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Angle Measures and Bisectors in Minkowski Planes

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Abstract. We prove that a Minkowski plane is Euclidean if and only if Busemann's or Glogovskij's definitions of angular bisectors coincide with a bisector defined by an angular measure in the sense of Brass. In addition, bisectors defined by the area measure coincide with bisectors defined by the circumference (arc length) measure if and only if the unit circle is an equiframed curve.

1 Introduction and Results

In a *Minkowski plane* $\mathbb{M}^2$, *i.e.,* a two-dimensional real linear space with a metric $l(\cdot\,,\cdot)$ induced by the norm $\|\cdot\|$, there is no natural definition of a unique angular measure as in the Euclidean plane with the Euclidean metric l_e. In fact, there are several possibilities of defining such a measure. We will study the measures μ_a and μ_l which are proportional to the area and to the arc length of the corresponding sector of the unit circle, respectively, as well as a class of further measures satisfying certain axioms.

Different definitions of angular measures yield different possibilities of defining *angular bisectors*. As long as this bisector is defined by an angular measure, there is a one-to-one correspondence between angular measures and angular bisectors. But generalizations of geometric properties of Euclidean angular bisectors yield definitions of angular bisectors in $\mathbb{M}^2$ which are independent of an angular measure. By means of (angular) bisectors in normed linear spaces, various deep characterizations of special Minkowski spaces can be obtained, *cf.* the survey [10, §4]. For the planar case, we will give further such characterization theorems.

We denote the origin of $\mathbb{M}^2$ by $\mathfrak{o}$, its *unit circle* by $C_0 := \{\mathfrak{x} \in \mathbb{M}^2 : \|\mathfrak{x}\| = 1\}$ and its *unit disc* by $B := \{\mathfrak{x} \in \mathbb{M}^2 : \|\mathfrak{x}\| \leq 1\}$. For a vector $\mathfrak{x} \in \mathbb{M}^2$, $\mathfrak{x} \neq \mathfrak{o}$, we denote by $\widehat{\mathfrak{x}} := \frac{1}{\|\mathfrak{x}\|}\mathfrak{x}$ the *normalization* of $\mathfrak{x}$. The distance of $\mathfrak{c} \in \mathbb{M}^2$ to a set $M \subset \mathbb{M}^2$ is denoted by $\varrho(c, M) := \inf_{m\in M} l(cm)$. See also the monograph [13] for more background.

In this paper we will characterize special Minkowski planes by properties of C_0. A *Radon curve* is affinely equivalent to a curve whose polar is a 90° rotation of the original curve, see [12] and [13, Chapter 4]. Minkowski planes with Radon curves as unit circle are exactly those with symmetric perpendicularity. *Equiframed curves* are precisely those centrally symmetric convex closed curves in the plane that are touched at each of their points by some circumscribed parallelogram of smallest area. These curves were introduced by Pełczyński and Szarek, see [11, 9].

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Inspired by Brass [2, Definition: p. 207], we define an angular measure for Minkowski planes:

Definition 1 An *angular measure* of the Minkowski plane $\mathbb{M}^2$ (in the sense of Brass) is a measure μ on the Borel sets of the unit circle C_0 which has the following properties:

(1) μ is normed, *i.e.*, $\mu(C_0) = 2\pi$,
(2) μ is centrally symmetric, *i.e.*, for $X \subset C_0$ we have $\mu(X) = \mu(-X)$,
(3) for each point $p \in C_0$ we have $\mu(\{p\}) = 0$,
(4) every arc ab on C_0 with $a \neq b$ has a positive measure $\mu(ab) > 0$.

The fourth property was not demanded by Brass but is necessary for defining a bisector (uniqueness).

Theorem 1 *Let μ_a and μ_l denote the angular measures which are proportional to the area and to the arc length of the corresponding sector of the unit disc B and the unit circle C_0, respectively,* i.e.,

$$\mu_a(X) = \frac{2\pi}{\lambda_2(B)} \lambda_2(\operatorname{cone}(X) \cap B), \quad X \subset C_0,$$

$$\mu_l(X) = \frac{2\pi}{\lambda_1(C_0)} \lambda_1(X), \quad X \subset C_0.$$

These measure coincide if and only if C_0 is an equiframed curve.

The question for the coincidence of μ_a and μ_l was posed by Helfenstein [7] in 1959, and two years later he himself gave a wrong answer, see [8]. In his solution Helfenstein erroneously assumed continuous differentiability of the radial function, yielding a restriction of the characterized class of unit circles from equiframed curves to Radon curves (which are more specific). To see that this restriction is wrong, one might consider the l_∞-norm where C_0 is the square. Then we have $\mu_a = \mu_l$, but C_0 is equiframed and not a Radon curve.

