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CONFORMAL GEOMETRY AND THE CYCLIDES OF DUPIN

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Introduction. A Riemannian manifold (M, g) is said to be *conformally flat* if every point has a neighborhood conformal to an open set in Euclidean space. Over the past thirty years, many papers have appeared attacking, with varying degrees of success, the problem of classifying the conformally flat spaces which occur as hypersurfaces in Euclidean space. Most of these start from the following pointwise result of Schouten.

THEOREM (Schouten [**24**]). *Let M be a hypersurface immersed in Euclidean space E^{n+1}, $n \geqq 4$. Then M is conformally flat in the induced metric if and only if at least $n - 1$ of the principal curvatures coincide at each point of M.*

This characterization fails when $n = 3$. An example of a conformally flat hypersurface in E^4 having three distinct principal curvatures was given by Lancaster [**14**, p. 6].

In this paper, we determine the conformally flat hypersurfaces in E^{n+1}, $n \geqq 4$, which are tautly embedded. As a by-product, we provide examples which demonstrate the incompleteness of some previous classification results (see concluding remarks).

The cyclides of Dupin play an important role both as examples of tautly embedded hypersurfaces and as counterexamples to previous erroneous assertions in the literature. The compact cyclides (which in this paper we call the *ring* cyclides, see next section), were characterized in terms of their focal sets in the first theorem of [**4**]. The analogous characterization of the non-compact parabolic cyclides is presented in this paper as Theorem 1.

The two main theorems of this paper relate several concepts which are invariant under conformal transformations of the ambient Euclidean space. These are conformal flatness, the tautness of the embedding, the structure of the focal set and, of course, the cyclides themselves. Thus our results may be regarded as belonging to conformal geometry.

All manifolds and maps are assumed to be C^∞ unless stated otherwise. Notation generally follows Kobayashi and Nomizu [**11**].

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1. Preliminaries. We begin by reviewing the formal definition of a taut immersion (see [**2**] for more detail). Suppose $\phi: M \to \mathbf{R}$ is a Morse function on a manifold M. If for all real r, $M_r = \phi^{-1}(-\infty, r]$ is compact, the following (Morse) inequalities

(1) $\mu_k \geqq \beta_k$

hold, where μ_k is the number of critical points of index k which ϕ has on M_r, and β_k is the k-th Betti number of M_r over any field F. The function ϕ is called a *T-function* if there exists a field F such that (1) is an equality for all r and k.

Notation. Let $f: M \to E^m$ be an immersion and let $p \in E^m$. The function

$$x \in M \to |f(x) - p|^2$$

is denoted by L_p.

Definition. An immersion $f: M \to E^m$ is said to be *taut* if every Morse function of the form L_p, $p \in E^m$, is a T-function.

Remark 1. If $f: M \to E^m$ is a taut immersion, it is not necessary for M to be compact. However, f must be proper, and thus M is complete in the induced metric. Carter and West [**2**] showed that a taut immersion must, in fact, be an embedding. This is in contrast with the situation for *tight* (minimal total absolute curvature) immersions on which every non-degenerate linear height function is required to be a T-function. In that case M must be compact, and the immersion need not be an embedding. Indeed, many of the most significant examples (Kuiper's non-orientable tight surfaces in E^3) have self-intersections. We refer the reader to [**2**] for a detailed comparison of the two conditions, including a proof that for compact M, tautness implies tightness. Finally, we note that for 2-dimensional surfaces, tautness is equivalent to the spherical two-piece property of Banchoff [**1**].

Tautness is a conformal property, i.e., it is preserved under conformal transformations of the ambient space [**2**, p. 703]. It is a restriction which can force the manifold into some rather special shapes. For example, a tautly embedded sphere S^n in E^m must be a round sphere $S^n \subset E^{n+1} \subset E^m$ [**20**], while a product manifold $S^k \times S^{n-k}$ which is taut in E^{n+1} must be a cyclide of Dupin [**4**, p. 184].

The round cyclides are analogous to tori of revolution in E^3, and an explicit description of them can be found in [**4**, p. 180]. The image of a round cyclide M under an inversion is called a *ring cyclide* if the center of inversion does not lie on M. Otherwise, it is called a *parabolic cyclide.* A parabolic cyclide is not compact but is tautly embedded. The ring cyclides are diffeomorphic to $S^k \times S^{n-k}$, for some $k < n$, while the parabolic cyclides are diffeomorphic to punctured round cyclides. Alternatively, the cyclides may be considered as the image under stereographic projection of a standard product of spheres $S^k \times S^{n-k}$ in a Euclidean sphere S^{n+1} of arbitrary radius. The image is a ring cyclide if the pole of the projection is not on $S^k \times S^{n-k}$, and a parabolic cyclide, otherwise. The names of the cyclides are taken from the classical names for their two-dimensional analogues [**15**, pp. 290–293]. In the classical literature our parabolic cyclides were actually called parabolic ring cyclides to distinguish them from the parabolic horn and spindle cyclides. (Horn and spindle cyclides are surfaces with singularities obtained by inverting circular cylinders and cones. The unbounded cyclides are called parabolic since the focal sets are parabolas. For more detail see [**8**, p. 312], [**10**, p. 217], [**15**].)

