose (ii) and set $\rho := \mathrm{injdim}_A \mathrm{Hom}_A(A/\mathfrak{m}^\nu, \mathfrak{E})$. It follows that the module $\mathrm{Ext}^{\rho+1}_A(M, \mathrm{Hom}_A(A/\mathfrak{m}^\nu, \mathfrak{E}))$ vanishes for every A-module M. In particular,

$$\mathrm{Ext}^{\rho+1}_A(\Omega_{A/k}, \mathrm{Hom}_A(A/\mathfrak{m}^\nu, \mathfrak{E})) = 0$$

and thus A must be regular by Proposition 6.

Let us show that (iii)$\Rightarrow$(i). As we have recalled in the proof of Proposition 6, the module $\mathfrak{m}^\nu$ is strongly rigid for any $\nu \geq 1$. By (iii) and [34, Proposition 3.6], we get

$$\mathrm{injdim}_A \mathfrak{m}^\nu \otimes_A \omega_A < \infty,$$

and then the regularity of A follows by applying [34, Corollary 6.11] (with $C = \omega_A$).

Next, we verify that (iv)$\Rightarrow$(i). Because $\mathrm{Ext}^i_A(\mathfrak{m}^\nu, \Omega_{A/k}) = 0$ for some $i \geq 2 = \dim A + 1$, and since $\mathfrak{m}^\nu$ is strongly rigid, we have $\mathrm{injdim}_A \Omega_{A/k} < \infty$ by [34, Proposition 3.6]. Now, since A is not Artinian, the asymptotic vanishing of $\mathrm{Ext}^j_A(\mathfrak{m}^\mu, A)$ is equivalent to A being Gorenstein, according to [8, Proposition 1.6]. By a well-known fact (see [6, Exercise 3.1.25]), we get $\mathrm{projdim}_A \Omega_{A/k} < \infty$, which in the present setting, as we have seen, implies (i).

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In order to prove (v)⇒(i), note we must have, as above, $\mathrm{injdim}_A \Omega_{A/k} < \infty$. By virtue of [34, Corollary 3.9], the vanishing of $\mathrm{Ext}^j_A(\mathfrak{a}, A)$ for some $j \geq 1 = \dim A$ forces $\mathrm{injdim}_A A < \infty$, that is, A is Gorenstein. It follows, as above, that A is regular.

Finally, assume that (vi) or (vii) takes place. In either case, we once again have $\mathrm{injdim}_A \Omega_{A/k} < \infty$. If (vi) holds, then since A is reduced (hence generically Gorenstein) and $\dim A = 1$, the condition $\mathrm{Ext}^1_A(\omega_A, A) = 0$ is equivalent to A being Gorenstein (see [16, Corollary 2.2]). Now if (vii) holds, then as A is a reduced excellent local ring (hence analytically unramified) and $\dim A = 1$, the vanishing of $\mathrm{Ext}^1_A(A', A)$ is equivalent to A having the Gorenstein property, by [32, Corollary 2.3]. Thus, (i) follows. □

3.3. On the Gorenstein case and more characterizations

In order to tackle Berger's Conjecture in the case where A is Gorenstein, one option is to show that the torsion-freeness of $\Omega_{A/k}$ yields an A-module isomorphism $\mathrm{H}_{A/k} \cong A$. This follows from a more general fact: if S is a one-dimensional Gorenstein local ring and M is a finite S-module such that $\mathrm{Hom}_S(M, M)$ is free, then M is free (see [33, Theorem 3.1]). A different proof, in the context of Berger's Conjecture, is contained in the following byproduct of Theorem 4.

Corollary 9. *Assume the setting and hypotheses of Berger's Conjecture. The following assertions are equivalent:*

(i) A *is regular.*
(ii) A *is Gorenstein and* $\mathrm{H}_{A/k}$ *is free.*
(iii) $\mathrm{injdim}_A \mathrm{H}_{A/k} < \infty$.