An *angle* of $\mathbb{M}^2$ is a closed convex subset T of $\mathbb{M}^2$ whose boundary ∂T is the union of two rays r_1, r_2 not on a line (called the *legs*) with common endpoint, called the *apex* of the angle. The two limit cases of a single ray and of a halfplane are not called angles. Thus the angle T is uniquely determined by its legs r_1, r_2, and we denote it by $\angle(r_1 r_2) = T = \operatorname{conv}(r_1 \cup r_2)$. The closed linear segment from x to y is denoted by $[x, y]$, the straight line through x and y by $\langle x, y \rangle$ and the ray with origin x passing through y by $[x, y\rangle$. Further on, we use the notation $\angle \mathfrak{bac} := \angle\big([\mathfrak{a}, \mathfrak{b}\rangle\, [\mathfrak{a}, \mathfrak{c}\rangle\big)$. An angular *bisector* of an angle T is a ray r such that there are two angles T_1, T_2 with $T_1 \cup T_2 = T$ and $\partial T_1 \cap \partial T_2 = r$. In this case we say that r *divides* T into T_1 and T_2. A *system* of angular bisectors is a function A mapping each angle T to a corresponding bisector $r = A(T)$. The *normalized representation* $\widehat{A}$ of A is the function $\widehat{A} : (\mathfrak{x}, \mathfrak{y}) \mapsto \mathfrak{r} \in C_0$, where $\mathfrak{r} \in A(\angle \mathfrak{x} \mathfrak{0} \mathfrak{y})$ for $\mathfrak{x}, \mathfrak{y} \in C_0$ with $\mathfrak{x} \neq \pm\mathfrak{y}$. We define a system A of angular bisectors by its normalized representation $\widehat{A}$ in the following way: $A(\angle \mathfrak{bac}) := [\mathfrak{a}, \mathfrak{a} + \widehat{A}(\widehat{\mathfrak{b} - \mathfrak{a}}, \widehat{\mathfrak{c} - \mathfrak{a}})\rangle$.

Following Busemann [3], we give

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Definition 2 The system $A_B = A_{B,\mathbb{M}^2}$ of angular bisectors of $\mathbb{M}^2$ given by

$$\widehat{A}_B\,(a,b) = \widehat{a+b}$$

for $a, b \in C_0, a+b \neq \mathfrak{o}$, is called the *system of Busemann angular bisectors*, and $A_B(T)$ is said to be the *Busemann angular bisector* of T.

The following definition is due to Glogovskij [6].

Definition 3 The system $A_G = A_{G,\mathbb{M}^2}$ of angular bisectors of $\mathbb{M}^2$ given by

$$A_G(\angle\,(r_1\,r_2)) = \{c \in \angle\,(r_1\,r_2) : \varrho(c, \operatorname{aff} r_1) = \varrho(c, \operatorname{aff} r_2)\},$$

i.e., the set of all points of the angle which are equidistant to the lines carrying the legs, is called the *system of Glogovskij angular bisectors*, and $A_G(T)$ is said to be the *Glogovskij angular bisector* of T.

In [5] the Minkowski planes with equivalence of Busemann's and Glogovskij's definition of angular bisectors are characterized: *In a Minkowski plane* $\mathbb{M}^2$ *we have* $A_{B,\mathbb{M}^2} = A_{G,\mathbb{M}^2}$ *if and only if the unit circle* C_0 *is a Radon curve.*

Given an angular measure μ we can measure every angle in an obvious manner: $\mu(\angle \mathfrak{bac}) := \mu(\angle(\mathfrak{b}-\mathfrak{a})\mathfrak{o}(\mathfrak{c}-\mathfrak{a}) \cap C_0)$.

Definition 4 Given an angular measure μ of $\mathbb{M}^2$, the system of angular bisectors such that $A_\mu(T)$ divides T into T_1 and T_2 with $\mu(T_1) = \mu(T_2) = \frac{1}{2}\mu(T)$, is called the *system of μ-bisectors*, and $A_\mu(T)$ is the *μ-bisector* of T.

There is exactly one μ-bisector for every angle of $\mathbb{M}^2$. (The uniqueness follows from Definition 1(4), the existence from 1(3).)