We now summarize some basic results on taut immersions. The proofs of (i) and (iii) may be found in [**2**]. Statement (ii) follows from (i) since a complete embedded hypersurface is orientable. (See [**23**] for an elementary proof.)

LEMMA 1. *Let $f: M^n \to E^{n+1}$ be a taut immersion of a connected manifold M. Then,*

(i) *f is a proper embedding and thus M is complete in the induced metric.*

(ii) *M is orientable.*

(iii) *If any point of M is an umbilic, then $M = S^n$ or E^n, and f embeds M as a round sphere or hyperplane.*

2. A characterization of the parabolic cyclide in terms of its focal set. In our previous paper [**4**], the ring cyclides (there referred to as the compact cyclides of Dupin) were characterized as the only compact connected hypersurfaces embedded in Euclidean space whose focal set consists of two distinct submanifolds of codimension greater than 1. This result generalizes the classical theorem for surfaces in E^3 which states that if the focal set consists of a pair of curves, then M is a cyclide of Dupin.

We now state our characterization of the parabolic cyclides which generalizes Theorem 1 of [**4**] to the non-compact case.

THEOREM 1. *Let M be a connected complete, non-compact hypersurface embedded in E^{n+1}. If the focal set consists of two distinct submanifolds* (*in the sense explained below*) *of codimension greater than* 1, *then M is a parabolic cyclide.*

The hypotheses of the theorem, while intuitively appealing, need a precise formulation. We now state and justify the appropriate technical hypotheses.

Hypotheses of Theorem 1. *M is a connected, complete, non-compact hypersurface embedded in E^{n+1} such that:*

(a) *At each point of M, there are exactly two distinct principal curvatures, neither of which is identically zero.*

(b) *These principal curvatures are constant along the leaves of their principal foliations.*

For a manifold M satisfying the above hypotheses, each principal curvature function λ_i determines a leaf space $M_i = U_i/T_i$ with a natural $(n - \nu_i)$-dimensional manifold structure. (Here U_i is the open set of M on which $\lambda_i \neq 0$, and ν_i is the constant multiplicity of λ_i). Furthermore, the focal map

$$f_i : U_i \to E^{n+1}$$

defined by

$$f_i(x) = f(x) + (1/\lambda_i(x))\xi(x)$$

factors through an immersion $g_i : M_i \to E^{n+1}$ for each i. ($f : M \to E^{n+1}$ is the embedding determining the hypersurface.) The above statements follow immediately from Theorem 3.1 (if $\nu_i > 1$) and Theorem 3.3 (if $\nu_i = 1$) of [**5**]. Here we are using the fact that a complete embedded hypersurface is orientable.

As the above discussion indicates, the constant multiplicity implied by hypothesis (a) is necessary for each sheet $f_i(U_i)$ of the focal set to be a submanifold in any reasonable sense, while (b) is necessary for the codimension of the focal set to be greater than one. It is important to note that (b) is automatically satisfied if $\nu_i > 1$. However, if $\nu_i = 1$, Theorem 2.1 of [**5**] shows that f_i has rank n at any point where λ_i is non-constant in the direction of its principal vector. In a neighborhood of such a point, f_i will be a codimension 1 embedding. Hence, it is necessary to assume (b) in the case $\nu_i = 1$.

We now summarize some results about the leaves of the principal foliations which we will need in our proof of Theorem 1.

LEMMA 2. *Let M be a connected, complete hypersurface embedded in E^{n+1} which satisfies hypotheses* (a) *and* (b). *Then,*

(1) *if $\lambda_i \neq 0$ on a leaf L of T_i, then L is a small or great ν_i-sphere of the n-sphere centered at the λ_i-focal point determined by L. Moreover, L is a great sphere if and only if each point of L is a critical point of λ_i;*

(2) *if $\lambda_i = 0$ on L, then L is a ν_i-sphere or ν_i-plane lying in the common tangent plane to M at each of its points.*

Proof. The fact that the leaves are ν_i-spheres or ν_i-planes follows from Proposition 3.2 of [**5**] if $\nu_i > 1$ and from Theorem 3.2 of [**5**] if $\nu_i = 1$. In case (1), since $(f_i)_* \equiv 0$ on T_i by Theorem 2.1 of [**5**], f_i is constant on L and L lies on the n-sphere in question. The fact that L is a great sphere if and only if λ_i has a critical value along L is just Lemma 2 of [**4**]. We note that this is a local result and does not require the compactness assumption in [**4**]. In case (2), one easily shows that the unit normal vector to M is Euclidean parallel along L, and hence L lies in the common tangent plane determined by its points.