Proof. The implications (i)⇒(ii) and (i)⇒(iii) are clear. Now set $\mathrm{H} := \mathrm{H}_{A/k}$ and note this A-module has rank 1, which follows by Lemma 2. Assume first that A is Gorenstein and H is free. In particular, because of the identification $\mathrm{H} = \mathfrak{h}$ as an ideal (see Notation 3), we have $\mathfrak{h} = (b)$ for some parameter $b \in \mathfrak{m}$. Now, for any parameter $a \in \mathfrak{m}$, the local ring $A/a\mathfrak{h} = A/(ab)$ is Gorenstein since so is A. By Theorem 4, A must be regular.

Finally, suppose (iii) holds, and recall that H is torsion-free because so is $\Omega_{A/k}$ by assumption. Hence, in the present setting, we must have $\operatorname{depth} \mathrm{H} = 1 = \operatorname{depth} A$. Thus, by [14, Proposition 3.2], the condition $\mathrm{injdim}_A \mathrm{H} < \infty$ forces A to be Gorenstein. Using [6, Exercise 3.1.25], we get $\mathrm{projdim}_A \mathrm{H} < \infty$ and hence H is necessarily free. It follows that (ii) holds, and so (i) holds by the preceding part. □

Remark 10. Recall that, for an A-module N, a *k-derivation of A with values in N* is a k-linear map $\delta\colon A \to N$ satisfying $\delta(ab) = a\delta(b) + b\delta(a)$ for all $a, b \in A$. Such derivations are collected in an A-module denoted $\mathrm{Der}_k(A, N)$. Now we recall that $\Omega_{A/k}$ comes equipped with a universal derivation $\delta_{A/k}\colon A \to \Omega_{A/k}$, that is, a k-derivation with the property that the A-linear map $\mathrm{Hom}_A(\Omega_{A/k}, N) \to \mathrm{Der}_k(A, N)$ given by composition with $\delta_{A/k}$ is an isomorphism. In particular, taking $N = \Omega_{A/k}$,

$$\mathrm{H}_{A/k} \cong \mathrm{Der}_k(A, \Omega_{A/k}).$$

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It follows that the assertions of Corollary 9 are also equivalent to the following one:

(iv) A is Gorenstein and $\mathrm{Der}_k(A, \Omega_{A/k}) = A\delta_{A/k}$.

The proof of Corollary 14 – where A is not required to be Gorenstein *a priori* – will crucially rely on Corollary 9 together with suitable Gorensteiness criteria from the literature. First, we provide a lemma which characterizes (e.g., by means of the length of a certain Artinian module) the vanishing

$$\mathrm{Ext}^1_A(\Omega_{A/k}, A) = 0$$

in the context of Berger's Conjecture; see also Remark 13. The characterizations will automatically provide some of the equivalences to be presented in Corollary 14.

Before stating the lemma, we introduce for convenience a piece of notation.

Notation 11. For a non-zero-divisor $a \in \mathfrak{m}$, we consider the relative derivation module $\mathrm{Der}_k(A, A/(a))$, which has finite length since $A/(a)$ is Artinian. We put

$$\mathfrak{d}(A, a) := \mathrm{length}\,\mathrm{Der}_k(A, A/(a)).$$

Notice for completeness that, since $\dim A = 1$, the Hilbert–Samuel multiplicity $\mathfrak{e}(A)$ of A is simply the leading coefficient of the Hilbert–Samuel polynomial of A. Also we recall that the property of a parameter $a \in \mathfrak{m} \setminus \mathfrak{m}^2$ being *superficial* in the one-dimensional local ring A – note such an element does exist as A has infinite residue field – forces (and is in fact equivalent to) the principal ideal (a) being a minimal reduction of $\mathfrak{m}$. For more information, see [20, Remark 1.5], also [24, 8.6].