Now we are interested in Minkowski planes in which two of the introduced systems of angular bisectors A_B, A_G, A_μ and $A_{\mu'}$ (defined for another angular measure μ' on the same unit circle C_0) are equal. In the following we will prove Theorem 1 and the following characterizations of Minkowski planes in which some systems of angular bisectors coincide.

Theorem 2 *In a Minkowski plane* $\mathbb{M}^2$ *we have* $A_B = A_\mu$ *if and only if* $\mathbb{M}^2$ *is the Euclidean plane and* μ *denotes its standard angular measure.*

Theorem 3 *In a Minkowski plane* $\mathbb{M}^2$ *we have* $A_G = A_\mu$ *if and only if* $\mathbb{M}^2$ *is the Euclidean plane and* μ *denotes its standard angular measure.*

For later use we still notice the following lemma which is an easy consequence of standard arguments from analysis.

Lemma 4 *For two angular measures* μ_1, μ_2 *of* $\mathbb{M}^2$ *we have* $A_{\mu_1} = A_{\mu_2}$ *if and only if* $\mu_1 = \mu_2$.

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2 Properties of Angular Bisectors

In a Minkowski plane $\mathbb{M}^2$, Definition 2 yields a bisector $[0, c\rangle$ satisfying Property 1 below. This property was used by Busemann [3] to define an angular bisector in a more general sense.

Property 1 Given an angle T with apex P and legs a, b. The angular bisector c of T has the *Busemann bisector property* if and only if for every segment $[x, y]$ joining a point x from $a \setminus \{P\}$ with one point y from $b \setminus \{P\}$ the ray c divides $[x, y]$ in the ratio of the lengths $l(Px)$ and $l(Py)$, *i.e.*, for $\{z\} := [x, y] \cap c$ one has

$$\frac{l(xz)}{l(zy)} = \frac{l(Px)}{l(Py)}. \tag{1}$$

Lemma 5 *An angular bisector c of T has the Busemann bisector property in $\mathbb{M}^2$ if and only if it is the Busemann angular bisector of T, $c = A_B(T)$.*

Proof This follows from the fact that the Euclidean angular bisector satisfies Property 1 in the Euclidean plane by comparing the lengths with a Euclidean background metric, see for example [5].

Definition 5 Let

$$u\colon [0, U] \to C_0$$

be a *parameterization of the unit circle by arc length* in the positive orientation. For an angular measure μ we define the *angle function* $w = w_\mu\colon [0, U] \to [0, 2\pi]$ by

$$w_\mu(t) = \mu(u([0, t])).$$

The angle function w_μ is strictly monotone increasing, namely,

$$\begin{aligned} w_\mu(t + dt) &= w_\mu(t) + \mu(u((t, t + dt])) \\ &= w_\mu(t) + \mu(u([t, t + dt])) && \text{by Definition 1(3)} \\ &> w_\mu(t) \quad \text{for } 0 < dt < U - t && \text{by Definition 1(4),} \end{aligned}$$

and, because of Definition 1(4), continuous, with

$$w(0) = 0, \quad w(U) = \mu(C_0) = 2\pi, \qquad \text{and}$$

$$w\Big(t + \frac{1}{2}U\Big) = w(t) + \mu\Big(u\Big(\Big[t, t + \frac{1}{2}U\Big]\Big)\Big) = \pi + w(t) \qquad \text{for } 0 \le t \le \frac{1}{2}U.$$

(Note that $u([t, t + \frac{1}{2}U]) \cup -u([t, t + \frac{1}{2}U]) = C_0$, and the intersection has empty measure.)

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Definition 6 Let $u\colon [0,U] \to C_0$ be a parameterization of the unit circle by arc length in the positive orientation, μ be an angular measure and $w = w_\mu$ be the corresponding angle measure. Then the *parameterization by the angle* is the function

$$m\colon [0,2\pi] \to C_0, \quad \phi \mapsto u(w^{-1}(\phi)),$$

where $w^{-1}\colon [0,2\pi] \to [0,U]$ denotes the inverse to the angle function $w = w_\mu$. The *extended parameterization by the angle* is the function

$$m\colon \mathbb{R} \to C_0, \quad \phi \mapsto u(w^{-1}(\phi \bmod 2\pi)).$$

3 Extensions of Systems of Angular Bisectors

Definition 7 For a system A of angular bisectors of $\mathbb{M}^2$ with normalized representation $\widehat{A}$ we define for every half-plane H and unit vector $\mathfrak{a} \in C_0$ with $\partial H = \langle -\mathfrak{a}, \mathfrak{a}\rangle$ the *inner limit of* $\widehat{A}$ *with fixed leg* $[0, \mathfrak{a}\rangle$, if it exists, by

$$\widehat{A}_{\mathfrak{a}}(H) := \lim_{\substack{\mathfrak{b}\to -\mathfrak{a},\\ \angle\mathfrak{a}0\mathfrak{b}\subset H}} \widehat{A}(\mathfrak{a},\mathfrak{b})\,.$$