Before we begin the proof of Theorem 1, we give a full description of the focal sets of the cyclides. This should aid the reader in understanding the proof. Such a description in the case $n > 2$ has not been given previously.

Remark 2. The focal sets of the cyclides.

We first consider the round cyclide $C(k, a, b)$ defined in [**4**, p. 180]. One sheet of the focal set is the core $(n - k)$-sphere of radius a centered at the origin in E^{n-k+1}. The second sheet is the k-plane E^k orthogonal to E^{n-k+1} at the origin. Each leaf of one family gives a point on the sphere while each point of the focal k-plane comes from two leaves (one inner, one outer) of the cyclide.

The focal set of a non-round ring cyclide is less degenerate. In visualizing this construction, the diagram in [**4**, p. 183] may help. Assume that λ_1 is the principal curvature of multiplicity k which is never zero, while λ_2 assumes both positive and negative values. There are four distinguished extreme leaves (two from each family corresponding to the extrema of the associated principal curvatures) and these are totally geodesic in the hyperspheres centered at the focal points they determine. There is exactly one line which is simultaneously a diameter of all four leaves. On this line, let p_1 and p_2 be the centers of the extreme λ_1-leaves and q_1, q_2 be the centers of the extreme λ_2-leaves. The sheet of the focal set coming from λ_1 is the ellipsoid of revolution consisting of all points x in the $(n - k + 1)$-plane spanned by the extreme λ_2-leaves satisfying

$$d(q_1, x) + d(q_2, x) = d(p_1, p_2).$$

In the usual terminology of conic sections, q_1 and q_2 are foci of the ellipsoid, and p_1 and p_2 are vertices along the major axis.

The second focal submanifold is the hyperboloid of revolution consisting of all points x in the $(k + 1)$-plane determined by the two extreme λ_1-leaves which satisfy

$$d(p_1, x) - d(p_2, x) = \pm\, d(q_1, q_2).$$

The points p_1 and p_2 are foci of this hyperboloid while q_1 and q_2 are its vertices. This focal submanifold has two connected components. This explains the "degeneracy" of the second focal submanifold of the round cyclide which "splits" into two components when the round cyclide is deformed to a non-round cyclide. In the classical case, an ellipse and hyperbola situated in this way were called *focal conics* (see, for example, [**8**, p. 226 and p. 312] and [**10**, p. 20]).

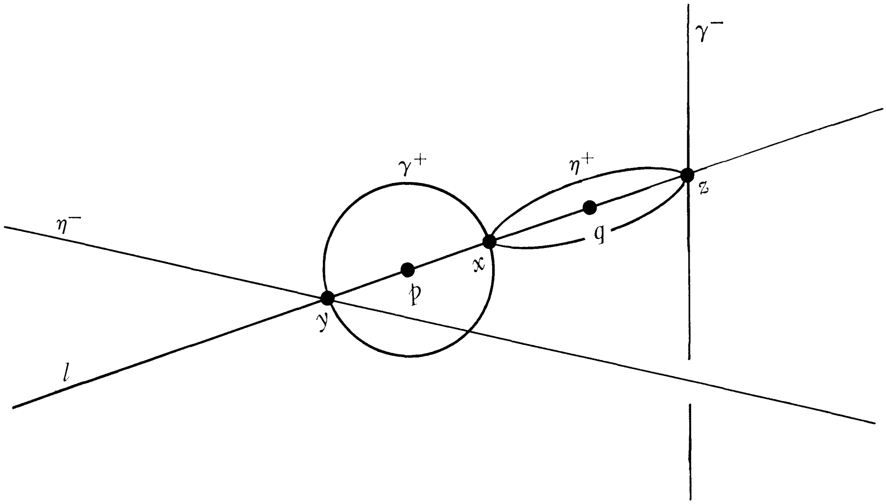

FIGURE 1

Finally, the parabolic cyclide C is the image of a round cyclide under an inversion in a sphere whose center lies on the round cyclide. As before assume λ_1 has constant multiplicity k and λ_2 has multiplicity $n - k$. In this case, $\lambda_1 \geqq 0$ and $\lambda_2 \leqq 0$ on C, and there is one non-compact leaf in each principal foliation which we denote by γ^- for T_1 and η^- for T_2. There is one other compact extreme leaf γ^+ for T_1 and η^+ for T_2 on which λ_i assumes it maximum, respectively minimum, for $i = 1, 2$. Thus γ^+ and η^+ are totally geodesic in the hyperspheres centered at the corresponding focal points p of γ^+ and q of η^+. The distinguished T_1-leaves γ^+ and γ^- lie in the same $(k + 1)$-plane E^{k+1}, while the T_2-leaves η^+ and η^- lie in an orthogonal $(n - k + 1)$-plane E^{n-k+1}. The two spaces intersect in the line l determined by p and q. The leaves γ^+ and γ^- intersect η^+ at antipodal points x and z, and the leaves η^+ and η^- intersect γ^+ at antipodal points x and y (see figure 1). The sheet of the focal set of C determined by λ_1 is an $(n - k)$-dimensional paraboloid of revolution defined as follows. The focus of the paraboloid is the point q and the directrix is the $(n - k)$-plane V parallel to η^- in E^{n-k+1} such that $d(p, q) = d(p, V)$. The paraboloid is then the set of x in E^{n-k+1} such that