Lemma 12. *Assume the setting and hypotheses of Berger's Conjecture. The following assertions are equivalent:*

(i) $\mathrm{Ext}^1_A(\Omega_{A/k}, A) = 0$.
(ii) $\Omega_{A/k} \otimes_A \omega_A$ *is torsion-free.*
(iii) $\mathfrak{d}(A, a) = \mathfrak{e}(A)$ *for some* (*any*) *superficial element* a.

Proof. Under the present hypotheses, $\Omega_{A/k}$ is a maximal Cohen–Macaulay A-module possessing a rank, and the one-dimensional local ring A, being reduced, is Gorenstein locally at the prime ideals of height zero (hence, Gorenstein locally on its punctured spectrum). Now, the equivalence (i)$\Leftrightarrow$(ii) follows directly by [16, Lemma 2.1].

In order to prove that (i) and (iii) are equivalent, we apply once again [16, Lemma 2.1], which gives that (i) holds if and only if there is a length equality

$$\mathrm{length}\,\mathrm{Hom}_{A/(a)}(\Omega_{A/k}/a\Omega_{A/k}, A/(a)) = \mathrm{length}\,\Omega_{A/k}/a\Omega_{A/k},$$

where a is any given parameter of A. Now, by standard homological algebra, there are $A/(a)$-module isomorphisms

$$\mathrm{Hom}_{A/(a)}(\Omega_{A/k}/a\Omega_{A/k}, A/(a)) \cong \mathrm{Hom}_A(\Omega_{A/k}, \mathrm{Hom}_{A/(a)}(A/(a), A/(a))),$$

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$$\cong \mathrm{Hom}_A(\Omega_{A/k}, A/(a)) \cong \mathrm{Der}_k(A, A/(a)),$$

where the last isomorphism is due to Remark 10 with $N = A/(a)$. Thus, we have shown that (i) holds if and only if $\mathfrak{d}(A, a) = \operatorname{length} \Omega_{A/k}/a\Omega_{A/k}$. On the other hand, because $\Omega_{A/k}$ is Cohen–Macaulay and, by Lemma 2(i), has a well-defined rank equal to 1, we can write (see [17])

$$\operatorname{length} \Omega_{A/k}/a\Omega_{A/k} = \operatorname{length} A/(a).$$

But $\operatorname{length} A/(a) = \mathfrak{e}(A)$ whenever (a) is a minimal reduction of $\mathfrak{m}$ (see [24, Proposition 11.2.2]), that is, if a is superficial. It follows that (i) is equivalent to $\mathfrak{d}(A, a) = \mathfrak{e}(A)$. $\square$

Remark 13. Consider the setting and hypotheses of Berger's Conjecture, and in addition, suppose A is complete and k is algebraically closed. The local ring A is said to be *rigid* if it admits no infinitesimal deformations, which is known to be equivalent to the condition $\mathrm{Ext}^1_A(\Omega_{A/k}, A) = 0$ characterized in Lemma 12. The *rigidity conjecture* predicts that A is rigid if and only if A is regular. Now, by Lemma 12, the algebra A is rigid if and only if $\Omega_{A/k} \otimes_A \omega_A$ is torsion-free, which can alternatively be seen by means of an isomorphism

$$\mathrm{Ext}^1_A(\Omega_{A/k}, A) \cong \mathrm{Hom}_A(\tau_A(\Omega_{A/k} \otimes_A \omega_A), \mathfrak{E}),$$

where the latter is the Matlis dual of the A-torsion of $\Omega_{A/k} \otimes_A \omega_A$. It follows that, in the Gorenstein case, Berger's Conjecture is equivalent to the rigidity conjecture. We refer to [19, p. 11, first paragraph].

Recall that the embedding dimension $\operatorname{edim} A$ of a local ring $(A, \mathfrak{m})$ is the minimal number of generators of $\mathfrak{m}$. The corollary is as follows.