Definition 8 For a system A of angular bisectors of X, we define the following binary relation in the set of nonzero vectors $x, y \in X \setminus \{0\}$: *$x$ is A-normal to y* if and only if there is a half-plane H with $\partial H = \langle -x, x\rangle$ and $y \in [0, \widehat{A}_{\mathfrak{a}}(H)\rangle$. For this A-normality we write $x \dashv_A y$.

4 The Equivalence of the Angular Measures μ_l and μ_a (Proof of Theorem 1)

For $0 \le t_1 \le t_2 < U$ and $t_2 - t_1 < \frac{U}{2}$ we have that

$$\mu_l(\angle u(t_1)u(t_2)) = 2\pi\frac{t_2 - t_1}{U} = \int_{t=t_1}^{t_2} \frac{2\pi}{U}\,dt.$$

Next we denote by $u'_+\colon [0,U) \to \mathbb{R}^2$ the *right tangent vector*

$$u'_+(t) := \lim_{h\downarrow 0} \frac{u(t+h) - u(t)}{h}$$

of u and write $\alpha(t) := \det[u(t), u'_+(t)]$. This yields

$$\mu_a(\angle u(t_1)u(t_2)) = \frac{2\pi}{\lambda_2(B)} \int_{t=t_1}^{t_2} \alpha(t)\,dt = \int_{t=t_1}^{t_2} \frac{2\pi\alpha(t)}{\lambda_2(B)}\,dt.$$

Thus we have $\mu_l \equiv \mu_a$ if and only if

$$\frac{2\pi}{U} = \frac{2\pi\alpha(t)}{\lambda_2(B)} \quad \forall t \in [0,U)$$

if and only if $\alpha(t) = \frac{\lambda_2(B)}{U}$ (note that $\alpha(t)$ is almost everywhere continuous). Thus we have that $\alpha(t)$ is constant for $\mu_l \equiv \mu_a$, and therefore [9, Proposition 1] C_0 is equiframed.

Now if C_0 is equiframed, then we have $\alpha(t) = c$ for some constant c, and so

$$2\pi = \int_{t=0}^{U} \frac{2\pi\alpha(t)}{\lambda_2(B)}\,dt = U\frac{2\pi c}{\lambda_2(B)}$$

and

$$\alpha(t) = c = \frac{\lambda_2(B)}{U},$$

yielding $\mu_l \equiv \mu_a$.

Summarizing we have Theorem 1.

5 The Equivalence of Busemann's Definition With That of a μ-Bisector

We want to show that the equivalence of any Busemann bisector with the corresponding μ-bisector (for some fixed measure μ) implies that the plane under consideration is Euclidean. In a first step, we mainly consider the inner limit of $\widehat{A_B}$ with a fixed leg to obtain that for the equivalence $A_B = A_\mu$ the plane is Radon (Lemma 7 states the relevant condition). Theorem 2 then affirms our characterization. Its proof will use induction to show that our norm is Euclidean.

Lemma 6 *Using the orientation of a parameterization u of C_0 by arclength, we define the* half-plane H_a spanned by $a = u(t)$ *as the half-plane with boundary $\langle -a, a\rangle$ also containing the arc $u([t, t + \frac{1}{2}U])$* (mod U).

Then for the inner limit of $\widehat{A}_B$ of H_a with fixed legs $[0, a\rangle$ and $[0, -a\rangle$, respectively, we have

$$(\widehat{A}_B)_a(H_a) = u'_-(t) \quad \text{and} \quad (\widehat{A}_B)_{-a}(H_a) = u'_+(t).$$

Proof Let us assume that $a = u(t)$. Then

$$\begin{aligned}(\widehat{A}_B)_{-a}(H_a) &= \lim_{\epsilon\downarrow 0} \widehat{A}_B\big(u(t+\epsilon), u(t + \tfrac{1}{2}U)\big)\\ &= \lim_{\epsilon\downarrow 0} u(t \widehat{+ \epsilon) -} u(t) = \lim_{\epsilon\downarrow 0} \frac{u(t+\epsilon) - u(t)}{\epsilon} = u'_+(t).\end{aligned}$$