$$d(x, q) = d(x, V).$$

Note that p is the vertex of the paraboloid. Similarly, the λ_2-focal set is a k-dimensional paraboloid in E^{k+1}. The focus is p and the directrix is the k-plane W parallel to γ^- in E^{k+1} such that $d(q, p) = d(q, W)$. The paraboloid consists of all points $x \in E^{k+1}$ satisfying

$$d(x, p) = d(x, W).$$

Thus q is the vertex of this paraboloid. In the classical case, such a pair of parabolas were called *focal parabolas*, and the reader is referred to the same sources given for the ellipse and hyperbola.

We close this discussion by noting that the above facts can be verified by observing that the extreme leaves mentioned are totally geodesic, the principal curvatures are constant along the leaves, and that each compact leaf of one family intersects each leaf of the other family in exactly one point.

The proof of Theorem 1. For the remainder of this section, assume that M satisfies the hypotheses of Theorem 1. The principal curvature λ_i has constant multiplicity ν_i, and we assume that the unit normal field ξ has been chosen so that $\lambda_1 > \lambda_2$ on M. Since neither principal curvature is identically zero, each principal foliation has some compact leaves.

LEMMA 3. *Each compact T_i-leaf meets each T_j-leaf, $j \neq i$, in exactly one point.*

Proof. Let γ be a compact leaf of T_i. We begin by showing that γ intersects any leaf of T_j, $j \neq i$, in at most one point. First suppose $\lambda_i \neq 0$ on γ. For $x \in \gamma$, let η_x be the T_j-leaf through x. Since $\lambda_i \neq 0$ on γ, the leaf γ lies on a hypersphere S centered at the λ_i-focal point $p = f_i(x)$. If $\lambda_j(x) \neq 0$, then η_x lies on a hypersphere centered at $q = f_j(x)$. Since p, q and x are distinct and collinear, the hyperspheres have only x in common and $\gamma \cap \eta_x = \{x\}$. If $\lambda_j(x) = 0$, then η_x lies in the tangent plane to M at x. Since this hyperplane intersects S only at x, we again have $\gamma \cap \eta_x = \{x\}$.

If $\lambda_i = 0$ on γ, then $\lambda_j(x) \neq 0$ for all $x \in \gamma$. Thus γ lies in the tangent hyperplane to M at each $x \in \gamma$, whereas η_x lies in a hypersphere which intersects the tangent hyperplane only at x. As above, $\gamma \cap \eta_x = \{x\}$.

We next show that γ does indeed intersect every leaf of T_j. Let N be the union of η_x as x ranges over γ. Using coordinate systems which arise naturally in the theory of foliations and the continuation theorem for such coordinate systems of [**21**, p. 10], one easily shows that N is an open submanifold of M. We now show that $N = M$ by proving that N is closed.

Let $\{x_n\}$ be a sequence of points in N converging to $x \in M$. If an infinite number of x_n lie on the same leaf L of T_j, then x is also on L, since L is complete. Hence $x \in N$. If not, let L_n be the leaf of T_j containing x_n. The leaf space N/T_j is diffeomorphic to γ, and hence it is compact and Hausdorff. Thus $\{L_n\}$ has an accumulation point $L \in N/T_j$. Obviously, x must belong to L and hence to N.

COROLLARY 4. *Each T_i has a non-compact leaf.*

Proof. If all the leaves of T_i were compact, then M would be a ν_i-

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sphere bundle over a compact leaf η of T_j. Hence, M itself would be compact, a contradiction.

COROLLARY 5. (i) $\lambda_1 \geqq 0 \geqq \lambda_2$ *on* M.
(ii) *If* $\lambda_i = 0$ *on a* T_i*-leaf* L, *then* L *is non-compact.*

Proof. (i) Let γ be a non-compact T_1-leaf. Then $\lambda_1 = 0$ on γ. By Lemma 3, every value of λ_2 except 0 is assumed on γ, and since $\lambda_1 > \lambda_2$ on M, we have $\lambda_2 \leqq 0$. By considering a non-compact T_2-leaf, one gets $\gamma_1 \geqq 0$.

(ii) Suppose $\lambda_i = 0$ on a compact leaf L. Then L intersects a non-compact T_j-leaf, $j \neq i$, on which $\lambda_j = 0$. The point of intersection is an umbilic point, contradicting hypothesis (a) of Theorem 1.

Since neither λ_i is identically zero, we get immediately the following.