Corollary 14. *Assume the setting and hypotheses of Berger's Conjecture. Let* ω_{A} *denote the canonical module of* A. *The following assertions are equivalent:*

(i) A *is regular.*
(ii) $\mathrm{H}_{A/k}$ *is free,* $\mathfrak{d}(A, a) = \mathfrak{e}(A)$ *for a superficial element* a *and* $\mathfrak{e}(A) \leq 2 \operatorname{edim} A - 1$.
(iii) $\mathrm{H}_{A/k}$ *is free,* $\Omega_{A/k} \otimes_A \omega_A$ *is torsion-free and* $\mathfrak{e}(A) \leq 2 \operatorname{edim} A - 1$.
(iv) $\mathrm{H}_{A/k}$ *is free,* $\mathrm{Ext}^1_A(\Omega_{A/k}, A) = 0$ *and* $\mathfrak{e}(A) \leq 2 \operatorname{edim} A - 1$.
(v) $\mathrm{H}_{A/k}$ *is free, and* $\mathrm{Ext}^i_A(\mathfrak{m}^\nu, A) = 0$ *for some* $\nu \geq 1$ *and all* $i \gg 0$.
(vi) $\mathrm{H}_{A/k}$ *is free, and* $\mathrm{Ext}^i_A(\mathfrak{a}, A) = 0$ *for some* $i \geq 1$ *and some integrally closed* $\mathfrak{m}$*-primary ideal* $\mathfrak{a}$ *of* A.
(vii) $\mathrm{H}_{A/k}$ *is free, and* $\mathrm{Ext}^1_A(\omega_A, A) = 0$.
(viii) $\mathrm{H}_{A/k}$ *is free, and* $\mathrm{Ext}^1_A(A', A) = 0$.

Proof. First, suppose (i). Then $\mathfrak{e}(A) = 1$ and $\operatorname{edim} A = \dim A = 1$, so that $\mathfrak{e}(A) = 2 \operatorname{edim} A - 1$. Moreover, $\Omega_{A/k} \cong A$; hence, $\mathrm{H}_{A/k} \cong A$ and, for any $a \in \mathfrak{m}$, we have $\mathrm{Der}_k(A, A/(a)) \cong \mathrm{Hom}_A(\Omega_{A/k}, A/(a)) \cong A/(a)$, so that $\mathfrak{d}(A, a) = \operatorname{length} A/(a)$, which

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by [24, Proposition 11.2.2] is equal to $\mathfrak{e}(A)$ if for instance $a \in \mathfrak{m} \setminus \mathfrak{m}^2$ is any uniformizing parameter of A. In addition, the A-modules ω_A and $\mathfrak{m}^\nu$ (for any $\nu \geq 1$) are free as well, and in particular, we can take $\mathfrak{a} = \mathfrak{m} \cong A$ in (vi). Finally, $A' = A$ if A is regular. Therefore, all items follow.

Lemma 12 immediately gives (ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv).

Now we prove (iv)$\Rightarrow$(i). Recall that $\Omega_{A/k}$ is a rank 1 A-module (see Lemma 2(i)), which is, in addition, minimally generated by $\operatorname{edim} A$ elements; for the latter fact in the algebraic (respectively, algebroid) case, see [27, Corollary 6.5(b)] (respectively, [27, Corollary 13.15]). Thus, condition $2\operatorname{edim} A > \mathfrak{e}(A)$ along with the vanishing of $\operatorname{Ext}^1_A(\Omega_{A/k}, A)$ put us in a position to apply [32, Theorem 3.1], which gives that A is Gorenstein. Now the regularity of A follows by Corollary 9.

Finally, in each of the assertions (v), (vi), (vii) and (viii), we observe that, apart from the condition of $\mathrm{H}_{A/k}$ being free, the remaining hypotheses guarantee that A is Gorenstein, as explained in the proof of Proposition 8. Once again, we apply Corollary 9 to conclude that (i) holds. $\square$

Remark 15. If in addition A is a domain, we can add some equivalent statements to the list of Corollary 14 without supposing the freeness of $\mathrm{H}_{A/k}$. Indeed, from the proof above and [15, Satz 6] (see also [5, Theorem 10]), we have for instance the following assertions:

(ix) $\mathfrak{d}(A, a) = \mathfrak{e}(A)$ for a superficial element a, and $\mathfrak{e}(A) \leq \min\{13, 2\operatorname{edim} A - 1\}$.