Here we use the fact that $\|u(t+\epsilon) - u(t)\| \to |\epsilon|$ for $\epsilon \to 0$. Analogously, we have

$$\begin{aligned}(\widehat{A}_B)_a(H_a) &= \lim_{\epsilon\downarrow 0} \widehat{A}_B\big(u(t), u(t + \tfrac{1}{2}U - \epsilon)\big) = \lim_{\epsilon\downarrow 0} \widehat{A}_B\,(u(t), -u(t-\epsilon))\\ &= \lim_{\epsilon\downarrow 0} u(t) \widehat{- u(t} - \epsilon) = \lim_{\epsilon\downarrow 0} \frac{u(t-\epsilon) - u(t)}{-\epsilon}\\ &= \lim_{\epsilon\uparrow 0} \frac{u(t+\epsilon) - u(t)}{\epsilon} = u'_-(t).\end{aligned}$$

■

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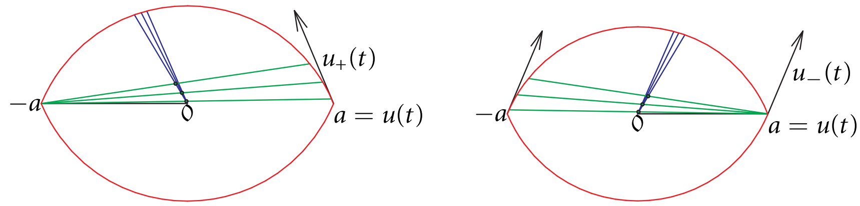

Figure 1: $(\widehat{A}_B)_{-a}(H_a) = u'_+(t)$ and $(\widehat{A}_B)_a(H_a) = u'_-(t)$

Lemma 7 *If $A_B = A_\mu$, then $u \dashv v$ if and only if $u \dashv_{A_\mu} v$.*

Proof Since we have $A_B = A_\mu$, Lemma 6 shows that for all t

$$u'_+(t) = (\widehat{A}_B)_{-a}(H_a) = (\widehat{A}_\mu)_{-a}(H_a) = (\widehat{A}_\mu)_a(H_a) = (\widehat{A}_B)_a(H_a) = u'_-(t).$$

Therefore the unit circle C_0 has only regular points. Thus $u(t) \dashv v$ is equivalent to $v = \lambda u'_+(t)$ for some (nonzero) $\lambda \in \mathbb{R}$. Furthermore, this is equivalent to $[0, v\rangle = \pm A_\mu(H_a) = A_\mu(\pm H_a)$ as well as $u(t) = a \dashv_{A_\mu} v$. By scaling, this result extends to all nonzero vectors $u \in \mathbb{M}^2$. ■

Proof of Theorem 2 Obviously, in the Euclidean plane $\mathbb{E}^2$ the Busemann-angular bisector of any two rays coincides with the μ-bisector where μ denotes the standard angular measure in $\mathbb{E}^2$.

So we can assume that in a given Minkowski plane $\mathbb{M}^2$ we have $A_B = A_\mu$.

Since the A_μ-normality is symmetric (we have $u \dashv_{A_\mu} v$ if and only if $\mu(\angle u0v) = \frac{\pi}{2}$), by Lemma 7 also the normality $\dashv$ is symmetric; thus C_0 is a Radon curve. Therefore C_0 is an equiframed curve, and we have that the function $\alpha(t)$ is constant (this fact, taken from [9], we used already in Section 4).

Let us now define a Euclidean metric by using $x := m(0)$ and $y := m(\frac{1}{2}\pi)$ as orthogonal unit vectors. We will show that for every $k, l \in \mathbb{N}$ we have for $\phi = \frac{l}{2^k}\pi$ that $m(\phi)$ coincides with the Euclidean unit vector $m_2(\phi) := \cos(\phi)x + \sin(\phi)y$ obtained by rotating (in the Euclidean sense) x by the angle ϕ in positive orientation. Thus our metric is the Euclidean one (since both functions $m(\cdot)$ and $m_2(\cdot)$ are continuous, they must coincide). Without loss of generality we can measure areas in this Euclidean metric, giving $\alpha(t) = \alpha(0) = 1$ for all $0 \le t < U$.

We use induction over k to show that for $0 \le l \le 2^{k+1}$ the vector $m(\frac{l}{2^K}\pi) \in C_0$ coincides with $m_2(\frac{l}{2^k}\pi)$.