COROLLARY 6. *There is a compact* T_1*-leaf on which* λ_1 *assumes its maximum. There is a compact* T_2*-leaf on which* λ_2 *assumes its minimum.*

Let γ^+ be a compact leaf of T_1 on which λ_1 assumes its maximum and η^+ a compact leaf of T_2 on which λ_2 assumes its minimum. Let x be the point of intersection of γ^+ and η^+ whose existence and uniqueness is established by Lemma 3. Lemma 2 guarantees that γ^+ and η^+ are totally geodesic in the respective n-spheres centered at the focal points $p = f_1(x)$ and $q = f_2(x)$. Since λ_1 and λ_2 have opposite signs at x, the points p and q lie on different sides of the tangent hyperplane T_xM (for simplicity, we will denote the image $f_*(T_xM)$ by T_xM). γ^+ lies in a Euclidean $\nu_1 + 1$-plane E^{ν_1+1} determined by $T_1(x)$ and $\xi(x)$, while η^+ lies in an E^{ν_2+1} determined by $T_2(x)$ and $\xi(x)$. E^{ν_1+1} and E^{ν_2+1} intersect orthogonally along the normal line l to M at x. The following lemma demonstrates that the extreme leaves of M have the same configuration as shown in figure 1 for the parabolic cyclide.

LEMMA 7 (i) *The* T_2*-leaf* η^- *through the point* y *antipodal to* x *in* γ^+ *is non-compact. Moreover,* η^- *lies in the* E^{ν_2+1} *determined by* η^+, *and the tangent hyperplane* T_yM *is disjoint from the* n*-sphere* S^+ *determined by* η^+.
(ii) *The* T_1*-leaf* γ^- *through the point* z *antipodal to* x *in* η^+ *is non-compact. Moreover,* γ^- *lies in the* E^{ν_1+1} *determined by* γ^+, *and the tangent hyperplane* T_zM *is disjoint from the* n*-sphere determined by* γ^+.

Proof. (i) We wish to show that η^- is non-compact. Suppose first that the T_1-leaf γ^- through z is compact. We know that there is a non-compact leaf η of T_2 through some point v of γ^+. Suppose $v \neq y$. Since γ^- is compact, it must intersect η in a unique point u by Lemma 3 and $u \neq v$ since the T_1-leaf through v is γ^+ not γ^-. Note that the normal line to M at u is orthogonal to the ν_2-plane η and hence cannot meet the $(\nu_1 + 1)$-plane E^{ν_1+1} containing γ^+ which is also orthogonal to η. On the other hand, they both contain the common λ_1-focal point of z and u, whose existence is guaranteed by Corollary 5 (ii) and the compactness of γ^-. We conclude that if γ^- is compact, then the only possibility for a non-compact leaf of T_2 through a point on γ^+ is the leaf η^- through y. Hence η^- is non-compact.

Now, we suppose γ^- is non-compact and show that η^- must be non-compact. The argument is similar to the preceding one. If η^- is compact, then η^- and γ^- intersect in a unique point $w \neq z$. Since y and w lie on the same compact leaf η^-, the normal lines to M at y and w must intersect at their common λ_2-focal point whose existence is known from Corollary 5(ii). The normal line to M at y is l which lies in E^{ν_2+1} which is orthogonal to the ν_1-plane γ^- at z. On the other hand, the normal line at w is orthogonal to γ^- at w. Consequently, these two normal lines must be disjoint, and we obtain a contradiction which proves that η^- is non-compact.

To show the remainder of (i), observe that $\eta^- = T_2(y)$ is orthogonal to the $(\nu_1 + 1)$-plane spanned by γ^+, and hence it lies in the $(\nu_2 + 1)$-plane spanned by η^+. Finally, T_yM is Euclidean parallel to T_xM and lies on the same side of T_xM as the focal point $f_1(x) = p$. In contrast, the n-sphere S^+ determined by η^+ is tangent to T_xM at x and lies on the opposite side of T_xM (since $\lambda_1(x)$ and $\lambda_2(x)$ have opposite signs) determined by $q = f_2(x)$. Thus, T_yM is disjoint from S^+, as desired. Obviously, by reversing the roles of the principal curvatures, one proves (ii) in an identical manner.

LEMMA 8. *For y as in Lemma 7, M lies in a closed half-space determined by the hyperplane T_yM.*

Proof. Let H be the half-space containing η^+. Every leaf of T_1 intersects η^+ by Lemma 3. If γ is a non-compact leaf of T_1 intersecting η^+, then γ is a ν_1-plane parallel in the Euclidean sense to the ν_1-plane γ^-. Hence γ is parallel to a ν_1-plane in T_yM, and γ lies in H. If γ is a compact leaf of T_1, then γ intersects η^- in a point w. Since $\lambda_2 \equiv 0$ on η^-, $\lambda_1(w) > 0$ and thus γ lies on an n-sphere tangent to T_yM at w and lying in H, since γ has a point in common with η^+. Thus every leaf of T_1 lies in H and so M is contained in H.