(x) $\mathfrak{e}(A) \leq 13$ and $\operatorname{Ext}^i_A(\mathfrak{a}, A) = 0$ for some $i \geq 1$ and some integrally closed $\mathfrak{m}$-primary ideal $\mathfrak{a}$ of A.

A natural question is whether, in item (vi) of Proposition 6, the ideal $\mathfrak{m}^\nu$ can be replaced with $\mathfrak{h}_{A/k}$. The answer is affirmative if we require suitable extra conditions. To see this, let $(A, \mathfrak{m})$ be as above and assume in addition that it is a domain. Recall that, for a non-zero ideal $\mathfrak{a}$ of A, we can consider

$$A^{\mathfrak{a}} := \bigcup_{s \geq 1} \mathfrak{a}^s : \mathfrak{a}^s$$

as a subring of the integral closure of A in its fraction field (note that taking $\mathfrak{a} = \mathfrak{m}$, we get the quadratic transform $A^{\mathfrak{m}}$ of A). The ideal $\mathfrak{a}$ is said to be *stable* if $A^{\mathfrak{a}} = \mathfrak{a} : \mathfrak{a}$.

Corollary 16. *Assume the setting and hypotheses of Berger's Conjecture, and in addition that* A *is a domain. Suppose the following conditions hold:*

(i) $\operatorname{Ext}^1_A(\mathfrak{F}, A) = 0$ *for some non-zero fractional ideal* $\mathfrak{F}$ *in* $A^{\mathfrak{m}}$.

(ii) $\mathfrak{h}_{A/k}$ *is stable.*

(iii) $\mathfrak{h}_{A/k} \otimes_A \operatorname{Hom}_A(\mathfrak{h}_{A/k}, A)$ *is torsion-free.*

Then, A *is regular.*

Proof. By [32, Corollary 3.2], condition (i) is equivalent to A being Gorenstein. Now, under the hypotheses (ii) and (iii), the ideal $\mathfrak{h}$ must be principal by [9, Proposition 3.1]. This means precisely that $\mathrm{H}_{A/k}$ is free as an A-module. Thus, Corollary 9 applies. $\square$

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Remark 17. The above corollaries deal – directly or indirectly – with the module $\mathrm{H}_{A/k} = \mathrm{Ext}_A^0(\Omega_{A/k}, \Omega_{A/k})$. Thus, it is natural to ask whether $\mathrm{Ext}_A^1(\Omega_{A/k}, \Omega_{A/k})$ also plays a role in the problem. First, it has been predicted in [25, Conjecture 1.2] that if S is a one-dimensional Gorenstein local domain and M is a torsion-free finite S-module such that $\mathrm{Ext}_S^1(M, M) = 0$, then M must be a free S-module. Now, assuming the validity of this conjecture, then in order to settle Berger's Conjecture in case A is a Gorenstein domain, it suffices to prove that the torsion-freeness of $\Omega_{A/k}$ forces

$$\mathrm{Ext}_A^1(\Omega_{A/k}, \Omega_{A/k}) = 0.$$

This vanishing condition is, in turn, equivalent to the surjectivity of a certain map involving the torsion-free module $\Omega_{A/k}$ and the conormal module of the ideal $\mathfrak{A}$ defining A as a quotient of either $k[X_1, \ldots, X_n]_{(X_1,\ldots,X_n)}$ or $k[[X_1, \ldots, X_n]]$. More precisely, as recalled in the proof of Lemma 2(i), we have first a short exact sequence

$$0 \longrightarrow \mathfrak{A}/\mathfrak{A}^{(2)} \longrightarrow A^n \longrightarrow \Omega_{A/k} \longrightarrow 0,$$