For $k = 0$ there is nothing to show: $m(0) = x = \cos(0)x + \sin(0)y = m_2(0)$, $m(\pi) = -x = \cos(\pi)x + \sin(\pi)y = m_2(\pi)$ (by properties of the function w). For $k = 1$ we have $m\left(\frac{1}{2}\pi\right) = y = m_2\left(\frac{1}{2}\pi\right)$, $m\left(\frac{3}{2}\pi\right) = -y = m_2\left(\frac{3}{2}\pi\right)$.

Now assume that for $k \ge 2$ we have $m(\phi) = m_2(\phi)$ for all $\phi = \frac{l}{2^{k-1}}\pi = \frac{2l}{2^k}\pi$, $l = 0, 1, \dots, 2^k$. We will show that for $0 \le l < 2^k$ and $\phi = \frac{2l+1}{2^k}\pi$ we have $m(\phi) = m_2(\phi)$, too.

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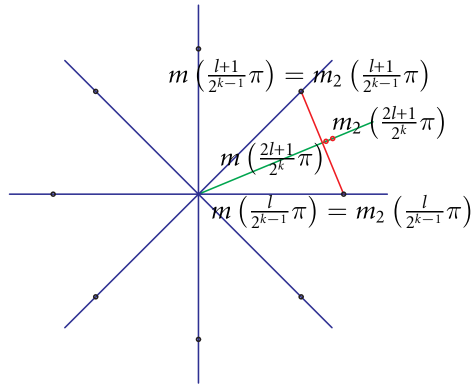

Figure 2: Proof of Theorem 2: For $\phi = \frac{2l+1}{2^k}\pi$ the vectors $m(\phi)$ and $m_2(\phi)$ have the same direction($k = 3, l = 0$).

By our induction hypothesis and $A_B = A_\mu$, we have for $0 \leq l < 2^k$ that

$$\begin{aligned} m\left(\frac{2l+1}{2^k}\pi\right) &\in A_\mu\left(\angle m\left(\frac{l}{2^{k-1}}\pi\right)\mathcal{O} m\left(\frac{l+1}{2^{k-1}}\pi\right)\right) \\ &= A_B\left(\angle m_2\left(\frac{l}{2^{k-1}}\pi\right)\mathcal{O} m_2\left(\frac{l+1}{2^{k-1}}\pi\right)\right) \\ &= \left[\mathcal{O}, m_2\left(\frac{l}{2^{k-1}}\pi\right) + m_2\left(\frac{l+1}{2^{k-1}}\pi\right)\right\rangle \\ &= \left[\mathcal{O}, m_2\left(\frac{2l+1}{2^k}\pi\right)\right\rangle. \end{aligned}$$

This means that for $\phi = \frac{2l+1}{2^k}\pi$ the vectors $m(\phi)$ and $m_2(\phi)$ have the same direction and

$$m(\phi) = \|m(\phi)\|_2 \cdot m_2(\phi) \ ,$$

see also Figure 2.

Next we show that for $0 < l < 2^k$

$$\left\|m\left(\frac{2l-1}{2^k}\pi\right)\right\|_2 = \left\|m\left(\frac{2l+1}{2^k}\pi\right)\right\|_2$$

holds, giving $\left\|m\left(\frac{l}{2^k}\pi\right)\right\|_2 = c$ for all odd l with $0 < l < 2^{k+1}$. We use the abbreviations

$$\phi_1 := \frac{2l-1}{2^k}\pi, \quad \phi_2 := \frac{2l+1}{2^k}\pi \quad \text{and} \quad \phi := \frac{l}{2^{k-1}}\pi.$$

Using Property 1 we get with $\{C\} := [\mathcal{O}, m(\phi)\rangle \cap [m(\phi_1), m(\phi_2)]$ that

$$1 = \frac{l(m(\phi_1)\,\mathcal{O})}{l(m(\phi_2)\,\mathcal{O})} = \frac{l(m(\phi_1)\,C)}{l(Cm(\phi_2))} = \frac{l_e(m(\phi_1)\,C)}{l_e(Cm(\phi_2))} = \frac{l_e(m(\phi_1)\,\mathcal{O})}{l_e(m(\phi_2)\,\mathcal{O})} = \frac{\|m(\phi_1)\|_2}{\|m(\phi_2)\|_2}.$$

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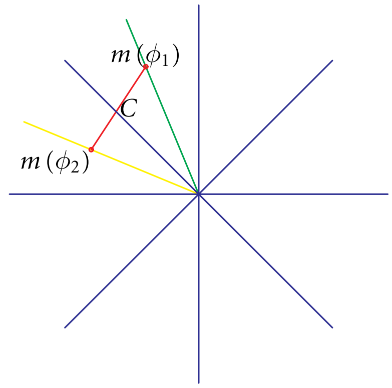

Figure 3: Proof of Theorem 2: For $\phi_1 := \frac{2l-1}{2^k}\pi$ and $\phi_2 := \frac{2l+1}{2^k}\pi$ the vectors $m(\phi_1)$ and $m(\phi_2)$ have the same Euclidean length. The figure shows why it is not possible that $\|m(\phi_1)\|_2 \neq \|m(\phi_2)\|_2$ for $k = 3, l = 3$: then also $l(m(\phi_1)C) \neq l(Cm(\phi_2))$.