We now complete the proof of Theorem 1 by producing an inversion of E^{n+1} which takes M to a punctured round cyclide. Let w be the unique point on l not in the closed half-space H such that $d(w, y) = r$, where

$$r^2 = d(y, q)^2 - d(x, q)^2.$$

One can check that any inversion I centered at w takes T_yM and S^+ to concentric n-spheres. Of course, $I(T_yM)$ is punctured at w.

It is now clear how to construct the punctured round cyclide $C(\nu_1, a, b)$ which coincides with $I(M)$. $I(\eta^-)$ and $I(\eta^+)$ determine the outer and inner rims of $C(\nu_1, a, b)$. In a similar fashion to Lemma 3 of [**4**], one can easily

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check that $C(\nu_1, a, b) - \{w\}$ is the union of ν_1-spheres (one of which, $I(\gamma^-)$, is punctured at w). We leave the details to the reader.

3. Remarks on the compact case. The existence of non-compact leaves actually makes the proof of Theorem 1 easier than the proof of our previous characterization of the ring cyclides in [**4**]. In fact, the result in the compact case can be proven most simply by an application of Theorem 1. We outline the proof below.

THEOREM 1′. *Let M be a connected, compact hypersurface embedded in E^{n+1}. If the focal set consists of two distinct (non-empty) submanifolds of codimension greater than one, then M is a ring cyclide.*

Proof. Let J be any inversion of E^{n+1} whose center p lies on M. Let $M' = J(M - \{p\})$. Then M' is a connected complete non-compact hypersurface embedded in E^{n+1}. By assumption, M satisfies hypotheses (a) and (b) of Theorem 1. Since J is conformal, M' also satisfies (a) and (b). The fact that property (b) is preserved for a principal curvature of multiplicity 1 by J is proven in almost identical fashion to Proposition 3.4 of [**5**, p. 36], which shows that (b) is preserved by stereographic projection.

We can now apply Theorem 1 to M' and obtain the result that M' is a parabolic cyclide. Thus there is an inversion I which takes M' to a punctured round cyclide. Now consider

$$I \circ J : M - \{p\} \rightarrow E^{n+1}.$$

$I \circ J$ is a diffeomorphism of $M - \{p\}$ onto a punctured round cyclide $C(\nu_1, a, b) - \{w\}$. Up to a similarity of E^{n+1}, $I \circ J$ is another inversion K or the identity. Since similarities preserve round cyclides, $M - \{p\}$ is related to some punctured round cyclide by the inversion K. The center of this inversion cannot be p (since the image under K of $M - \{p\}$ is bounded). By continuity, K must take p to the puncture of the round cyclide, and thus the original manifold M is a ring cyclide.

4. The classification of taut conformally flat hypersurfaces.

THEOREM 2. *Let M^n, $n \geqq 4$, be a connected manifold tautly embedded in E^{n+1}. Then M^n is conformally flat in the induced metric if and only if it is one of the following:*

1) *a hyperplane or round sphere (these are umbilic);*
2) *a cyclinder over a circle or round $(n-1)$-sphere;*
3) *a ring cyclide (diffeomorphic to $S^1 \times S^{n-1}$);*
4) *a parabolic cyclide (diffeomorphic to $S^1 \times S^{n-1}$ with a point removed).*

Proof. The previous discussion and calculations make it clear that all of the hypersurfaces listed in (1)–(4) are conformally flat and tautly embedded. Furthermore, the only taut connected hypersurfaces which have umbilics are the hyperplane and the sphere, by Lemma 1.

In the remainder of this proof, we assume M has no umbilics. By Schouten's result, M has two distinct principal curvatures of respective multiplicities $n - 1$ and 1 at each point. Since M is orientable, a smooth choice of unit normal field can be made giving rise to principal curvature functions λ (of multiplicity $n - 1$) and μ (of multiplicity 1) defined on all of M. It is well-known that λ and μ and their corresponding principal distributions are smooth and integrable. Furthermore, because λ has multiplicity $n - 1 > 1$, λ is constant along the leaves of its principal distribution T_λ. Proofs of these statements may be found in [**22**, pp. 371–373]. We now show that (because of tautness) μ is constant along the leaves of its one-dimensional principal distribution T_μ. Classically these leaves are called *lines of curvature.*

LEMMA 9. *The principal curvature μ of multiplicity one is constant along its lines of curvature.*

Proof. This result is essentially the same as Proposition 6 of [**3**] which was proven for surfaces in E^3. Recall that a point $p \in E^{n+1}$ is called a *focal point* of (M, x), $x \in M$, if $p = x + (1/\beta)\xi$, where ξ is a unit normal to M at x, and β is a principal curvature of A_ξ. Since our M has two distinct principal curvatures at each point, there are at most two focal points of (M, x) along either normal ray to M at x. By Lemma 3.2 of [**2**, p. 708], if p is a first, respectively second, focal point of (M, x) along a normal ray to M at x, then L_p has an absolute minimum, respectively maximum, at x. (In the terminology of [**2**], a focal point is the image of a critical normal under the exponential map from the normal bundle $N(M)$ into E^{n+1}.) Now the argument of Proposition 6 of [**3**] can be used, with very little modification, on the principal curvature μ. If one assumes that μ is not constant along one of its lines of curvature, then one obtains a contradiction to the fact that for the focal point $p = x + (1/\mu)\xi$, the function L_p has an absolute minimum or maximum, as the case may be, at a point x along the line of curvature in question. The details are left to the reader.