to which we can apply the functor $\mathrm{Hom}_A(-, \Omega_{A/k})$. Identifying $\mathrm{Hom}_A(A^n, \Omega_{A/k}) \cong \Omega_{A/k}^{\oplus n}$ and noticing that $\mathrm{Hom}_A(\mathfrak{A}/\mathfrak{A}^{(2)}, \Omega_{A/k}) \cong \mathrm{Hom}_A(\mathfrak{A}/\mathfrak{A}^2, \Omega_{A/k})$ (since $\Omega_{A/k}$ is torsion-free and $\mathfrak{A}^{(2)}/\mathfrak{A}^2$ is the torsion of $\mathfrak{A}/\mathfrak{A}^2$), we get an exact sequence

$$\Omega_{A/k}^{\oplus n} \xrightarrow{\Phi} \mathrm{Hom}_A(\mathfrak{A}/\mathfrak{A}^2, \Omega_{A/k}) \longrightarrow \mathrm{Ext}_A^1(\Omega_{A/k}, \Omega_{A/k}) \longrightarrow 0.$$

It follows that the module $\mathrm{Ext}_A^1(\Omega_{A/k}, \Omega_{A/k})$ vanishes if and only if Φ is an epimorphism. Also notice that $\ker \Phi \cong \mathrm{H}_{A/k}$.

3.4. A connection to Rees algebras

We close the paper by pointing out a quite surprising link to the theory of blowup rings. First, as a preparation for Corollary 19 below, recall that the *Rees algebra* of an ideal $\mathfrak{a}$ in a ring A is the graded algebra

$$\mathfrak{R}_A(\mathfrak{a}) = \bigoplus_{i \geq 0} \mathfrak{a}^i t^i \subset A[t],$$

where t is an indeterminate over A. In terms of generators, if $\mathfrak{a} = (a_1, \ldots, a_r)$ then $\mathfrak{R}_A(\mathfrak{a}) = A[a_1 t, \ldots, a_r t]$.

Notation 18. Given $a, b \in A$, we consider the monic polynomial $t^2 + at + b \in A[t]$. Now, following [1], we set

$$\mathrm{R}(\mathfrak{a})_{a,b} := \mathfrak{R}_A(\mathfrak{a})/(t^2 + at + b)A[t] \cap \mathfrak{R}_A(\mathfrak{a}).$$

It follows from [1, p. 138] that if A is a one-dimensional reduced local ring and the ideal $\mathfrak{a}$ contains a non-zero-divisor, then $\mathrm{R}(\mathfrak{a})_{a,b}$ is Cohen–Macaulay. The next result reveals that the Gorensteiness of this algebra plays a role in regard to Berger's Conjecture.

Corollary 19. *Assume the setting and hypotheses of Berger's Conjecture. The following assertions are equivalent:*

(i) A *is regular.*
(ii) $\mathrm{R}(\mathfrak{h}_{A/k})_{a,b}$ *is Gorenstein for all* $a, b \in A$.
(iii) $\mathrm{R}(\mathfrak{h}_{A/k})_{a,b}$ *is Gorenstein for some* $a, b \in A$.

Proof. If A is regular, then $\Omega_{A/k}$ is free and consequently $\mathfrak{h} \cong A$ as A-modules. In this case, the ring $\mathfrak{R}_A(\mathfrak{h})$ can be identified with the regular domain $A[t]$. Thus, for any pair $a, b \in A$, we have $\mathrm{R}(\mathfrak{h})_{a,b} = A[t]/(t^2 + at + b)$, which clearly is Gorenstein. This shows (i)⇒(ii)⇒(iii).

Finally, by [1, Corollary 3.3], the ring $\mathrm{R}(\mathfrak{h})_{a,b}$ is Gorenstein for some $a, b \in A$ if and only if $\mathfrak{h}$ is isomorphic to a canonical module of A. In this case, the proof of Theorem 4 applies and we thus conclude that A is regular if (iii) holds. □

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