Here we use the facts that $[0, m(\phi)\rangle$ is the Euclidean angular bisector of the two rays $[0, m(\phi_1)\rangle, [0, m(\phi_2)\rangle$, thus satisfying Property 1 using the Euclidean metric l_e, and that it is also the Busemann angular bisector of the same rays, see Figure 3.

Thus we know that $m(\phi) = c \cdot m_2(\phi)$ for $\phi = \frac{l}{2^k}\pi$ with odd l, where c denotes a constant.

Then we have that for $u := m\left(\frac{1}{2^k}\pi\right)$ and $v := m\left(\frac{1+2^{k-1}}{2^k}\pi\right)$, the relation $u \dashv_{A_\mu} v$ holds. For $u = u(t)$ we have that $u'_+(t) = v$. Namely, by Lemma 6, $u \dashv v$ follows; thus we have by $\|v\| = 1$ that $u'_+(t) = \pm v$. That $u'_+(t) = -v$ is impossible one can see a few lines below. As shown above, u and v are also orthogonal in the Euclidean sense. Thus $1 = \alpha(t) = \det[u(t), u'_+(t)] = \det[u, v] = c \cdot c$. Finally we conclude that $c = 1$. Hence our induction argument is completed, showing that $m(\phi) = m_2(\phi)$ for all $0 \leq \phi \leq 2\pi$, since m and m_2 are both continuous.

Thus the Minkowskian metric coincides with the introduced Euclidean metric, and μ coincides with the standard angular measure in the Euclidean plane. ■

6 The Equivalence of Glogovskij's Definition with That of a μ-Bisector

The following lemma says that the Glogovskij angular bisector coincides with the Busemann angular bisector in the plane $\mathbb{M}^2_I$ with the isoperimetrix I of the plane $\mathbb{M}^2 = \mathbb{M}^2_B$ as unit ball. Now the isoperimetrix of the introduced Minkowski plane $\mathbb{M}^2_I$ is homothetic to the unit ball B of $\mathbb{M}^2$. Thus this lemma also holds if we interchange the roles of I and B, *i.e.*, if we interchange the roles of A_B and A_G: $A_{G,\mathbb{M}^2_I} = A_{B,\mathbb{M}^2}$.

Lemma 8 *If the (original) Minkowski plane $\mathbb{M}^2$ has the isoperimetrix I and the Minkowski plane $\mathbb{M}^2_I$ has the unit ball I, then we have*

$$A_{B,\mathbb{M}^2_I} = A_{G,\mathbb{M}^2}.$$

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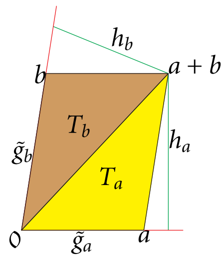

Figure 4: Sketch to Proof of Lemma 8

Proof Let us consider two vectors a and b which are non-collinear with 0 and which have unit length in $\mathbb{M}_I^2$, *i.e.*,

$$\|a\|_I = \|b\|_I = 1 \ .$$

We must show that the two angular bisectors of the angle $T = \angle a0b$, namely

$$A_{G,\mathbb{M}^2}(T) = \{c \in T : \varrho(c, \langle 0, a\rangle) = \varrho(c, \langle 0, b\rangle)\}$$

and

$$A_{B,\mathbb{M}_I^2}(T) = [0, a+b\rangle,$$

coincide since for fixed a, b there are unique bisectors. Since both these bisectors are rays, it is sufficient to show that

$$\varrho(a+b, \langle 0, a\rangle) = \varrho(a+b, \langle 0, b\rangle). \tag{2}$$

The parallelogram with vertices $0, a, b$ and $a+b$ contains two triangles T_a and T_b with vertices $0, a, a+b$ and $0, b, a+b$, respectively, which both have the same area (see also Figure 4). As in the Euclidean case, there is a nice formula for the area of a triangle in a Minkowski plane, see [1, Theorem 7]:

$$v = \frac{1}{2} h\tilde{g},$$

where v is the area measured in a fixed Euclidean background metric, h is the Minkowskian height of the triangle with respect to a side of it (*i.e.*, the distance of the opposite vertex to the line containing this side) and $\tilde{g}$ is the length of this side measured in the corresponding plane with I as unit ball. (I is obtained from the polar reciprocal of B at the Euclidean unit disc after a rotation by $\pi/2$.)