We now summarize the results of this section as follows.

LEMMA 10. *Let M be a taut conformally flat hypersurface in $E^{n+1} (n \geqq 4)$ which is not totally umbilic.*

(a) *At every point of M, there are two distinct principal curvatures λ and μ of respective multiplicities $n - 1$ and 1.*

(b) *The principal distributions T_λ and T_μ are integrable and λ and μ are constant along the leaves of their corresponding principal distributions.*

Lemma 10 yields that if M is compact, the hypotheses of Theorem 1′ are satisfied, and hence M is a ring cyclide diffeomorphic to $S^1 \times S^{n-1}$.

If M is non-compact, there are three cases to consider. As before λ is the principal curvature of multiplicity $n - 1$ and μ of multiplicity 1. The cases are:

(i) λ is identically zero, $\mu > 0$.
(ii) μ is identically zero, $\lambda > 0$.
(iii) Neither λ nor μ is identically zero.

Note that in cases (i) and (ii), the fact that M has no umbilics implies that the principal curvature which is not identically zero, is never zero.

In case (i), the Hartman-Nirenberg theorem [**9**] shows that M is a cylinder over a complete plane curve. (See [**19**, p. 57] for a proof not requiring the simple connectivity of M.) Since T_μ is well-defined on all of M, the complete plane curve is just a leaf of T_μ. But such leaves are all circles, and so M is a circular cylinder.

In case (ii) M satisfies the hypothesis

(2) $R(X, Y)\cdot R = 0$ for all X, Y tangent to M,

discussed by Nomizu [**19**], where R is the Riemann curvature tensor of M. Although this algebraic condition is an interesting one which has been extensively studied, for our purposes it is only necessary to use the fact [**19**, pp. 47–48] that a hypersurface in E^{n+1} satisfies (2) if and only if at each $x \in M$, the principal curvatures $\{\lambda_k\}_{k=1}{}^n$ satisfy

(3) $\lambda_i\lambda_j\lambda_l(\lambda_j - \lambda_i) = 0$

for any triple of distinct indices (i, j, l). Clearly, our case satisfies (3) and hence (2). Furthermore, since $n \geqq 4$, the number of non-zero principal curvatures (called the *type number* in [**19**]) is at least three on M. We can thus apply the following theorem of Nomizu to obtain the desired result for case (ii).

THEOREM (Nomizu [**19**]). *Let M^n be a connected immersed hypersurface in E^{n+1} which satisfies* (2) *and is complete in the induced metric. If the type number is greater than* 2 *at least at one point, then $M = S^k \times E^{n-k}$ embedded as a cylinder over a round k-sphere.*

Of course, the type number of the embedding is the constant k. Thus, in our case, $k = n - 1$.

Finally, in case (iii), Lemma 10 proves that M satisfies the hypotheses of Theorem 1, and thus M is a parabolic cyclide.

5. Concluding remarks. The classification of conformally flat hypersurfaces still remains a challenging problem. A review of the literature reveals two distinct approaches. First, one can attempt to describe the local intrinsic geometry by finding canonical forms for the metric. Specifically, one tries to determine the functions σ such that a neighborhood of the origin in Euclidean space E^n with the metric

$$ds^2 = e^{2\sigma} \sum_{i=1}^{n} (dx^i)^2,$$

can be isometrically immersed in E^{n+1}. For details along this line, the reader may consult [**13**] and the articles cited there.

The second approach, which has both local and global aspects, derives extrinsic geometrical properties which any conformally flat hypersurface must possess with the goal of providing a concrete list of examples. Schouten's result is the first step in this direction. In this spirit, J. D. Moore [**16**] has recently obtained topological restrictions on conformally flat submanifolds in arbitrary codimension. However, in light of Morse theory, his result reduces to that of Schouten in codimension 1.

Various authors have asserted further restrictions on conformally flat hypersurfaces. The most explicit of these classification results is the following assertion published independently by Nishikawa [**17**] and Kulkarni [**12**].

ASSERTION 1 ([**17**] and [**12**]). *Every complete analytic conformally flat hypersurface of dimension greater than* 3 *in Euclidean space is one of the following:*

(1) *a flat hypersurface,*

(2) *a tube, i.e., the normal sphere bundle (of sufficiently small fixed radius) of a curve in* E^{n+1},

(3) *a hypersurface of revolution, i.e., the envelope of a family of hyperspheres whose locus of centers lies on a straight line.*

It is easy to see that the non-round ring cyclides and the parabolic cyclides are not included in this classification, since both tubes and hypersurfaces of revolution have at least one constant principal curvature. Neither principal curvature is constant on any open set of one of these cyclides. On the other hand, cyclides of both types which have principal curvatures of multiplicity 1 and $n - 1$ are conformally flat.