Applied to the triangles T_a and T_b and sides $0a$ and $0b$, respectively, this yields

$$v_a = h_a\tilde{g}_a \quad = \quad v_b = h_b\tilde{g}_b,$$

where $\tilde{g}_a = \|a\|_I = 1 = \|b\|_I = \tilde{g}_b$. Thus the Minkowskian heights h_a and h_b are equal which, in fact, is equation (2). ■

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Proof of Theorem 3 We have that $A_{G,\mathbb{M}^2} = A_{\mu,\mathbb{M}^2}$ in the Minkowski plane $\mathbb{M}^2$ with the unit ball B if and only if $A_{B,\mathbb{M}_I^2} = A_{G,\mathbb{M}^2} = A_{\mu,\mathbb{M}^2} = A_{\mu',\mathbb{M}_I^2}$, where $\mathbb{M}_I^2$ is meant as above and the angular measure μ' defined on I is derived from μ (such that for any angle T we have with our extended notion that $\mu(T) = \mu'(T)$, see Lemma 8). This holds if and only if $\mathbb{M}_I^2$ is a Euclidean plane and μ' its standard angular measure due to Theorem 2. The last condition is equivalent to the case that $\mathbb{M}^2$ is a Euclidean plane with $\mu = \mu'$ as its standard angular measure (we can assume that $B = I$). ■

7 Summary

In this paper we considered three different definitions for angular bisectors in a Minkowski plane.

The first two types of bisectors (Busemann's and Glogovskij's angular bisectors) are uniquely determined by the metric of the plane. The third definition involves an angular measure as parameter.

We answered the question when two bisector definitions coincide for the whole plane. See Table 1, where μ_1 and μ_2 are arbitrary angular measures, μ_a and μ_l are the angular measures induced by area and arc length.

$=$	A_G	A_{μ_2}
A_B	$\mathbb{M}^2$ is Radon, see [5]	$\mathbb{M}^2$ is Euclidean $\mu_2 = \mu_a = \mu_l$, see Section 5
A_{μ_1}	$\mathbb{M}^2$ is Euclidean $\mu_1 = \mu_a = \mu_l$, see Section 6	$\mu_1 = \mu_2$, see Lemma 4 for $\mu_l = \mu_a$, $\mathbb{M}^2$ has an equiframed unit circle, see Section 4

Table 1: Characterization of equivalences for angular bisectors in Minkowski planes

Now we can extend our definitions to define angular bisectors to arbitrary Minkowski spaces $\mathbb{M}^d$, $d \geq 3$. Given an angle with apex 0, consider the 2-plane spanned by the two legs, *i.e.,* take the restriction of the space to a 2-plane. Then one can determine the different angular bisectors. For simplicity, we assume that for a μ-bisector we are given a separate angular measure for every two-dimensional subspace. This measure may be the trace of a measure defined on the unit sphere or something like that, but it does not matter.

Using the simple fact that a Minkowski space is Euclidean if and only if each two-dimensional subspace is Euclidean and the corresponding result for Radon-like subspaces (see [5]), Table 2 summarizes the classification of equivalences of two such bisectors.

Open was the question for the equivalence $A_{\mu_l} = A_{\mu_a}$, *i.e.,* if each intersection of the unit sphere (= set of all unit vectors) with a 2-plane through 0 is an equiframed curve. In a separate paper [4] the author will show that all Minkowski spaces with this property are really Euclidean spaces.

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$=$	A_G	A_{μ_2}
A_B	$\mathbb{M}^d$ is Euclidean, see [5]	$\mathbb{M}^d$ is Euclidean $\mu_2 = \mu_a = \mu_l$, see Section 5
A_{μ_1}	$\mathbb{M}^d$ is Euclidean $\mu_1 = \mu_a = \mu_l$, see Section 6	$\mu_1 = \mu_2$, see Lemma 4 for $\mu_l = \mu_a$, ?

Table 2: Characterization of equivalences for angular bisectors in Minkowski spaces $\mathbb{M}^d$, $d \geq 3$

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