Thus these cyclides provide counterexamples to the assertion above and to the corresponding local result [**18**]. More specifically, let x be a non-umbilic point of a conformally flat hypersurface M. Let λ and μ be the principal curvatures of respective multiplicities $n - 1$ and 1 on a neighborhood of x. It is incorrectly asserted in [**17**, p. 565] and [**18**, p. 165] that

(4) $\quad (Y\lambda)(X\mu) = 0$ for all $X \in T_\lambda$, $Y \in T_\mu$

near x. This, of course, cannot be true on the non-round cyclides since one

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could use (4) together with

(5) $X\lambda = 0$ and $Y\mu = 0$ for $X \in T_\lambda$, $Y \in T_\mu$,

to show that either λ or μ is constant on this neighborhood of x.

Geometrically speaking, a tube is the envelope of a 1-parameter family of n-spheres of fixed radius in E^{n+1}. A non-round ring cyclide is the envelope of a 1-parameter family of n-spheres, but of varying radii. Essentially, the assertion of [**17**] and [**12**] fails because the set of all tubes is not invariant under conformal transformations of E^{n+1}.

The results in this paper are also relevant to attempts by Chen and Yano ([**6**], [**7**]) to characterize conformally flat hypersurfaces in terms of their intrinsic geometry.

Let S and $s = \text{trace } S$ denote the (1, 1) Ricci tensor and the scalar curvature of a Riemannian manifold M^n. Set

$$L = -\frac{1}{n-2}\left(S - \frac{s}{2(n-1)}\,I\right),$$

where I is the identity (1, 1) tensor. In [**6**], a conformally flat manifold M was defined to be *special* if there exist functions α and β with $\alpha > 0$ such that for all vector fields X,

$$LX = -\tfrac{1}{2}\alpha^2 X + \beta\langle X, \operatorname{grad} \alpha\rangle \operatorname{grad} \alpha.$$

Thus, if M is special, then at any $x \in M$ either S is a scalar multiple of the identity or S has eigenvalues of multiplicities 1 and $n - 1$. One can show through some straightforward calculations that if M^n is a special conformally flat hypersurface in E^{n+1}, then the function α must equal the principal curvature λ of multiplicity at least $n - 1$ on M^n. Furthermore, grad α cannot vanish at a non-umbilic point of M^n.

In Theorem 1 of [**6**], Chen and Yano prove that every simply connected special conformally flat space can be isometrically immersed in Euclidean space as a hypersurface. The proof uses the assumption $\alpha > 0$ on M. The statement of this theorem also includes:

ASSERTION 2 [**6**]. *Every conformally flat hypersurface in Euclidean space is special.*

Every taut conformally flat hypersurface M^n in E^{n+1}, with the exception of round spheres, is a counterexample to this assertion. First, hyperplanes and cylinders diffeomorphic to $S^1 \times \mathbf{R}^{n-1}$ are not special, because the principal curvature λ of multiplicity $n - 1$ is identically zero on M^n. On the cylinders diffeomorphic to $S^{n-1} \times \mathbf{R}$, λ is constant, and hence each point of M^n is a non-umbilic point at which grad $\alpha = 0$. Similarly, if M^n is a conformally flat cyclide, then grad $\alpha = 0$ along the extreme λ-leaves. Any open set in the above examples which contains a non-umbilic point where grad $\alpha = 0$ serves as a local counterexample to Assertion 2.

In a subsequent paper presented in the book [**7**, p. 155], Chen and Yano gave a new definition of a special conformally flat space. In this case, they defined an index $i(M)$ and proved that if $k < i(M)$, then M (if simply connected) can be isometrically immersed in a space form of constant curvature k.

It is asserted [**7**, p. 157] that every conformally flat hypersurface in a space form is special in this new sense. The taut conformally flat hypersurfaces are not counterexamples to this assertion. However, the new definition states that α (which is the principal curvature of multiplicity at least $n - 1$) is differentiable on M. This is certainly true on the set U of non-umbilic points of M on which α has constant multiplicity $n - 1$. The function β in the definition is easily shown to satisfy

$$(6) \quad \beta = \text{trace } A - n\alpha,$$

where A is the shape operator of M. In [**7**, p. 147] the author states: "It is clear that α is differentiable on M and that β is continuous on M."

For the two-dimensional monkey saddle, one can verify that the function α is not differentiable at the isolated umbilic. While we do not know of a conformally flat counterexample having $n \geqq 3$, we believe that an explicit proof that α is differentiable in this case must be given. We note that (6) implies that if α is differentiable, then β is also differentiable, not merely continuous.

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